Reconstruction of Finite Signal Derivatives from Multiscale Extrema Representations: Application to Transient Estimation and Signal Approximation

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Abstract- In this paper, we propose a novel approach to the reconstruction of finite signal derivatives from the extrema of a multiscale representation, first introduced by Berkner. We show that this reconstruction procedure naturally leads to the definition of a signal characteristic scale. We also focus on signal approximation from the multiscale extrema representation of one of its derivatives. Numerical applications involve the use of the signal characteristic scale for transient estimation and also the study of the new signal approximation method we propose.

Keywords- Berkner Decomposition, Multiscale Extrema Representation, Transient Estimation, Signal Approximation.

EDICS Category: RECO,WAVL

I. INTRODUCTION

Image and Signal approximation from wavelet transform modulus maxima (WTMM) representation or from zero-crossing representation has received a lot of interest in the past decade [8][9][10]. These representations are based on irregular sampling of the continuous dyadic scale wavelet transform at points which correspond to singularities or edges in the signal. However, Berman [2] and Meyer [12] showed that the corresponding (discrete-time or continuous) signal is not, in general, unique, the uniqueness being related to the completeness of the wavelet sampling basis in the signal subspace [7]. In spite of the non-uniqueness, it is possible to build a consistent signal having the same properties as those of the original signal described by the WTMM representation. A number of reconstruction methods were proposed for the 1-D WTMM reconstruction problem such as the projection-based method [8] [7][3], the conjugate gradient error minimization method [4] and the least square eigenspace method [6]. For the 2-D WTMM reconstruction problem, different methods were proposed based on projection onto a convex space [8][5]. In [13], a consistent discrete-time signal reconstruction from discrete-time wavelet transform extrema and zero-crossings is proposed.

All these multiscale extrema representations are based on the dyadic scale wavelet transform but in many applications, such as transient estimation, this scale representation is much too sparse. Furthermore, extrema of wavelet coefficients are known (for a specific choice of wavelet) to make up maxima lines in the time-scale space. These maxima lines cannot be constructed from the dyadic scale wavelet transform but are very useful to compensate for time-shifts of extrema with scale [11].

In this paper, we use a multiscale representation that behaves asymptotically like a wavelet transform [1] (section II). The corresponding multiscale extrema representation leads to a simple construction of maxima lines which were recently used for transient estimation [11]. We show here that a perfect finite signal derivative reconstruction is achieved with this multiscale extrema representation. The reconstruction procedure leads to a characteristic scale which we use for transient estimation (section III). We then introduce an approximation to compute the (k-1)th order signal derivative from the kth order signal derivative multiscale extrema representation. From this, any signal can be approximated from the multiscale extrema representation of any of its derivatives (section IV). Numerical applications involve the use of the characteristic scale mentioned above for transient estimation and also the study of our new signal approximation method which we compare to another method based on WTMM at dyadic scales (section V).

II. Shifted Berkner Decomposition

The decomposition we use is close to the discrete B-spline decomposition. Let us recall that the discrete B-spline of order $n > 0$, at scale m, is defined by [14]:

$$b_m^n = \overbrace{b_m^0 * b_m^0 * \cdots * b_m^0}^{n+1}, \tag{1}$$

where $b_m^0 = \frac{1}{m}\{1,\ 1, \cdots, 1\}$ is the normalized sampled pulse of width $m \geq 2$, with support $\left\{-\frac{m-1}{2}, \cdots, \frac{m-1}{2}\right\}$ when m is odd and $\left\{-\frac{m}{2}+1, \cdots, \frac{m}{2}\right\}$ otherwise. We also define $b_m^{-1} = \delta_0$ where $(\delta_l[p])_{p\in\mathbb{Z}}$ is the sequence that equals 1 if $p = l$ and zero otherwise. Using probability theory we obtain the approximation for large n:

$$b_m^n[p] \approx \sqrt{\frac{6}{\pi(n+1)(m^2-1)}} \exp\left(\frac{6(p-\epsilon\frac{n+1}{2})^2}{(n+1)(1-m^2)}\right),$$

where $\epsilon = 1$ if m is even and 0 otherwise [11]. We suppose that this is the case for the rest of the paper. To derive approximations of the derivatives of the Gaussian function at discrete points location, we consider $\rho = \{1, -1\}$, $\rho_k = \overbrace{\rho * \cdots * \rho}^{k}$, and $b_{m,k}^n = \rho_k * b_m^n$. With these notations and for large n we have:

$$b_{m,k}^n[p] \approx \sqrt{\frac{6}{\pi(n+1)(m^2-1)}} \left[\exp(\frac{6x^2}{(n+1)(1-m^2)})\right]^{(k)} (p - \frac{\epsilon(n+1)+k}{2}). \tag{2}$$

