Estimation of Transients, Signal Approximation and Blob Detection with Maxima Lines

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General Presentation

- First Part: Definition of the signal decomposition and highlight on some properties of the extrema of such a decomposition (*maxima lines*).

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- Second Part: Estimation-detection of transients with the *maxima lines* of this signal decomposition.
- Third Part: Signal approximation from extrema, comparison with existing wavelet based methods.
- Fourth Part: Blob detection with *maxima lines* of wavelet coefficients.

First Part:Signal Decomposition and Extrema properties

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- The continuous B-spline of order $n > 0$ is:

$$\beta^n(x) = \overbrace{\beta^0 * \beta^0 * \cdots * \beta^0}^{n+1}(x), \tag{4}$$

where β^0 is the characteristic function of the interval $[0, 1]$. The discrete B-spline of order $n > 0$, at scale m, is defined as:

$$b_m^n = \overbrace{b_m^0 * b_m^0 * \cdots * b_m^0}^{n+1}, \tag{5}$$

where $b_m^0 = \frac{1}{m}\{1,\ 1, \cdots, 1\}$ and $m \geq 2$. We also define $b_m^{-1} = \delta_0$. The link between discrete and continuous B-splines is the dilation equation:

$$\frac{1}{m}\beta^n\left(\frac{x}{m}\right) = b_m^n * \beta^n(x). \tag{6}$$

Limit Properties of b_m^n

- Continuous B-splines decomposition at different scales m involve a basis function with a regularity given by the order n of the spline. Here, we are going to filter a discrete signal with b_m^n with fixed m and increasing n. Now we consider that b_m^n is the filter obtained by centering b_m^0 in 0 if m is odd, and in $1/2$ otherwise and then convolving

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- For large n, b_m^n are approximations of a Gaussian function (i.e. C^∞)
  Let X denote a discrete random variable with uniform distribution over the set $\{-\frac{m-1}{2}, \cdots, \frac{m-1}{2}\}$ (resp. $\{-\frac{m}{2}+1, \cdots, \frac{m}{2}\}$) for m odd (resp. even). Then $b_m^n[p]$ is the probability that the sum of $n+1$ independent identically distributed variables X_i is equal to p (convolution of their probability distributions). The mean of the variable X_i is 0 (resp. $\frac{1}{2}$) if m is odd (resp. even), while its standard deviation is $\sqrt{\frac{m^2-1}{12}}$. Applying the central limit theorem we get:

$$\frac{\sum_{i=1}^{n+1} X_i - \epsilon\frac{n+1}{2}}{\sqrt{(n+1)(m^2-1)/12}} \underset{n\to+\infty}{\longrightarrow} N(0,1)$$

in distribution, where $\epsilon = 1$ if m is even and 0 otherwise. For large n:

$$b_m^n[p] \approx \sqrt{\frac{6}{\pi(n+1)(m^2-1)}} \exp\left(\frac{6(p-\epsilon\frac{n+1}{2})^2}{(n+1)(1-m^2)}\right).$$

Filters and Coefficients Definitions

- Let us define $\rho = \{1, -1\}$, $\rho_k = \overbrace{\rho * \cdots * \rho}^{k}$, $b^n_{m,k} = \rho_k * b^n_m$. The following approximation holds, for large n:

$$b^n_{m,k}[p] \approx \sqrt{\frac{6}{\pi(n+1)(m^2-1)}} \left[\exp\left(\frac{6x^2}{(n+1)(1-m^2)}\right)\right]^{(k)} \left(p - \frac{\epsilon(n+1)+k}{2}\right).$$

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- An approximation of the kth derivative of the Gaussian function is obtained by shifting the filter $b^n_{m,k}$ properly :

$$\alpha^n_{m,k}[p] = b^n_{m,k}[p + \lfloor \frac{\epsilon(n+1)+k}{2} \rfloor]$$

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- We consider the correlation of the sequence $f[j]$ with $\alpha^n_{m,k}[j]$:

$$\begin{aligned} \forall p \in \mathbb{Z} \quad c^n_{m,k}[p] &= \sum_{j\in\mathbb{Z}} \alpha^n_{m,k}[j] f[j+p] \\ &= \sum_{j\in\mathbb{Z}} \alpha^n_{m,k}[j-p] f[j] \text{ for } n \geq -1 \end{aligned}$$

