Application of the Convergence of the Control Points of B-splines to Wavelet Decomposition at Rational Scales and Rational Location

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Abstract- In this letter, we first recall the relation between discrete B-splines and the control points of B-splines. We then use some convergence properties of the control points of B-splines to derive a new algorithm that approximates the wavelet decomposition at rational scales and rational location. We conclude the paper by some numerical experiments illustrating the behavior of our algorithm and by a discussion on an efficient implementation of the proposed method.

Keywords- Discrete B-spline Decomposition, B-splines.

EDICS Category:TFSR,FAST

I. INTRODUCTION

B-splines are a common tool for signal representation, since they are good approximations of Gaussian functions [3] and have a compact support. Many studies have been carried out to implement efficiently digital filters associated with B-spline decomposition through the use of what are called discrete sampled B-splines [4] [5]. In [5], an algorithm is proposed to compute an approximation of the continuous wavelet transform at rational scales and integer location using discrete sampled B-splines.

B-splines can also be more efficiently approximated by sampling the spline of order zero and then by making repeated discrete convolutions [1]. The filter associated with such a construction is often called a discrete B-spline. A discrete B-spline can be viewed as the vector of the control points of B-splines for which convergence properties are well known [2]. We use these convergence properties to build an approximation of the wavelet decomposition at rational scales and rational location following the same kind of approach as that of Wang [5]. We establish the link between a good approximation and the length of the filters, which we also relate to the computational cost of the algorithm.

The outline of the paper is as follows. First, we recall the definition of discrete B-splines, their relation with the control points of B-splines and some convergence properties. Section III is devoted to the comparison of B-splines and of discrete B-splines while, in section IV, we derive a new approximation for the wavelet decomposition at rational scales and rational location. In the last part of the paper, we give some numerical applications of our algorithm and we suggest future development on the subject.

II. CONVERGENCE PROPERTIES OF DISCRETE B-SPLINES

The B-spline of order $n > 0$ with knots $a_k = k$ is defined by:

$$N^n(x) = \overbrace{N^0 * N^0 * \cdots * N^0}^{n+1} \tag{1}$$

where N^0 is the characteristic function of the interval $[0, 1]$. The discrete B-spline of order $n > 0$, at scale m, is defined by [5]:

$$b_m^n = \overbrace{b_m^0 * b_m^0 * \cdots * b_m^0}^{n+1} \tag{2}$$

where $b_m^0 = \frac{1}{m}\{1,\ 1, \cdots, 1\}$ is the normalized sampled pulse of width $m \geq 2$ with support $\{0, \cdots, m-1\}$. The link between discrete and B-splines is the following refinement equation:

$$\frac{1}{m}N^n(\frac{t}{m}) = b_m^n * N^n(t) = \sum_{k\in\mathbb{Z}} b_m^n[k]N^n(t-k) \tag{3}$$

which can be rewritten as:

$$N^n(t) = \sum_{k\in\mathbb{Z}} mb_m^n[k]N^n(mt-k) = \sum_{k\in\mathbb{Z}} mb_m^n[k]N_k^n(t)$$

where $(N_k^n)_{k\in\mathbb{Z}}$ are the B-splines with knots $(\frac{k}{m})_{k\in\mathbb{Z}}$. Consequently, $(mb_m^n[k])_{k\in\mathbb{Z}}$ are the control points of N^n with knots $(\frac{k}{m})_{k\in\mathbb{Z}}$. It was shown by Schaback [2] that the control points converge quadratically to N^n in the following way:

$$\max_k |N^n(\frac{k}{m}+\frac{n+1}{2m}) - mb_m^n[k]| = O(\frac{1}{m^2}). \tag{4}$$

Let us now define the piecewise constant function:

$$\begin{array}{lll} M_m^n(t) = mb_m^n[k] & \frac{k}{m}+\frac{n+1}{2m} \leq t < \frac{k+1}{m}+\frac{n+1}{2m} & \text{for } n \text{ odd} \\ M_m^n(t) = mb_m^n[k] & \frac{k}{m}+\frac{n}{2m} \leq t < \frac{k+1}{m}+\frac{n}{2m} & \text{for } n \text{ even} \end{array} \tag{5}$$

A Taylor expansion of N^n leads to $\|M_m^n - N^n\|_\infty = O(\frac{1}{m})$, the convergence being faster where the derivative of N^n is small.