For a proof see [11]. To get an approximation of the kth order derivative of the Gaussian function we shift the filter $b_{m,k}^n$ properly ; we define:

$$\alpha_{m,k}^n[p] = b_{m,k}^n[p + \lfloor\frac{\epsilon(n+1)+k}{2}\rfloor] \tag{3}$$

where $\lfloor . \rfloor$ denotes the integer part, to obtain unshifted approximations of the derivatives of the Gaussian function. Therefore, for large N and $k \geq 1$ the filtering of a signal f by $\alpha_{m,k}^n$ is like a wavelet transform. By analogy with wavelet decompositions that are convolutions with reverse time version of wavelets, the decomposition we use is the correlation of the signal f with $\alpha_{m,k}^n$ defined in (3):

$$\begin{aligned} \forall p \in \mathbb{Z} \quad c_{m,k}^n[p] &= \sum_{j\in\mathbb{Z}} \alpha_{m,k}^n[j] f[j+p] \\ &= \sum_{j\in\mathbb{Z}} \alpha_{m,k}^n[j-p] f[j] \text{ for } n \geq -1. \end{aligned} \tag{4}$$

In the following, as n is a dilation factor, we will call it "scale" n.

III. Extrema Definition and Signal Derivatives Reconstruction from Multiscale Extrema Representation

A. Extrema Definition and Properties

We define the maxima (respectively minima) of the sequence $c^n_{m,k}$ as the strictly positive (respectively negative) coefficients $c^n_{m,k}[p]$ such that $c^n_{m,k}[p-1] \leq (\text{ resp. } <) c^n_{m,k}[p] > (\text{ resp. } \geq) c^n_{m,k}[p+1]$ (resp. $c^n_{m,k}[p-1] \geq (\text{ resp. } >) c^n_{m,k}[p] < (\text{ resp. } \leq) c^n_{m,k}[p+1]$). The specificity of the case $m=2$ is that an extremum at scale n arises from a unique extremum of the same nature at scale $n-1$ [1] [11]. These extrema define curves in the time-scale space which are called maxima lines. Let us note that, when $m=2$, the decomposition satisfies

$$\begin{aligned} \text{if } n+k \text{ odd, } c^n_{2,k}[p] &= \frac{1}{2}(c^{n-1}_{2,k}[p-1] + c^{n-1}_{2,k}[p]) \\ \text{else } c^n_{2,k}[p] &= \frac{1}{2}(c^{n-1}_{2,k}[p] + c^{n-1}_{2,k}[p+1]) \end{aligned} \quad (5)$$

which enables us to prove the following result on extrema:

Theorem 1: If $n+k$ is odd (resp even) and if there is an extremum at p in scale n then there is an extremum of the same nature either at p or $p-1$ (resp. at p or $p+1$) at scale $n-1$.

PROOF: We consider the case $n+k$ odd and we assume there is a maximum at p in scale n of the form:

$$\begin{aligned} & c^n_{2,k}[p] > 0 \quad c^n_{2,k}[p-1] \leq (\text{ resp. } <) c^n_{2,k}[p] > (\text{ resp. } \geq) c^n_{2,k}[p+1] \\ & \qquad \Leftrightarrow \begin{cases} c^{n-1}_{2,k}[p-1] > (\text{ resp. } \geq) c^{n-1}_{2,k}[p+1] \\ c^{n-1}_{2,k}[p-2] \leq (\text{ resp. } <) c^{n-1}_{2,k}[p]. \end{cases} \end{aligned} \quad (6)$$

Then, using (5), if $c^{n-1}_{2,k}[p] \leq 0$ we deduce $c^{n-1}_{2,k}[p-1] > 0$; the second inequality of (6) permits us to conclude that $c^{n-1}_{2,k}[p-1]$ is a local maximum. On the contrary, if $c^{n-1}_{2,k}[p] > 0$, when $c^{n-1}_{2,k}[p-1] \leq 0$ there is a local maximum at p, using the first inequality of (6) ; when $c^{n-1}_{2,k}[p-1] > c^{n-1}_{2,k}[p]$ there is a local maximum at $p-1$ (second inequality of (6)) ; finally, when $c^{n-1}_{2,k}[p] \geq c^{n-1}_{2,k}[p-1]$ there is a local maximum at p (first inequality of (6)). The same reasoning holds for the other kind of maxima and also for minima. The proof is the same for $n+k$ even $\blacksquare$.

Theorem 1 tells us that multiscale extrema make up maxima lines in the time-scale space. An illustration is given in Figure 1, where we display a noisy sin function and corresponding maxima lines for $k=1$. The graph ordinate for the maxima line plot is $\sqrt{\frac{n+1}{2}}$ (cf (2)), which corresponds to the definition of scale in the usual sense. Each singularity (i.e an extremum of the discrete derivative) belongs to a single maximal line (one must not misinterpret the graphic representation ; it may seem that there are sometimes two maxima lines for one singularity but this is due to a change in the extrema position from one scale to another).