Definition of the Extrema, Study for $m = 2$

- The maxima (resp. minima) of the sequence $c^n_{m,k}$ are the strictly positive (resp. negative) coefficients $c^n_{m,k}[p]$ such that:

$$c^n_{m,k}[p-1] \leq (\text{ resp. } <) c^n_{m,k}[p] > (\text{ resp. } \geq) c^n_{m,k}[p+1]$$
$$(\text{resp. } c^n_{m,k}[p-1] \geq (\text{ resp. } >) c^n_{m,k}[p] < (\text{ resp. } \leq) c^n_{m,k}[p+1]).$$

  When $m = 2$, an extremum at scale n arises from a unique extremum of the same nature at scale $n-1$ (Berkner '99). These extrema define curves in the time-scale space which are called *maxima lines*.

- The way the extrema propagate is given by (M. et al '07):

  Theorem 1 *If $n+k$ is odd (resp even) and if there is an extremum at p in scale n then there is an extremum of the same nature either at p or $p-1$ (resp. at p or $p+1$) at scale $n-1$.*

Finite Signal Derivatives Reconstruction from Multiscale Extrema Representation

- With our notations, the kth order discrete derivative of f is $c_{2,k}^{-1}$.
- The multiscale extrema representation of the kth order signal derivative up to scale n enables us to compute many coefficients at smaller scales (M. et al '07):

Theorem 2 *Assume that $c_{2,k}^{n}[p]$ is an extremum in scale n and that the value of the coefficients along the corresponding maxima line is known up to scale n, then we can compute $c_{2,k}^{n-2q}[l]$ for $l \in \{p-q, \cdots, p+q\}$, $c_{2,k}^{n-2q+1}[l]$ for $l \in \{p-q, \cdots, p+q-1\}$, when $n+k$ is odd and for $l \in \{p-q+1, \cdots, p+q\}$ when $n+k$ is even.*

- Theorem 2 tells us that the knowledge of $c_{2,k}^{l}[q]$ along the maxima line corresponding to the extremum at p in scale n enables to compute $c_{2,k}^{-1}[l]$ for $l \in I_k^n[p]$.

Theorem 3 *Assume that f has a finite support, that it is not the null sequence and that $\mathcal{S}(c_{2,k}^{-1})$ is the support of $c_{2,k}^{-1}$. Let us denote M_k^n the set of indices corresponding to R_k^n extrema in scale n. Then the smallest scale for the reconstruction of $c_{2,k}^{-1}$ with our scheme is:*

$$N_k = \underset{N}{argmin} \left(\bigcup_{n=-1}^{N} \bigcup_{r=1}^{R_k^n} I_k^n \left[M_k^n[r] \right] \subset \mathcal{S}(c_{2,k}^{-1}) \right).$$

Second Part: Estimation-Detection of Transients

Definition 1 *We call transient some sudden variation in a signal which can be of different kind.*

Definition 2 *The estimation of transients is the problem of the determination of the time when the transient occurs.*

Definition 3 *The detection of transients is the problem of the existence of a transient.*

Remark: robust detection and good estimation are somehow contradictory since the former requires a rather large scale for time integration while the second requires a rather small scale for time integration

Definition of the variables of interest (M. et al ’05)

- As each extremum of the sequence $c_{2,k}^n$ belongs to a single *maxima line*, we associate with each *maxima line* $\mathcal{L}_k$ in the time-scale space (where the time is indexed by p and the scale by n) the variable $D_{\mathcal{L}_k,q}$, with $|\mathcal{L}_k| \geq q+2$ (where $|\mathcal{L}_k|$ is the length of $\mathcal{L}$) defined by:

$$D_{\mathcal{L}_k,q} = \sum_{(p,n)\in\mathcal{L}_k, n\leq q} \frac{\left(c_{2,k}^n[p]\right)^2}{\|\alpha_{2,k}^n\|_2^2}.$$