III. Discrete B-spline Decomposition of a Signal

In this section we investigate how to derive some properties of discrete B-splines from properties of B-splines. The interest for B-splines lies in the properties of the polynomial spaces:

$$S_h^n = \left\{ g(x) = \sum_{k\in\mathbb{Z}} c[k]\frac{1}{h}N^n(\frac{x}{h}-k), c \in l^2(\mathbb{Z}), h > 0 \right\}.$$

Indeed, they satisfy a stability property through dilation and their union is dense in $L^2(\mathbb{R})$ [5]:

$$S_{im}^n \subset S_m^n \quad \forall i \in \mathbb{N}^* \tag{6}$$

and

$$\overline{\bigcup_{h>0} S_h^n} = L^2(\mathbb{R})$$

Let us now study the discrete case. If we consider integer values for h denoted by m, and if we bear in mind the convergence property enounced in (4), it is natural to study the properties of the subspaces:

$$\mathcal{D}_m^n = \left\{ \alpha_m^n[l] = \sum_{k\in\mathbb{Z}} c[k]b_m^n[l-mk], c \in l^2(\mathbb{Z}), l \in \mathbb{Z} \right\}$$

First, we note that the Fourier series of α_m^n is in $L^2([-\frac{1}{2m},\frac{1}{2m}[)$ (the full demonstration is given in Appendix A). Secondly, as in the continuous case, the embedding property (6) holds. To prove it, it suffices to rewrite the refinement equation (3) in two different ways; the first one is:

$$\frac{1}{im}N^n\left(\frac{x}{im}\right) = \sum_{k\in\mathbb{Z}} b_{im}^n[k]N^n(x-k)$$

and the second:

$$\begin{aligned} \frac{1}{im}N^n\left(\frac{x}{im}\right) &= \sum_{k\in\mathbb{Z}} b_i^n[k]\frac{1}{m}N^n(\frac{x}{m}-k) \\ &= \sum_{k\in\mathbb{Z}} b_i^n[k]\sum_{l\in\mathbb{Z}} b_m^n[l]N^n(x-mk-l) \\ &= \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} b_i^n[k]b_m^n[l-mk]N^n(x-l) \end{aligned}$$

As a result, $b_{im}^n[l] = \sum_{k\in\mathbb{Z}} b_i^n[k]b_m^n[l-mk]$ which means that $\mathcal{D}_{im}^n \subset \mathcal{D}_m^n$.

The third property is that any signal $f \in l^2(\mathbb{Z})$ can be decomposed over $\mathcal{D}_m^0$. It suffices to prove how to reconstruct the Dirac sequence at zero (i.e. the sequence $\delta[k] = 1$ if and only if $k = 0$) to be able to reconstruct any function of $l^2(\mathbb{Z})$. In the case $n = 0$, for $l \in \{0, \cdots, m-1\}$, we have $b_m^0[l - mk] = \frac{1}{m}\delta[k]$, where δ is the Dirac sequence at 0. Consequently, we have $\delta[k] = \sum_{l=0}^{m-1} b_m^0[l - mk]$, which concludes the proof. So, for any $m \geq 2$, $\mathcal{D}_m^0$ can be viewed as different writings of $l^2(\mathbb{Z})$, while $\mathcal{D}_m^n$ is an approximation at scale n of $l^2(\mathbb{Z})$.