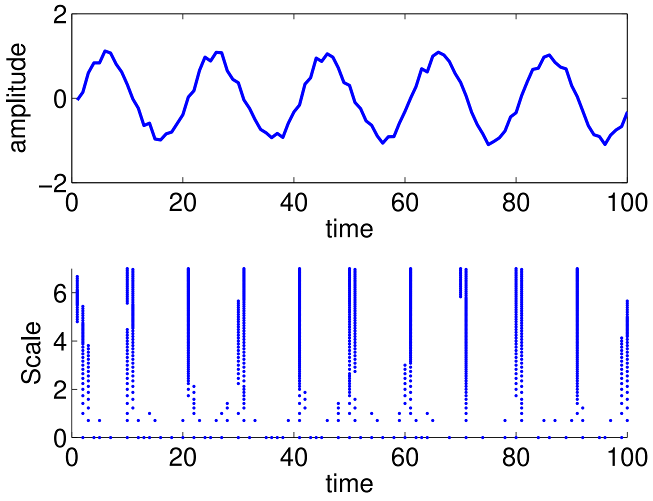


Fig. 1. Computation of the maxima lines with $k = 1$ (lower graph) corresponding to a noisy sin function (upper graph)

B. *Finite Signal Derivatives Reconstruction from Multiscale Extrema Representation*

Finite signal derivatives reconstruction from their multiscale extrema representation is an application of Theorem 1. With our notations, the kth order discrete derivative (which we call derivative for simplicity) of f is $c_{2,k}^{-1}$. Indeed, applying (4) and (3) we get:

$$c_{2,k}^{-1}[p] = \sum_{j \in \mathbb{Z}} \rho_k \left[j + \lfloor \frac{k}{2} \rfloor \right] f[p + j],$$

where $\rho_k[. + \lfloor \frac{k}{2} \rfloor]$ is the centered kth order finite difference operator.

We notice that the multiscale extrema representation of the kth order signal derivative up to scale n enables us to compute many coefficients at smaller scales. More precisely:

Theorem 2: Assume that $c_{2,k}^{n}[p]$ is an extremum in scale n and that the value of the coefficients along the corresponding maxima line is known up to scale n, then we can compute $c_{2,k}^{n-2q}[l]$ for $l \in \{p - q, \cdots, p + q\}$, $c_{2,k}^{n-2q+1}[l]$ for $l \in \{p - q, \cdots, p + q - 1\}$, when $n + k$ is odd and for $l \in \{p - q + 1, \cdots, p + q\}$ when $n + k$ is even.

PROOF: We assume $n + k$ odd, the case $n + k$ even can be treated in the same way. The proof is by induction on q. When $q = 1$, if $c_{2,k}^{n}[p]$ is an extremum then $c_{2,k}^{n-1}[p]$ is an extremum or $c_{2,k}^{n-1}[p - 1]$

is an extremum (cf Theorem 1), so one of these coefficients are known by hypothesis. Using (5), both $c_{2,k}^{n-1}[p-1]$ and $c_{2,k}^{n-1}[p]$ are known. Then, there is an extremum at $p-1, p$ or $p+1$ in scale $n-2$ (using Theorem 1). If it is at $p-1$, one computes $c_{2,k}^{n-2}[p]$ and then $c_{2,k}^{n-2}[p+1]$, both using (5). If it is at p (resp. $p+1$), we compute $c_{2,k}^{n-2}[p-1]$ and then $c_{2,k}^{n-2}[p+1]$ similarly (resp. $c_{2,k}^{n-2}[p]$ and then $c_{2,k}^{n-2}[p-1]$). This ends the proof for $q=1$.

We now assume that $c_{2,k}^{n-2q}[l]$ for $l \in \{p-q, \cdots, p+q\}$ and $c_{2,k}^{n-2q+1}[l]$ for $l \in \{p-q, \cdots, p+q-1\}$ are known. As there is an extremum in the sequence $c_{2,k}^{n-2q-1}[l]$ (Theorem 1), $l \in \{p-q-1, \cdots, p+q\}$, and as $c_{2,k}^{n-2q}[l]$, $l \in \{p-q, \cdots, p+q\}$ are known, we compute $c_{2,k}^{n-2q-1}[l]$ for $l \in \{p-q-1, \cdots, p+q\}$ with (5). Indeed, assume that an extremum is at l^* then, applying (5) iteratively, we compute $c_{2,k}^{n-2q-1}[l^*-1], \cdots, c_{2,k}^{n-2q-1}[p-q-1]$. Similarly, we compute iteratively $c_{2,k}^{n-2q-1}[l^*+1], \cdots, c_{2,k}^{n-2q-1}[p+q]$. The algorithm to compute these coefficients is written in the Appendix. At scale $n-2q-2$, we compute $c_{2,k}^{n-2q-2}[l]$ for $l \in \{p-q-1, \cdots, p+q+1\}$, noticing that $n-2q-1$ is even and using (5) (see the Appendix for details). This ends the proof $\blacksquare$.