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- In practice, the variable $D_{\mathcal{L}_k,q}$ is not sufficient to properly characterize frequencies changes in the signal. We therefore add another variable:

$$F_{\mathcal{L}_k,q} = O((\mathcal{L}_k)_+(q)) - O(\mathcal{L}_k),$$

  where $O(\mathcal{L}_k)$ is the origin of the maxima line $\mathcal{L}_k$ and $(\mathcal{L}_k)_+(q)$ is the maxima line that follows $\mathcal{L}_k$ at rank q.

Principle of the Estimation-Detection

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- For any probability Pr, the empirical distribution of $D_{\mathcal{L}_k,q}$ provides a_q and b_q such that $P(a_q < D_{\mathcal{L}_k,q} < b_q) = Pr$. For each maxima line $\mathcal{L}_k$ such that $|\mathcal{L}_k| \geq q+2$ we have the standard choice between:

$$H_0(q) : D_{\mathcal{L}_k,q} \text{ is in } [a_q, b_q]$$
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- The variable $F_{\mathcal{L}_k,q}$ takes integer values and we compute its distribution for each $q \leq N_k$. Any probability Pr defines a subset $A(q)$ of $\mathbb{N}$ such that $A(q) = \{x, P(F_{\mathcal{L}_k,q} = x) > 1 - Pr\}$. For each maxima line, we again have the choice between:

$$H_0'(q) : F_{\mathcal{L}_k,q} \text{ is in } A(q)$$
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- Note that from a detection point of view $1 - Pr$ is the probability of false alarm.

Non-Parametric Test for Estimation-Detection

- We build a non-parametric test to estimate the transients. We consider maxima lines $\mathcal{L}_k$ such that $O(\mathcal{L}_k)$ is inside $[T-d, T+d]$. If $|\mathcal{L}_k| \geq q+2$, four cases may occur:

i) $\mathcal{L}_k$ satisfies $H_0(q) \cup H_0'(q)$

ii) $\mathcal{L}_k$ satisfies $H_1(q) \cup H_0'(q)$

iii) $\mathcal{L}_k$ satisfies $H_0(q) \cup H_1'(q)$

iv) $\mathcal{L}_k$ satisfies $H_1(q) \cup H_1'(q)$

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- Scanning interval $[T-d, T+d]$,the first line $\mathcal{L}_k$ that satisfies hypothesis ii) or iv) in scale q corresponds to a transient $T_1(q) = O(\mathcal{L}_k)$, while the first line $\mathcal{L}_k$ that satisfies iii) or iv) corresponds to a transient $T_2(q) = O(\mathcal{L}_k)$.

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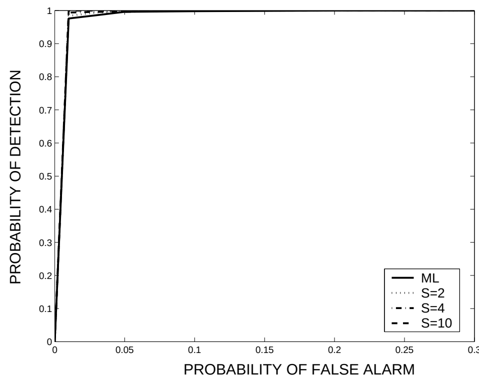
- We have two vectors T_1 and T_2 for which the best scales q are those that maximize the probability of transition. If we denote q_1 (resp. q_2) the scale associated with T_1 (resp. T_2), we choose between $T_1(q_1)$ and $T_2(q_2)$ taking the one with the highest probability of transition. The estimated transition $\hat{T}$ is then:

$$\hat{T} = argmax \left\{ P(\tilde{T}), \tilde{T} \in \{T_1(\underset{q \leq N_k}{argmax}\, P(T_1(q))), T_2(\underset{q \leq N_k}{argmax}\, P(T_2(q)))\} \right\}$$