IV. Derivation of a Discrete Wavelet Decomposition at Rational Scales and Rational Location

We use here the analogy between discrete B-splines and B-splines to derive a new algorithm to approximate wavelet decomposition at rational scales and rational location. We first recall the approach of Wang et al. [5] which consists in approximating the wavelet decomposition using B-splines. Let consider a signal f and a wavelet ψ both in $L^2(\mathbb{R})$ which are projected on a spline basis of respective order n_1 and n_2, to obtain the following approximations:

$$f(x) \approx \sum_{k \in \mathbb{Z}} c[k] N^{n_1}(x-k) \text{ and } \psi(x) \approx \sum_{k \in \mathbb{Z}} g[k] N^{n_2}(x-k).$$

In practice, we often consider an approximation for f such that $f(l) = c * N^{n_1}(l)$, c being computed recursively using causal and anti-causal filters [4]. Note that if f is in l^2 so is c. For filter g one uses a finite difference of order p to obtain an approximation of the pth derivative of the Gaussian function [5].

We now recall that the wavelet decomposition of f over ψ at scale s is defined by:

$$Wf(s,x) = \int f(t) \frac{1}{s} \psi(\frac{t-x}{s}) dt$$

Any real scale s can be approximated by a rational $\frac{m_1}{m_2}$, so that it is worth studying:

$$Wf(\frac{m_1}{m_2}, t) = \left(\psi_{\frac{m_1}{m_2}} * f\right)(t) \approx \frac{m_2}{m_1} \sum_k \sum_l g[k] c[l] N^{n_2}\left(\frac{m_2}{m_1} t - k\right) * N^{n_1}(t-l). \tag{7}$$

The term on the right side of equation (7) can be computed exactly at integer location [5] (i.e $t = p$ integer) and is equal to:

$$\frac{m_2}{m_1} \sum_k \sum_l g[k] c[l] N^{n_2}\left(\frac{m_2}{m_1} t - k\right) * N^{n_1}(t-l)[p] = m_2 \left(b^{n_1+n_2+1} * b_{m_2}^{n_1} * b_{m_1}^{n_2} * c_{\uparrow m_2} * g_{\uparrow m_1}\right)_{\downarrow m_2}[p],$$

where $b^{n_1+n_2+1}[k] = N^{n_1+n_2+1}[k]$. We now turn to the discrete case. As m_1 and m_2 can be chosen arbitrarily large provided their ratio is constant, the following approximation:

$$Wf(\frac{m_1}{m_2}, t) \approx \frac{m_2}{m_1} \sum_k \sum_l g[k] c[l] M_{m_1}^{n_2}\left(\frac{m_2}{m_1} t - k\right) * M_{m_2}^{n_1}(t-l).$$

makes sense for m_1 and m_2 sufficiently large, M_m^n being defined in (5). Note that this approximation converges uniformly to the expression on the right side of (7). We now prove that this new approximation of the wavelet decomposition leads to an efficient implementation of the wavelet decomposition at rational location. Let us rewrite formally the convolution product and then discuss the validity of the derived expression:

$$\begin{aligned}
M_{m_1}^{n_2}\left(\frac{m_2}{m_1}t-k\right) * M_{m_2}^{n_1}(t-l)(t) &= \frac{1}{m_2}\int_{\mathbb{R}} M_{m_1}^{n_2}(\frac{u}{m_1}-k)M_{m_2}^{n_1}(t-\frac{u}{m_2}-l)du \\
&= \frac{1}{m_2}\sum_{q\in\mathbb{Z}} m_1 b_{m_1}^{n_2}[q]\int_{q+\frac{n_2+\epsilon_2}{2}+km_1}^{q+1+\frac{n_2+\epsilon_2}{2}+km_1} M_{m_2}^{n_1}(t-\frac{u}{m_2}-l)du \\
&= \frac{1}{m_2}\sum_{q\in\mathbb{Z}} m_1 b_{m_1}^{n_2}[q]\int_{-q-1-\frac{n_2+\epsilon_2}{2}-km_1+(t-l)m_2}^{-q-\frac{n_2+\epsilon_2}{2}-km_1+(t-l)m_2} M_{m_2}^{n_1}(\frac{u}{m_2})du \\
&= \sum_{q\in\mathbb{Z}} m_1 b_{m_1}^{n_2}[q] b_{m_2}^{n_1}[-(q+1)-\frac{n_2+\epsilon_2}{2}-km_1+(t-l)m_2-\frac{n_1+\epsilon_1}{2}] \\
&= \sum_{q\in\mathbb{Z}} m_1 b_{m_1}^{n_2}[q-km_1-\frac{n_2+\epsilon_2}{2}] b_{m_2}^{n_1}[-q-1+(t-l)m_2-\frac{n_1+\epsilon_1}{2}].
\end{aligned}$$

where $\epsilon_1 = 1$ (resp. $\epsilon_1 = 0$) if n_1 (resp. n_2) is odd and zero otherwise. This expression only makes sense if $t = \frac{i}{m_2}$ where i is an integer. Using the previous expression, we have the following approximation for $Wf(\frac{m_1}{m_2}, \frac{i}{m_2})$:

$$\begin{aligned}
\forall i \in \mathbb{Z} \quad Wf(\frac{m_1}{m_2}, \frac{i}{m_2}) &\approx \sum_{q\in\mathbb{Z}} m_2 \sum_{k\in\mathbb{Z}} g[k] b_{m_1}^{n_2}[q-km_1-\frac{n_2+\epsilon_2}{2}] \sum_{l\in\mathbb{Z}} c[l] b_{m_2}^{n_1}[-q-1+i-lm_2-\frac{n_1+\epsilon_1}{2}] \\
&= m_2 \sum_{q\in\mathbb{Z}} \left(g_{\uparrow m_1} * b_{m_1}^{n_2}[q-\frac{n_2+\epsilon_2}{2}]\right)\left(c_{\uparrow m_2} * b_{m_2}^{n_1}[-q-1-\frac{n_1+\epsilon_1}{2}+i]\right) \\
&= m_2 g_{\uparrow m_1} * b_{m_1}^{n_2} * c_{\uparrow m_2} * b_{m_2}^{n_1}[i-1-\frac{n_1+\epsilon_1}{2}-\frac{n_2+\epsilon_2}{2}] = D(\frac{m_1}{m_2}, \frac{i}{m_2}).
\end{aligned}$$

If we refer to the study of section II A, we can associate with $g_{\uparrow m_1} * b_{m_1}^{n_2}$ (resp. $c_{\uparrow m_2} * b_{m_2}^{n_1}$) a function of $L^2([-\frac{1}{2m_1}, \frac{1}{2m_1}[)$ (resp. $L^2([-\frac{1}{2m_2}, \frac{1}{2m_2}[)$). We note that a first difference between the computation of the approximation at integer location [5] and at rational location that we propose is the absence of the factor $b^{n_1+n_2+1}$. We may say that we get more information with less computation. However, while we have an exact computation at integer location with the Wang et al. method, the results at rational location we obtain are only asymptotic. In other words, it imposes that m_1 and m_2 be sufficiently large. The question we have to answer is how large must m_1 and m_2 be to lead to a good approximation. We tackle this issue in the following results section.

V. RESULTS

According to the study made by Ichige [1] on the convergence of M_m^n to B-splines in L^2 for $n = 3$ or $n = 4$, m must be at least equal to 6 to provide a similar L^2 approximation as that using the discrete sampled B-splines. It is therefore wise to use filters of length at least equal to 6 and we are going to see the impact of this parameter on the quality of the approximation.