We derive from Theorem 2 a simple algorithm for the reconstruction of the kth order finite signal derivative (i.e $c_{2,k}^{-1}$) from its multiscale extrema representation. Theorem 2 tells us that the knowledge of $c_{2,k}^{l}[q]$ along the maxima line corresponding to the extremum at p in scale n enables to compute $c_{2,k}^{-1}[l]$ for $l \in I_k^n[p] = \{p - \frac{n+1}{2}, \cdots, p + \frac{n+1}{2}\}$, when n is odd, for $l \in I_k^n[p] = \{p - \frac{n}{2} - 1, \cdots, p + \frac{n}{2}\}$, when n is even and k odd and for $l \in I_k^n[p] = \{p - \frac{n}{2}, \cdots, p + \frac{n}{2} + 1\}$ otherwise. We can therefore state the following theorem:

Theorem 3: Assume that f is finite, that it is not the null sequence and that $\mathcal{S}(c_{2,k}^{-1})$ is the support of $c_{2,k}^{-1}$. Let us denote M_k^n the set of indices corresponding to R_k^n extrema in scale n. Then the smallest scale for the reconstruction of $c_{2,k}^{-1}$ with our scheme is:

$$N_k = \underset{N}{argmin} \left(\bigcup_{n=-1}^{N} \bigcup_{r=1}^{R_k^n} I_k^n \left[M_k^n[r] \right] \subset \mathcal{S}(c_{2,k}^{-1}) \right). \tag{7}$$

Proof: We first prove that N_k exists. $\bigcup_{r=1}^{R_k^n} I_k^n[M_k^n[r]]$ is the set of indices l corresponding to the coefficients known from the extrema that propagate up to scale n (Theorem 2).

Let us first assume that $c_{2,k}^{n}[l]$ has no extremum. Either it is strictly increasing or strictly decreasing or constant. As it has a finite support, if it were strictly increasing (resp. decreasing) it would have a maximum at the least upper (resp. lower) bound of its support. Now, if $c_{2,k}^{n}[l]$ were constant, it would necessarily be the null sequence, since it has a finite support. (5) then tells us that $c_{2,k}^{n-1}$ would be such that $c_{2,k}^{n-1}[p] = -c_{2,k}^{n-1}[p-1]$. Since $c_{2,k}^{n-1}$ has a finite support this would mean $c_{2,k}^{n-1}$ is the null sequence, and iteratively $c_{2,k}^{-1}$ would also be the null sequence. This would finally lead to f is the null sequence which contradicts the hypothesis. $c_{2,k}^{n}$ necessarily has one extremum. Let then p_n be the location of an extremum at scale n ; then $|I_k^n[p_n]|$ (where $|A|$ is the cardinal of A) is strictly increasing with n and thus covers $\mathcal{S}(c_{2,k}^{-1})$ for some n_0. As $I_k^{n_0}[p_{n_0}] \subset \bigcup_{r=1}^{R_k^{n_0}} I_k^{n_0}\left[M_k^{n_0}[r]\right] \subset \bigcup_{n=-1}^{n_0} \bigcup_{r=1}^{R_k^{n}} I_k^{n}\left[M_k^{n}[r]\right]$, N_k exists.

Now, N_k is the minimal scale for the reconstruction of the kth order signal derivative with this multiscale extrema representation, since for $N_k - 1$ there exists some indices q for which $c_{2,k}^{-1}[q]$ is undetermined if we only use the values of the extrema up to rank $N_k - 1$ $\blacksquare$.

The reconstruction of $c_{2,k}^{-1}$ is based on the application of the algorithm detailed in the Appendix, considering the extrema at scale N varying from 0 to N_k. The case $N_k = -1$ corresponds to the case where every point of $c_{2,k}^{-1}$ is an extremum and is treated separately. We will compare in the Numerical Applications section the characteristic scale N_k just obtained to that computed in [11] for transient estimation.

IV. Signal Approximation from kth Order Derivative Multiscale Extrema Representation

We here show how to use the kth order derivative multiscale extrema representation we have just introduced to approximate the signal. In the Numerical Applications section, we will draw a comparison between our method and an existing method based on WTMM at dyadic scales [8].

A. (k-1)th Order Signal Derivative Approximation from kth Order Signal Derivative Multiscale Representation

Formula (5) enables us to write, for $-1 \leq l \leq N-1$ [11]:

$$\begin{aligned} c_{2,k-1}^{l}[p] &= c_{2,k-1}^{N}[p + \lfloor\frac{N-l}{2}\rfloor] + \frac{1}{2}\sum_{n=l}^{N-1} c_{2,k}^{n}[p + \lfloor\frac{n-l+1}{2}\rfloor] \text{ l+k even} \\ c_{2,k-1}^{l}[p] &= c_{2,k-1}^{N}[p + \lfloor\frac{N-l+1}{2}\rfloor] + \frac{1}{2}\sum_{n=l}^{N-1} c_{2,k}^{n}[p + \lfloor\frac{n-l+2}{2}\rfloor] \text{ otherwise,} \end{aligned} \tag{8}$$

where $\lfloor . \rfloor$ denotes the integer part. We first establish a general property of the sequence $c_{m,k}^{n}$. Let us recall that:

$$c_{m,k}^{n}[p] = \frac{1}{m}\sum_{i\in I} c_{m,k}^{n}[p-i], \tag{9}$$

where I is a set of indices of length m centered at 0 modulo a shift by one (cf (3)). The following relation between $C^{n-1}_{m,k}$ and $C^{n}_{m,k}$ holds:

$$\begin{aligned}
C^n_{m,k} &= \sum_{p\in\mathbb{Z}} \left(c^n_{m,k}[p]\right)^2 = \frac{1}{m^2}\sum_{p\in\mathbb{Z}}\left(\sum_{i\in I} c^{n-1}_{m,k}[p-i]\right)^2 \\
&= \frac{1}{m^2}\left(\sum_{p\in\mathbb{Z}} m\left(c^{n-1}_{m,k}[p]\right)^2 + 2\sum_{p\in\mathbb{Z}}\sum_{(i,j)\in I^2, i>j} c^{n-1}_{m,k}[p-i]c^{n-1}_{m,k}[p-j]\right) \\
&\leq \frac{1}{m^2}\left(m\sum_{p\in\mathbb{Z}}\left(c^{n-1}_{m,k}[p]\right)^2 + 2\frac{m(m-1)}{2}\sum_{p\in\mathbb{Z}}\left(c^{n-1}_{m,k}[p]\right)^2\right), \text{ using the Cauchy-Schwarz inequality} \\
&= \sum_{p\in\mathbb{Z}}\left(c^{n-1}_{m,k}[p]\right)^2 = C^{n-1}_{m,k}. \qquad (10)
\end{aligned}$$

The inequality is an equality if and only if $c^{n-1}_{m,k}[p-i] = c^{n-1}_{m,k}[p-j], \quad i>j$. Consequently, the sequence $\left(C^n_{m,k}\right)_n$ converges (decreasing and positive) and $c^n_{m,k}[p]$ tends to zero with n whatever p. When $m=2$, and for large N, we deduce from formula (8) the following approximation:

$$\begin{aligned}
c^l_{2,k-1}[p] &\approx \frac{1}{2}\sum_{n=l}^{N-1} c^n_{2,k}[p+\lfloor\frac{n-l+1}{2}\rfloor] = d^{l,N}_{2,k-1}[p] \text{ k+l even} \\
c^l_{2,k-1}[p] &\approx \frac{1}{2}\sum_{n=l}^{N-1} c^n_{2,k}[p+\lfloor\frac{n-l+2}{2}\rfloor] = d^{l,N}_{2,k-1}[p] \text{ otherwise.} \qquad (11)
\end{aligned}$$

We now study the (k-1)th order signal derivative approximation obtained taking $l=-1$ in (11):

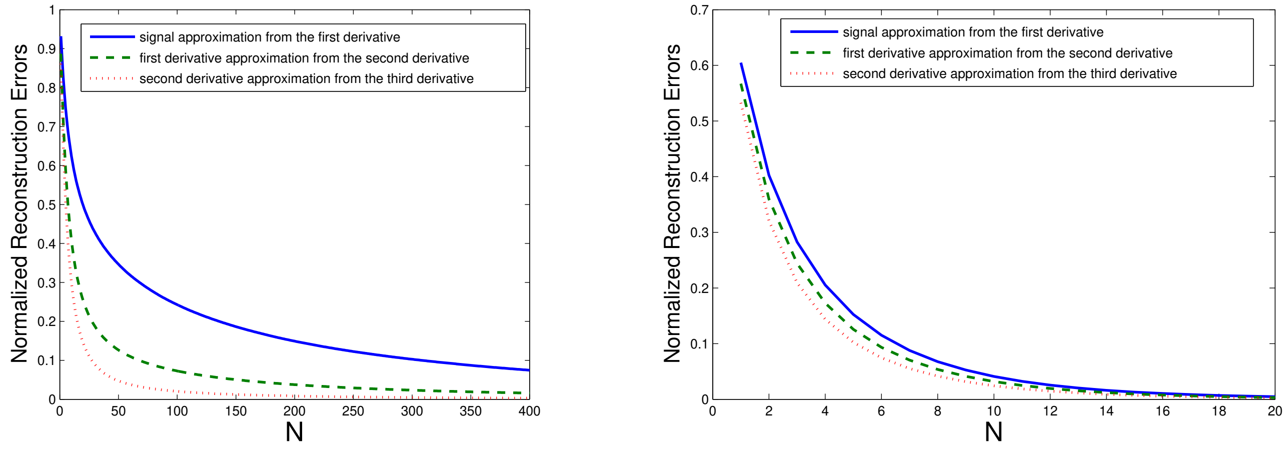
$$\begin{aligned}
c^{-1}_{2,k-1}[p] &\approx \frac{1}{2}\sum_{n=-1}^{N-1} c^n_{2,k}[p+\lfloor\frac{n+2}{2}\rfloor] = d^{-1,N}_{2,k-1}[p] \text{ k odd} \\
c^{-1}_{2,k-1}[p] &\approx \frac{1}{2}\sum_{n=-1}^{N-1} c^n_{2,k}[p+\lfloor\frac{n+3}{2}\rfloor] = d^{-1,N}_{2,k-1}[p] \text{ otherwise.} \qquad (12)
\end{aligned}$$

To assess the quality of the approximation, we compute the quantity

$$E^N_k = \sqrt{\frac{\sum_{p\in\mathbb{Z}}(c^{-1}_{2,k-1}[p]-d^{-1,N}_{2,k-1}[p])^2}{\sum_{p\in\mathbb{Z}}\left(c^{-1}_{2,k-1}\right)^2}} \qquad (13)$$

and then study the rapidity of convergence with N for different k and for different signal frequency bandwidths. The signals we use in simulations are generated through the filtering of a Gaussian white noise by a Butterworth filter with varying frequency bandwidths.