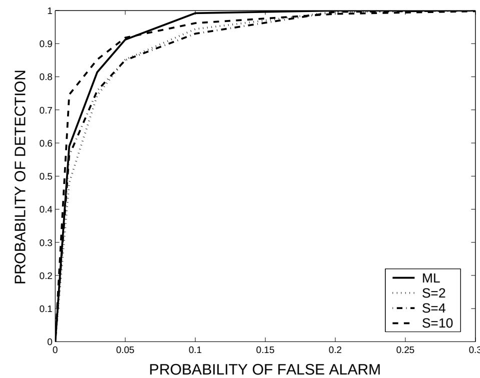
Comparison with Wavelet Based Methods

- Let us define $Y[b] = \left(Y_s[b] = \left(f(t), \frac{1}{s}\Psi(\frac{t-b}{s})\right)\right)_{s \leq S}$. If f is Gaussian, Y is a Gaussian vector with zero mean components characterized by its covariance matrix $\sum$. In such a case, $Z[b] = Y[b](\sum)^{-1}Y[b]$ is χ_2 distributed with S degrees of freedom.
- For Gaussian signals, we compare the wavelet approach to our approach for a variation in frequency or a variation in amplitude of the signal.
- We study both amplitude and frequency variations

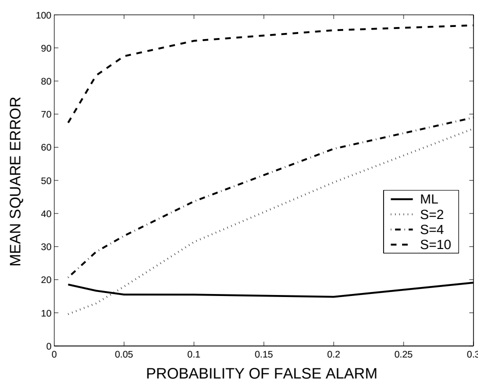
Results



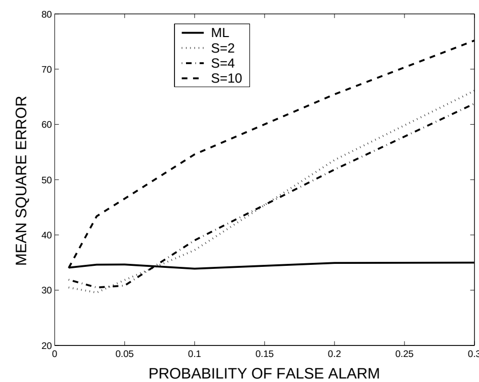
(A)



(B)

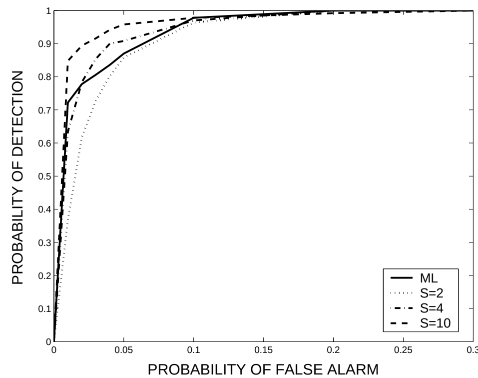


(C)



(D)

(A):ROC curves ($\beta = 3{,}20 - 400$ Hz) (B): idem but for $\beta = 1.5$. (C) Mean square error ($\beta = 3{,}20 - 400$ Hz) (D):idem but for $\beta = 1.5$



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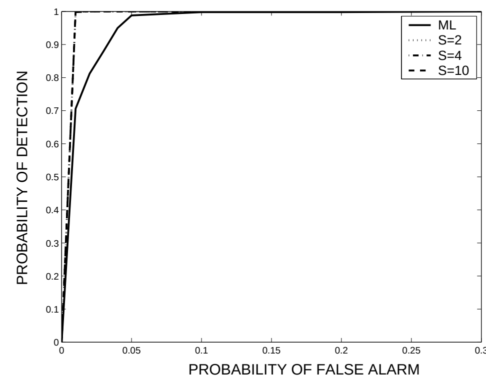