Furthermore, to parallelize the approach one should compute $c_{\uparrow m_2} * b_{m_2}^{n_1}$ and $g_{\uparrow m_1} * b_{m_1}^{n_2}$ separately. As the first term is related to the denominator of the rational scale, in our simulation we compute it once for $m_2 = 6$ and then compute the second filter for varying m_1. For our simulations, we take $n_1 = n_2 = 3$ while $g = [1 \quad -2 \quad 1]$ which means that the wavelet we use is an approximation of the second derivative of the Gaussian function.

Note that when $\frac{m_1}{m_2} = s < 1$, the length of the initial filter m_1 is lower than the desired length (i.e. 6) which creates little distortions of the results in such cases, so we multiply both m_1 and m_2 by a factor to get new m_1 and m_2 both larger than 6.

We now give two illustrations: the first one shows the importance of the renormalization factor when $s < 1$ in terms of l^2 error (we consider only integer location to compute the error) while the second illustrates the fact that our decomposition provides approximation of the wavelet decomposition at rational location and rational scales and that the convergence is indeed uniform.

To illustrate the first point, we consider a signal f for which we compute the proposed decomposition with m_1 and m_2 equal to 10 times the minimal values for m_1 and m_2 such that $\frac{m_1}{m_2} = s$. With filters of this length, we are very close to the asymptotic result we seek therefore we take this measure as a reference measure which we denote by $Dr(s, p)$, where s is the scale and p is a rational corresponding to location. To study the l^2 error with respect to scales and integer locations (i.e. p integer) we compute:

$$E(S) = \sum_{s \le S} \frac{\sqrt{\sum_{p \in \mathbb{Z}} (D(s,p) - Dr(s,p))^2}}{\sqrt{\sum_{p \in \mathbb{Z}} Dr(s,p)^2}}$$

We display in Figure 1 (B) the evolution of $E(S)$ when $m_2 = 6$ and $m_1 \in \mathbb{N}^*$ for $s \le 5$ and for the signal of Figure 1 (A). We notice that the approximation error is mainly related to scales $s < 1$ (solid line). With the proposed modification (dashed lines), i.e. to increase the length of the filters when the scale is smaller than 1, we get much better results with very little performance loss (the performance loss only involves scales $s < 1$). We also note that asymptotic convergence to the B-spline instead of exact computation at integer location provided by the discrete sampled B-spline is not a problem in practice since we have very fast convergence when m_1 and m_2 increase. We can now conclude on the relevance of the approach since it provides extra information about the value of the wavelet decomposition without any new computation. Indeed, with such a method it is no longer necessary to interpolate the values of the wavelet approximation at integer location to get an approximation of what happens elsewhere.

In figure 1 (C), we check that the normalized l^2 error defined by:

$$NE(S) = \frac{\sum_{s \le S} \sum_{p \in \mathbb{Z}} (D(s,p) - Dr(s,p))^2}{\sum_{s \le S} \sum_{p \in \mathbb{Z}} Dr(s,p)^2}$$

is also low with the filter size we consider. Once again, we notice that there is little impact on the approximation (solid line) and that multiplying the length of the filters by an appropriate factor (dashed line) still permits to reduce the error drastically.

We now illustrate the second point which is that our method provides approximation of the wavelet decomposition at rational location and rational scales and that we have uniform convergence of the approximation. We consider once again a step edge function and we compute the decomposition for $\frac{m_1}{m_2}$ in $\{\frac{3}{2}, \frac{6}{4}, \frac{12}{8}, \frac{24}{16}\}$. We focus on the behavior of the decomposition around the edge; the result is depicted in Figure 2. We notice that we indeed have