Numerical simulations show a faster convergence of E^N_k when k increases and also when the signal frequency bandwidth is shifted towards higher frequencies. These two results correspond to the same phenomenon since signal differentiation leads to a higher frequency signal. We give in Figure 2 an example of this phenomenon for signals with respective frequency bandwidth 20-200 Hz and 200-400 Hz (the frequency sampling is 1kHz and the results are averaged over 500 realizations).



(A) (B)

Fig. 2. (A): Computation of E_k^N, for $k = 1, 2, 3$, for a signal with frequency bandwidth 20-200Hz, (B): idem but for a signal with frequency bandwidth 200-400Hz. The sampling frequency is 1 kHz

B. Signal Approximation from the kth Order Signal Derivative Multiscale Extrema Representation

The signal approximation from the kth order signal derivative multiscale extrema representation we introduced is carried out using the following procedure: we first reconstruct the kth order signal derivative from its multiscale extrema representation (see section III) and we then iteratively build approximations of the derivatives of order $k-1, \cdots, 1$ and finally of the signal f using (12). In practice, the approximation of the signal f is better from the first derivative than from higher order derivatives. Indeed, in these later cases, only an approximation of the first derivative (obtained from the kth derivative) is used to approximate f. Note also that the quality of the approximation increases with N ; this will be discussed in the Numerical Applications section.

V. Numerical Applications

We showed that the reconstruction of the kth order signal derivative from its multiscale extrema representation leads to the definition of a scale N_k, which we now prove to be characteristic of signal frequency bandwidth. We also show how to use this characteristic scale for transient estimation following the same procedure as in [11]. Finally, we numerically compare our new signal approximation method to a WTMM at dyadic scales method [8].

A. *Computation of the Signal Characteristic Scale N_k, Application to Transient Estimation*

We investigate the evolution of N_k with signal frequency bandwidth and with the order k of the derivative. We carry out simulations for signals with frequency bandwidth $50-150$ Hz, $150-250$ Hz and $250-350$ Hz (with sampling frequency 1kHz). For each k and each bandwidth, we compute the mean and the variance of N_k over 500 realizations (each signal lasting 1.5 s). The results, depicted in Table I, show that N_k is characteristic of signal frequency bandwidth whatever the order k of the derivative. As signal differentiation increases the mean frequency, it results in a larger number of extrema when k increases leading to smaller a N_k. Also, shifting the frequency bandwidth towards higher frequencies with fixed k makes N_k decrease for the same reason. We must also stress that N_k increases with signal length, so N_k can discriminate different frequency bandwidth signals provided they have the same length.

		$k=1$	$k=2$	$k=3$
50-150 Hz	mean(N_k)	9.1	8	7.2
	variance(N_k)	1.3	0.9	0.5
150-250 Hz	mean(N_k)	4.5	4.3	4.2
	variance(N_k)	0.3	0.2	0.2
250-350 Hz	mean (N_k)	3	3	3
	variance(N_k)	0	0	0

TABLE I

COMPUTATION OF THE MEAN AND THE VARIANCE OF N_k OVER 500 REALIZATIONS FOR $k = 1, 2, 3$

In another direction, the characteristic scale N_k can be used for transient estimation purpose. One crucial issue in transient estimation problems in a multiscale framework is the scale of analysis. When the transient estimation is carried out with maxima lines methods this amounts to seeking up to which scale the coefficients on the maxima lines are relevant. The maxima lines method proposed in [11] is a particular case of the proposed scheme for $k = 1$. In that paper, the largest scale for the study of maxima lines corresponds to the minimum of the l_2 reconstruction error when the signal is approximated using formula (12) for $k = 1$ and keeping only the coefficients that are extrema. Once this characteristic scale (called N_0 in the paper) is computed, an estimator of transients is proposed, using features extracted from each maxima line $\mathcal{L}$ at rank $q \leq N_0$. These features were: $D_{\mathcal{L},q} = \sum_{(p,n)\in\mathcal{L}} \frac{(c^n_{2,1}[p])^2}{\|\alpha^n_{2,1}\|_2^2}$ and $F_{\mathcal{L},q} = O(\mathcal{L}_+(q)) - O(\mathcal{L})$, where $O(\mathcal{L})$ is the origin of the maxima line $\mathcal{L}$, and $\mathcal{L}_+(q)$ is the maxima line that follows $\mathcal{L}$ at rank q.

This approach suffers from the lack of a theoretical proof of the existence of N_0 for any signal. Furthermore, the transient estimator in [11] is based on the multiscale extrema of the first order signal derivative. We find it of interest to study the role of the order of the derivative for transient estimation.