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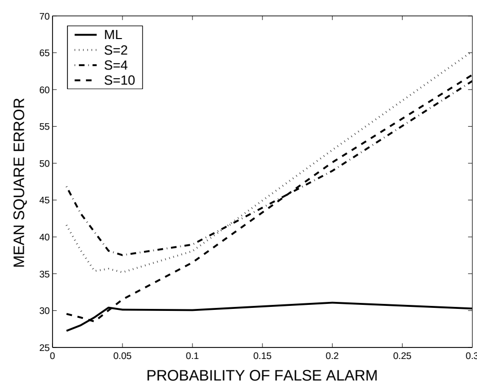
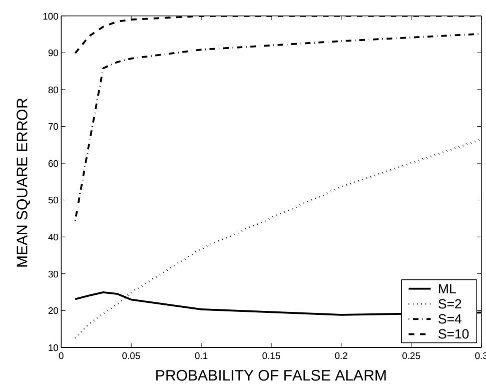


(C)

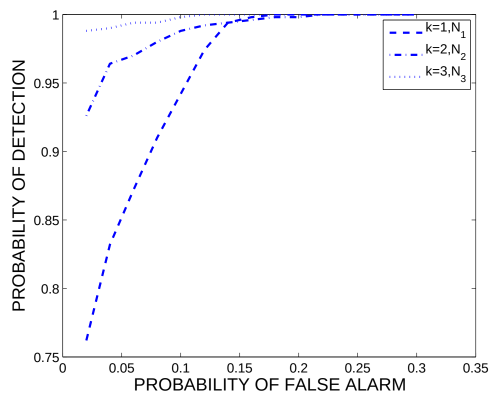


(D)

(A): ROC curves (variation of frequency from 20-200 Hz to 200-400 Hz), (B): idem for a variation of frequency from 200-400 Hz to 20-200 Hz, (C): Mean square error for a variation from 20-200Hz to 200-400 Hz, (D): idem for a variation from 200-400 Hz to 20-200 Hz

Perspective for Transients Estimation

- The choice of the order of the derivative for the estimation of transients should be further discussed. To use the *maxima lines* associated to higher order derivatives may be interesting in some instances.



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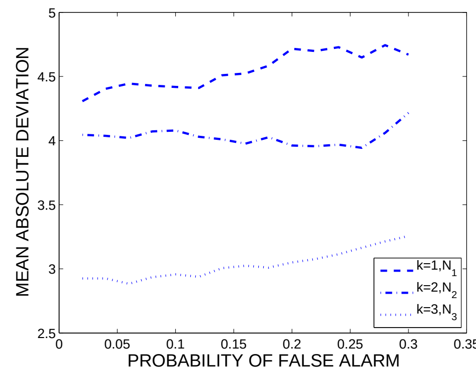

(B)

(A): Computation of ROC curves for $k = 1$ and N_0, and for $k = 1$ and N_1, for $k = 2$ and N_2 and for $k = 3$ and N_3, for a frequency bandwidth change from 20-200Hz to 200-400Hz, (B): Mean absolute deviation (in ms) for the same parameters and the same frequency bandwidth change as in A