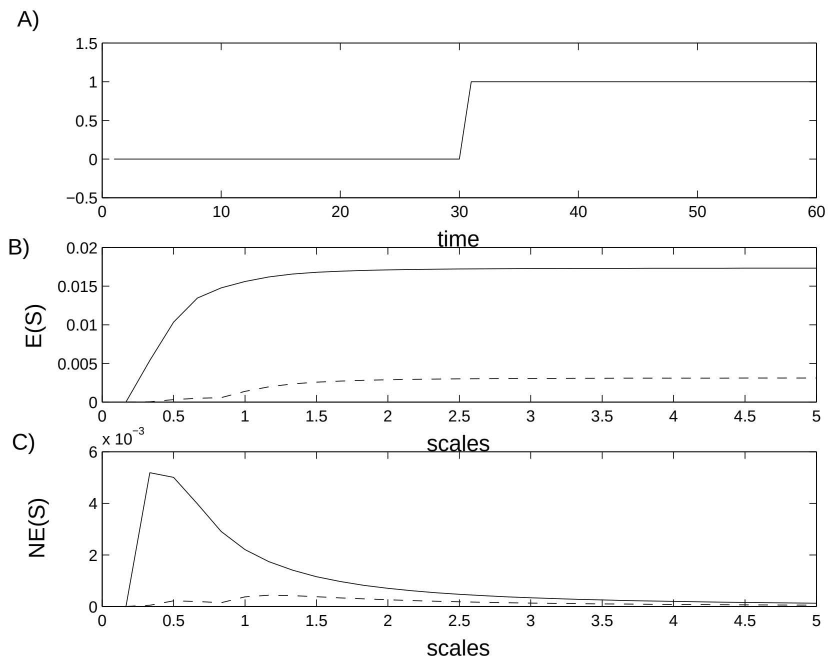

Fig. 1. A) signal f used for decomposition. B) The solid line represents the value of $E(S)$ for $m_1 \in \mathbb{N}^*$ and $m_2 = 6$ while the dashed line represents the value of $E(S)$ with the modified approach. C) The solid line represents the normalized l^2 error ($NE(S)$) for $m_1 \in \mathbb{N}^*$ and $m_2 = 6$ and the dashed line represents $NE(S)$ computed with the modified approach.

uniform convergence of the approximation at rational location and that the exact computation at integer location provided by discrete sampled B-splines is replaced by a converging approximation.

VI. Conclusion

The scope of this letter was to study the links between discrete B-splines, discrete-sampled B-splines and B-splines. We have shown that as far as continuous wavelet approximation is concerned, the discrete B-spline provides an efficient tool to compute an approximation of the wavelet decomposition at rational location with a lower computational cost than the approximation that uses discrete sampled B-splines. This approximation is mathematically sound due to uniform convergence of discrete B-splines to B-splines. The interest of the method we propose is that given a sampled signal we are able to get rapidly a good approximation of its wavelet decomposition at location that does not correspond to the sampling points. This is particularly interesting when one needs to get precise information about the behavior of the signal at a given point.

Obviously, this method extends to two dimensions using discrete separable multiresolution analysis of $l^2(\mathbb{Z}^2)$ and we keep the same kind of convergence we have in the one dimensional case. As far as the two dimension case is concerned, work is being carried out to show the interest of discrete B-spline decomposition for singularity analysis.