To investigate the influence of the characteristic scale and of the order of the derivative we slightly change the transient estimator proposed in [11]. For each maxima line $\mathcal{L}$ of the kth order derivative multiscale extrema representation and for $q \leq N_k$, we consider: $D^k_{\mathcal{L},q} = \sum_{(p,n)\in\mathcal{L}} \frac{(c^n_{2,k}[p])^2}{\|\alpha^n_{2,k}\|_2^2}$ and $F^k_{\mathcal{L},q} = O(\mathcal{L}_+(q)) - O(\mathcal{L})$. Applying the transient estimator described in [11], but with these new features, we compare the influence of the different parameters with frequency changes and variations of amplitude. We display in Figure 3, the ROC curves (Figure 3.A) to assess the quality of detection as well as the mean absolute deviation from the true transition (Figure 3. B) for a frequency bandwidth change from 20-200Hz to 200-400Hz. In this case, the improvement brought about by the use of higher order derivatives ($k = 3$) is both in terms of detection performance and in terms of better localization of the transition. When the transition is from 200-400 Hz to 20-200Hz, we also notice an improvement using higher order derivatives. When the change arises from a variation of amplitude, the order of derivative does not change much the estimation and the detection results. We do not display those results due to the lack space. Finally, one must remark that in any case, the behavior of the estimator with $k = 1$ and taking N_0 as the characteristic scale ([11]) is very close to the new estimator we propose for $k = 1$ and taking N_1 as the characteristic scale.

B. Signal Approximation: Comparison with WTMM at Dyadic Scales

We compare the signal approximation we propose to the WTMM at dyadic scales [8] implemented in the WaveLab software [15]. First, we recall that, as given by (2), the scale n of our decomposition corresponds to the scale $\sqrt{\frac{n+1}{2}}$ (since $m = 2$ in our case) in usual sense. We compare the average normalized reconstruction errors for our method when the signal is approximated from the first derivative (i.e. E^N_1), with the corresponding quantity for WTMM at dyadic scales. We notice that the approximation is very good with WTMM at dyadic scales for low frequency signals. For instance for 20-200Hz frequency signals (see Figure 4.A) a good reconstruction is obtained when we use only the dyadic scale $J = 2$.

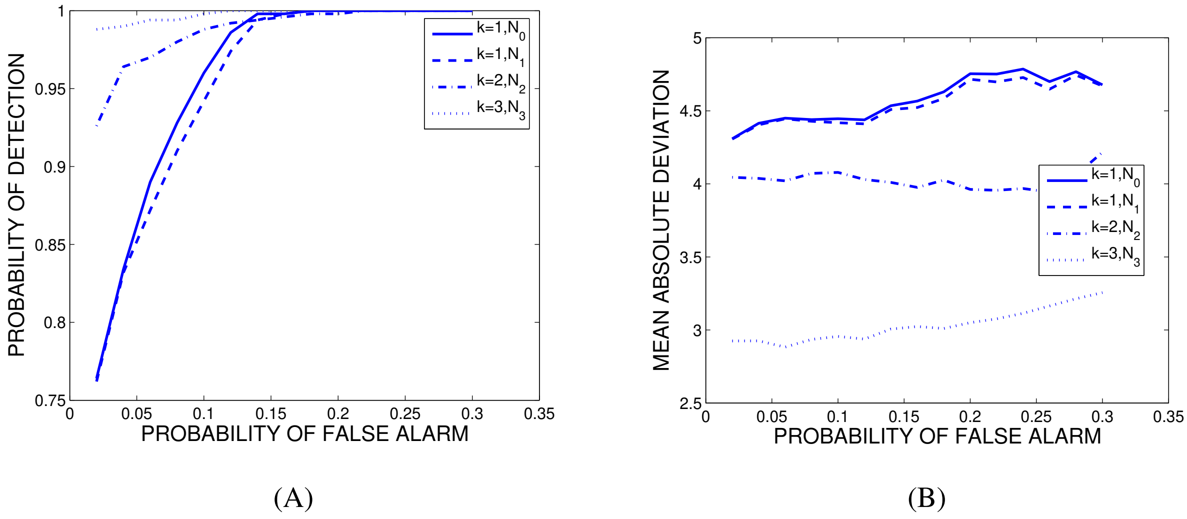


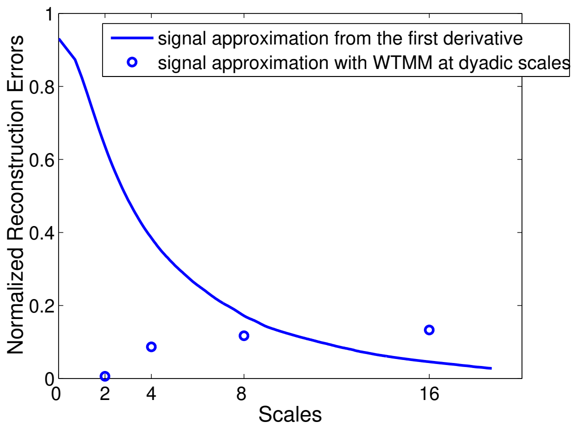
Fig. 3. (A): Computation of ROC curves for $k = 1$ and N_0, and for $k = 1$ and N_1, for $k = 2$ and N_2 and for $k = 3$ and N_3, for a frequency bandwidth change from 20-200Hz to 200-400Hz, (B): Mean absolute deviation (in ms) for the same parameters and the same frequency bandwidth change as in A

However, the quality of the approximation deteriorates drastically when the signal frequency bandwidth increases (see Figure 4.B). The reason for such a behavior is that WTMM at dyadic scale is designed to approximate signals with isolated singularities. Furthermore, the quality of the approximation also worsens when unnecessary dyadic scales are taken into account. To use dyadic scales corresponds to a too sparse representation in wavelet coefficients subspace. Furthermore, as already mentioned such representations do not allow for the construction of maxima lines that are useful for other purposes (section V A.).