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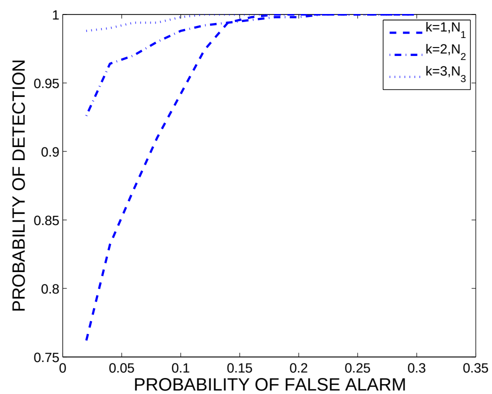

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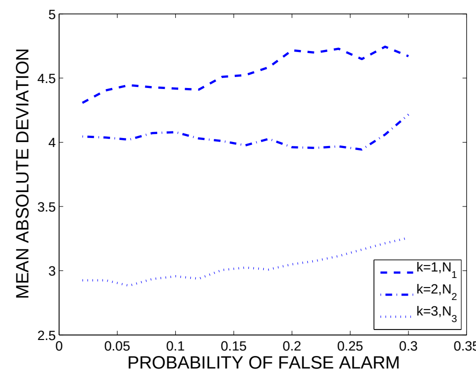

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- For bidimensional problems, one can simply consider *maxima lines* of the decomposition of the rows and of the columns.

Third Part: Signal Approximation from Multiscale Extrema

- For $m = 2$, the decomposition we use satisfies, for $-1 \leq l \leq N-1$ (M. et al '05):

$$c^l_{2,k-1}[p] = c^N_{2,k-1}[p + \lfloor \frac{N-l}{2} \rfloor] + \frac{1}{2}\sum_{n=l}^{N-1} c^n_{2,k}[p + \lfloor \frac{n-l+1}{2} \rfloor] \text{ l+k even}$$

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\end{aligned}
$$

- Whatever m one can show that $c^{n}_{m,k}$ tends to zero with n, leading to the approximation ($m = 2, l = -1$):

$$
\begin{aligned}
c^{-1}_{2,k-1}[p] &\approx \frac{1}{2}\sum_{n=-1}^{N-1} c^{n}_{2,k}[p + \lfloor \frac{n+2}{2} \rfloor] = d^{-1,N}_{2,k-1}[p] \text{ k odd} \\
c^{-1}_{2,k-1}[p] &\approx \frac{1}{2}\sum_{n=-1}^{N-1} c^{n}_{2,k}[p + \lfloor \frac{n+3}{2} \rfloor] = d^{-1,N}_{2,k-1}[p] \text{ otherwise.}
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- For $m = 2$, the decomposition we use satisfies, for $-1 \leq l \leq N-1$ (M. et al '05):

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- Whatever m one can show that $c^n_{m,k}$ tends to zero with n, leading to the approximation ($m = 2, l = -1$):

$$c^{-1}_{2,k-1}[p] \approx \frac{1}{2}\sum_{n=-1}^{N-1} c^n_{2,k}[p + \lfloor \frac{n+2}{2} \rfloor] = d^{-1,N}_{2,k-1}[p] \text{ k odd}$$

$$c^{-1}_{2,k-1}[p] \approx \frac{1}{2}\sum_{n=-1}^{N-1} c^n_{2,k}[p + \lfloor \frac{n+3}{2} \rfloor] = d^{-1,N}_{2,k-1}[p] \text{ otherwise.}$$

- This defines a converging approximation of the $k-1$th derivative from the kth derivative.

Algorithm for Signal Reconstruction

from the Multiscale Extrema of the k-th order Derivative

- The signal approximation from the kth order signal derivative multiscale extrema representation we introduced is carried out using the following procedure:
  - First, reconstruct the kth order signal derivative from its multiscale extrema representation
  - Then, iteratively build approximations of the derivatives of order $k-1, \cdots, 1$ and finally of the signal f
- In practice, the approximation of the signal f is better from the first derivative than from higher order derivatives. Indeed, in these later cases, only an approximation of the first derivative (obtained from the kth derivative) is used to approximate f.
- The quality of the approximation increases with N.