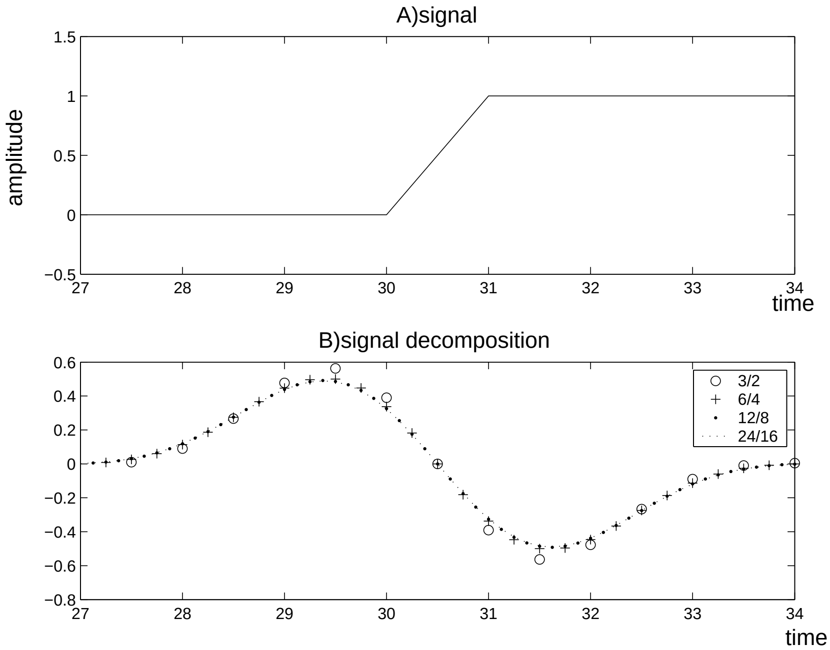

Fig. 2. A) studied signal, B) signal decomposition for values of $\frac{m_1}{m_2}$ in $\{\frac{3}{2}, \frac{6}{4}, \frac{12}{8}, \frac{24}{16}\}$.

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Appendix A

We show here that the Fourier series of α_m^n is a function of $L^2([-\frac{1}{2m}, \frac{1}{2m}[)$. Indeed, the Fourier series of α_m^n reads:

$$\hat{\alpha}_m^n(\nu) = \sum_{l\in\mathbb{Z}} \alpha_m^n[l]\exp(-2i\pi l\nu) = \sum_{k\in\mathbb{Z}} c[k]\exp(-2i\pi mk\nu)\sum_{l\in\mathbb{Z}} b_m^n[l]\exp(-2i\pi l\nu) = \hat{c}(m\nu)\hat{b}_m^n(\nu)$$

Since c is in $l^2(\mathbb{Z})$ we deduce that $\hat{c}$ is in $L^2([-\frac{1}{2}, \frac{1}{2}[)$. Consequently, $\hat{c}(m\nu)$ is in $L^2([-\frac{1}{2m}, \frac{1}{2m}[)$. To show that $\hat{b}_m^n(\nu)$ is bounded for $\nu \in [-\frac{1}{2m}, \frac{1}{2m}[$, we use the recurrence formula that arises from (2):

$$\hat{b}_m^n(\nu) = \left(\hat{b}_m^0(\nu)\right)^{n+1}$$

and then compute $\hat{b}_m^0$ to obtain:

$$\hat{b}_m^0(\nu) = \frac{\exp(-i\pi(m-1)\nu)}{m}\frac{\sin(\pi m\nu)}{\sin(\pi\nu)},$$

from which we deduce that $|\hat{b}_m^n(\nu)| = |\frac{\sin(\pi m\nu)}{m\sin(\pi\nu)}|^{n+1}$. We show that $h(\nu) = |\frac{\sin(\pi m\nu)}{m\sin(\pi\nu)}|$ is smaller than 1 on $[0, \frac{1}{2m}[$ (due to the parity of the function). For $\nu \in [0, \frac{1}{2m}[$, the function $\frac{\sin(m\pi\nu)}{m\sin(\pi\nu)}$ is positive. The sign of the derivative of the function over this interval is the sign of $m\tan(\pi\nu) - \tan(m\pi\nu)$, whose derivative is $m\pi(\tan^2(\pi\nu) - \tan^2(m\pi\nu))$

which is negative for $\nu \in [0, \frac{1}{2m}[$. As a result, the derivative of h is negative. We deduce that the maximum is reached at 0 and, as $h(0) = 1$, $\hat{b}_m^n(\nu)$ is bounded on $[-\frac{1}{2m}, \frac{1}{2m}[$. Consequently, $\hat{c}(m\nu)\hat{b}_m^n(\nu)$ belongs to $L^2([-\frac{1}{2m}, \frac{1}{2m}[)$.

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