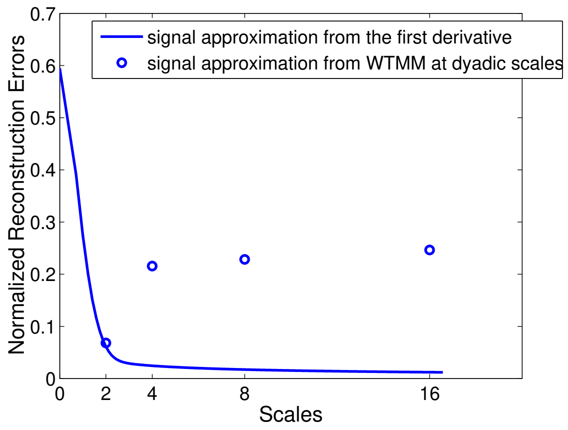
The signal approximation method we propose behaves better for high frequency signals (see Figure 4.B). In the case of Figure 4.B, the results are better from dyadic scale $J = 2$ which corresponds to $N = 7$ with our scheme. This means that the signal approximation improvement is achieved at a reasonable computational cost.

VI. Conclusion

In this paper, we proposed a new scheme to reconstruct the kth order signal derivative from its multiscale extrema. We first showed that this reconstruction enabled the definition of a characteristic scale that we used for transient estimation purpose. Second, the reconstruction procedure helped us to define a new signal approximation method. The use of the characteristic scale associated with the



(A)



(B)

Fig. 4. (A): Computation of E_1^N and the corresponding quantity for signal approximation with WTMM at dyadic scales for signals with frequency bandwidth 20-200Hz ; (B): idem but for a signal with frequency bandwidth 200-400Hz. The sampling frequency is 1 kHz. The results are averaged over 500 realizations

reconstruction procedure for transient estimation allowed us to clarify the role of the order of derivative for such problems. A remaining problem that we will address is how to deal with noise with the maxima lines approach we propose. As far as signal approximation is concerned, we showed that while our method is converging to the true signal, to the detriment of computational cost for low frequency signals, WTMM at dyadic scales does not converge to the true signal. Numerical applications suggested that WTMM at dyadic scales and our signal approximation method are complementary since the former is well adapted to low frequency bandwidth signal while the latter is better for high frequency bandwidth signal. Another points we will develop is the extension of the proposed method to the bidimensional case.

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Appendix

We detail the algorithm used for the reconstruction of the kth order signal derivative reconstruction from its multiscale extrema. We assume there is an extremum at scale n at p and that the extrema along the maxima line corresponding to this extremum is known for scales smaller or equal to n. We note

$M_k^n[p]$ the corresponding set of points $[l,q]$, l standing for the scale and q for the time. We also define a table E whose coefficient $E[l,q]$ is 1 when $c_{2,k}^l[q]$ is known and 0 otherwise.

Initialization:

for $[l,q] \in M_k^n[p], \ E[l,q] = 1;$

Iterations:

for $l = n-1, \cdots, 0,$

if $(l+1+k)$ odd

if $([l,q] \in M_k^n[p]), \ c_{2,k}^l[q-1] = 2c_{2,k}^{l+1}[q] - c_{2,k}^l[q]; \ E[l,q-1] = 1;$

else $c_{2,k}^l[q] = 2c_{2,k}^{l+1}[q] - c_{2,k}^l[q-1]; \ E[l,q] = 1;$

$q = p-1;$

while $(E[l+1,q] = 1), \ c_{2,k}^l[q-1] = 2c_{2,k}^{l+1}[q] - c_{2,k}^l[q]; \ E[l,q-1] = 1; \ q = q-1;$

$q = p+1;$

while $(E[l+1,q] = 1), \ c_{2,k}^l[q] = 2c_{2,k}^{l+1}[q] - c_{2,k}^l[q-1]; \ E[l,q] = 1; \ q = q+1;$

else

if $([l,q] \in M_k^n[p]), \ c_{2,k}^l[q+1] = 2c_{2,k}^{l+1}[q] - c_{2,k}^l[q]; E[l,q+1] = 1;$

else $c_{2,k}^l[q] = 2c_{2,k}^{l+1}[q] - c_{2,k}^l[q+1]; E[l,q] = 1;$

$q = p-1;$

while $(E[l+1,q] = 1), \ c_{2,k}^l[q] = 2c_{2,k}^{l+1}[q] - c_{2,k}^l[q+1]; \ E[l,q] = 1; \ q = q-1;$

$q = p+1;$

while $(E[l+1,q] = 1), \ c_{2,k}^l[q+1] = 2c_{2,k}^{l+1}[q] - c_{2,k}^l[q]; \ E[l,q+1] = 1; \ q = q+1;$

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