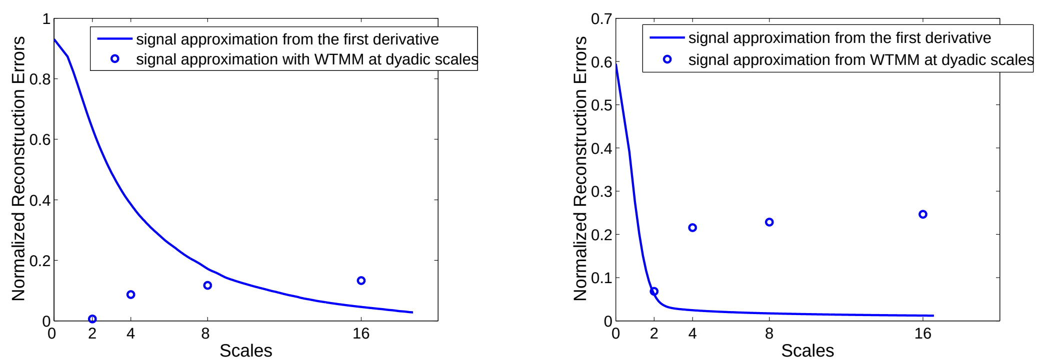
State of the Art on Signal Approximation from Wavelet Transform Modulus Maxima (WTMM)

- Image and signal approximation from wavelet transform modulus maxima (WTMM) representation or from zero-crossing representation is very popular. These representations are based on irregular sampling of the continuous dyadic scale wavelet transform at points which correspond to singularities in the signal.
- Berman (Berman '92) and Meyer (Meyer '91) showed that the corresponding (discrete-time or continuous) signal is not, in general, unique.
- The uniqueness is related to the completeness of the wavelet sampling basis in the signal subspace (Liew et al. '95).
- It is however possible to build a consistent signal having the same properties as those of the original signal described by the WTMM representation (projection-based methods (Mallat ' 97, Liew et al. '95, Cetin '94), the conjugate gradient error minimization method (Law et al. '97) and the least square eigenspace method (Liew et al. '00))).
- For the 2-D WTMM reconstruction problem, different methods were proposed based on projection onto a convex space (Mallat et al. '92, Liew et al. '97).
- Czetkovic (Czetkovic et al. '95) proposed a consistent discrete-time signal reconstruction from discrete-time wavelet transform extrema and zero-crossings.

Numerical Applications:

Comparison with WTMM implemented in WaveLab

- n of our decomposition corresponds to the scale $\sqrt{\frac{n+1}{2}}$ (since $m = 2$ in our case) in usual sense. Computation of the reconstruction error



(A): Computation of the reconstruction error for signals with frequency bandwidth 20-200Hz ;
(B): idem but for frequency bandwidth 200-400Hz.

- WTMM at dyadic scale is designed to approximate signals with isolated singularities.
- The quality of the approximation worsens when unnecessary dyadic scales are taken into account. WTMM do not allow for the construction of *maxima lines* that are useful for other purposes.
- Perspective: we are currently working on a bidimensional extension of our approximation scheme.

Fourth Part: Blob Detection with Wavelet Maxima Lines

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- We propose a novel approach to blob detection based on the study of wavelet transform modulus maxima along *maxima lines*.
- The algorithm we propose enables automatic blob detection and blob size determination (robustness to noise).

Linear Scale-Space, Scale selection and Wavelet Decomposition

- Let $g(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$ and $g_t(x) = \frac{1}{\sqrt{t}} g(\frac{x}{\sqrt{t}})$, the linear scale-space is $L(x,t) = g_t * f(x) = \int_{\mathbb{R}} \frac{1}{\sqrt{t}} g(\frac{x-u}{\sqrt{t}}) f(u) du.$

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$$\partial^m_{x,\gamma norm} L(x,t) = t^{\frac{\gamma m}{2}} \partial^m_x L(x,t)$$

$$= \int_{\mathbb{R}} t^{\frac{\gamma m}{2}} \frac{1}{\sqrt{t}} g\left(\frac{u}{\sqrt{t}}\right) f^{(m)}(x-u) du = t^{\frac{\gamma m}{2}} f^{(m)} * g_t(x).$$

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- Scale selection can be carried out studying the modulus maxima of the CWT along maxima lines of interest.

The Bidimensional Case

- $g(x_1, x_2) = \frac{1}{2\pi} \exp(-\frac{x_1^2+x_2^2}{2})$ and $g_t(x_1, x_2) = \frac{1}{t} g(\frac{x_1}{\sqrt{t}}, \frac{x_2}{\sqrt{t}})$.
- The bidimensional linear scale-space definition is identical to its one dimensional counterpart.
- Let us define $\partial_{x^\alpha} L = L_{x^\alpha} = \partial_{x_1^{\alpha_1} x_2^{\alpha_2}} L = L_{x_1^{\alpha_1} x_2^{\alpha_2}}$, with $\alpha = (\alpha_1, \alpha_2)$. We, then, consider differentiations of the linear scale-space of the form $\mathcal{D}L = \sum_{j=1}^{I} c_j L_{x^{\alpha^j}}$, where $|\alpha^j| = \alpha_1^j + \alpha_2^j = M$ is independent on j.
- For such differentiations of the linear scale-space, the appropriate normalization of the operator $\mathcal{D}$ is

$$\mathcal{D}_{x,\gamma norm} L = t^{\frac{M\gamma}{2}} \mathcal{D}L.$$

- The modulus maxima of $\mathcal{D}_{x,\gamma norm} L$ define maxima lines in scale and space.
- Blob detection is carried out stuying the maxima of $t|\Delta L|$.

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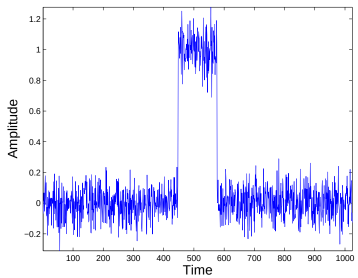
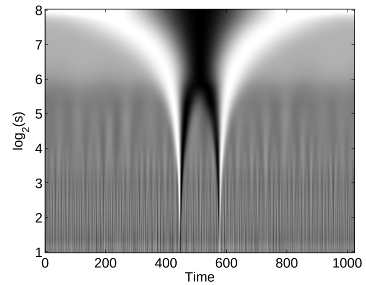
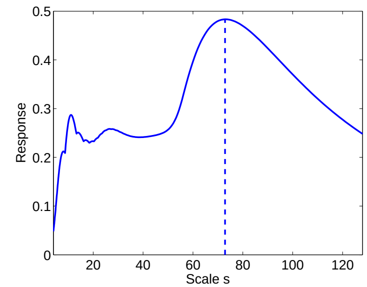
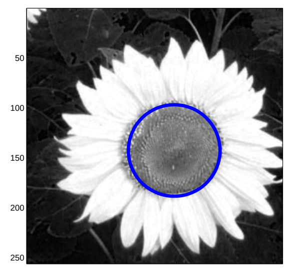
- The maxima line characteristic scale is $S^* = \sqrt{(x^* - x_0)^2 + (y^* - y_0)^2}$
- Determination of interest points: group the maxima lines $\{L_i, i \in I\}$ that join at (x^*, y^*, s). The interest point is finally given by its location (x^*, y^*) and its characteristic scale $S_m^* = median\{S_i^*, i \in I\}$. If the set I contains only one element, the interest point is (x^*, y^*, S_m^*).

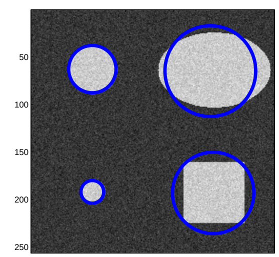
Examples











Perspective

- Maxima lines approach with nonlinear operator (determinant of the Hessian matrix of L for corner detection)
- Problem with the stability criterion when the blob are too close
- Definition of object descriptors based on *maxima lines* (Lipschitz regularity of edges, blob location and scale,..)