A Fast Algorithm for Bidimensional EMD

Christophe Damerval, Sylvain Meignen and Valérie Perrier

Abstract

In this paper, we describe a new method for bidimensional empirical mode decomposition (EMD). This decomposition is based on Delaunay triangulation and on piecewise cubic polynomial interpolation. A particular attention is devoted to boundary conditions that are crucial for the feasibility of the bidimensional EMD. The study of the behavior of the decomposition on different kind of images shows its efficiency in terms of computational cost and the decomposition of Gaussian white noises leads to bidimensional selective filter banks.

Index Terms

Empirical Mode Decomposition, Delaunay Triangulation.

EDICS Category: TFSR,IMMD

C. Damerval, S. Meignen and V. Perrier are with the LMC-IMAG Laboratory, University of Grenoble, France

I. INTRODUCTION

Empirical mode decomposition (EMD) was first introduced by Huang et al. [2] and provides a powerful tool for adaptive multiscale analysis of nonstationary signals. As far as the one dimensional case is concerned, studies were carried out to show the similarities of EMD with selective filter bank decompositions [7]. Its efficiency for signal denoising was also shown in [8]. These interesting aspects of the EMD motivate the extension of this method to bidimensional signals.

The basis of EMD (in 1D) is the construction of some intrinsic mode functions (IMFs) that are constructed through a so-called ”sifting” process (SP). A one-dimensional SP is an iterative procedure that depends both on an interpolation method and on a stopping criterion that ends the procedure. For bidimensional EMD, these two elements are still crucial ; we propose to focus on their influence over the construction of IMFs. As far as the interpolation is concerned, several techniques were proposed using, for instance, radial basis functions such as thin-plate splines [3][4][5][6]. These methods require the resolution of time consuming optimization problems which makes them hard to exploit especially in a noisy context as we will see.

In this paper, we propose a new bidimensional EMD where the SP is based on Delaunay triangulation and then cubic interpolation on triangles and also on a fixed number of iterations to build IMFs. The two major advantages of the proposed interpolation method over existing ones [5][6][3][4] is that it takes into account the geometry while preserving a low computational cost.

II. EMD BASIS

Here, we briefly describe the principle of EMD for a one-dimensional signal $(f[n])_{n\in\mathbb{Z}}$. It is based on the characterization of f through its decomposition in intrinsic mode functions (IMF) that are defined as follows [2]:

Definition 1: A function is an intrinsic mode function if the number of extrema equals the number of zero-crossings and if it has a zero local mean.
With this definition, we can describe the principle of EMD as follows:

1) Initialization : $r_0 = f$, $k = 1$
2) Computation of the kth IMF, d_k (SP)
   a) Initialization : $h_0 = r_{k-1}$, $j = 1$
   b) Identify all the local extrema of h_{j-1}
   c) Interpolate the local minima (resp. maxima) to get $Env_{\min,j-1}$ (resp $Env_{\max,j-1}$)

d) Compute the mean of these envelops:

$$Env_{mean,j-1}(t) = \frac{1}{2}\left(Env_{\min,j-1}(t) + Env_{\max,j-1}(t)\right)$$

e) $h_j[n] = h_{j-1}[n] - Env_{mean,j-1}(n)$

f) If the stopping criterion is fulfilled then $d_k = h_j$ else $j = j + 1$

3) $r_k[n] = r_{k-1}[n] - d_k[n]$

4) if r_k is not monotonic, go to step 2 otherwise the decomposition is complete.

When the decomposition is complete, we can write f as follows:

$$f[n] = \sum_{k=1}^{K} d_k[n] + r_K[n], \quad K \in \mathbb{N}^*$$

With this presentation, we notice that the key point of the algorithm is the SP entirely defined by an interpolation method (usually cubic spline interpolation [7] [2]) and by a stopping criterion. Note that to our knowledge, there is no mathematical proof of the convergence of the algorithm. The next section explains how to adapt the algorithm to bidimensional signals.

III. Bidimensional EMD: State of the Art and New Algorithm

For bidimensional signals, a similar algorithm as that of section II can be written, the key points still being: what interpolation technique to use in the SP and how many iterations to consider in the SP to build the IMFs? When we will have defined the SP, we will give a definition of the bidimensional IMFs we obtain and we will compare it to its 1D counterpart.

A. State of the Art

As far as interpolation is concerned, a natural extension of cubic spline to images is the thin-plate spline [1] used in [3], which is a particular case of radial basis functions used as interpolators [5]. In these cases, the envelop of the maxima (resp. minima) is the solution of a global optimization problem which requires the inversion of a linear system of size $q \times q$ [1], where q is the number of maxima (resp. minima). Such techniques are inappropriate for images that contain many extrema. Indeed, we numerically notice that for a bidimensional Gaussian white noise approximately 10 % of the points corresponds to maxima (resp. minima) ; if we assume that the size of the image is N^2, then these methods require the inversion of a $\frac{N^2}{10} \times \frac{N^2}{10}$ system which is prohibitive for large N. Note that this remark also holds for noisy images. A faster approach is proposed in [11] and uses tensor products to build the envelops. In spite of its rapidity, this method is based on one-dimensional envelops (along the columns and the rows of the image) and do not take the geometry into account. Furthermore, in all these approaches, the construction of IMFs is based on a stopping criterion in the SP.

B. New Bidimensional EMD

As explained earlier, we focus on the interpolation procedure and on the number of iterations that define the SP. For the interpolation part, we use Delaunay triangulation and then cubic interpolation on triangles and we replace the stopping criterion used to define IMFs [5][3] by a fixed number of iterations in the SP.

We now assume that $f[m,n]$ is a $N \times N$ image. We recall the definition of extrema we will use in the following:

Definition 1: $f[m,n]$ is a maximum (resp. minimum) if it is larger (resp. lower) than the value of f at the eight nearest neighbors of $[m,n]$.

Bidimensional Interpolation

We here assume that the values $f[m,n]$ correspond to an approximation of a continuous function $\tilde{f}$, defined on $[0,1] \times [0,1]$, at the points:

$$\mathcal{D} = \left\{ (\frac{m+\frac{1}{2}}{N}, \frac{n+\frac{1}{2}}{N}), [m,n] \in \{0, \cdots, N-1\}^2 \right\}.$$

The interpolation method we use to build the bidimensional IMFs is based on a Delaunay triangulation of the subset $\mathcal{D}_{\max}$ (resp. $\mathcal{D}_{\min}$) of $\mathcal{D}$ that corresponds to maxima (resp. minima) for f and then on piecewise cubic interpolation on triangles as explained in [10]. We use Delaunay triangulation since it is well adapted to the interpolation of scattered data of which the set $\mathcal{D}_{\max}$ (resp. $\mathcal{D}_{\min}$) is a typical example. As it is, the method is not satisfactory since the triangulation of the points of $\mathcal{D}_{\max}$ (resp. $\mathcal{D}_{\min}$) whose support is the convexhull of $\mathcal{D}_{\max}$ (resp. $\mathcal{D}_{\min}$) may not contain $\mathcal{D}$. To fulfill this condition, we symmetrize the maxima points (resp. the minima points) with respect to the boundary of the support of $\tilde{f}$. So as to reduce the computational cost of the algorithm, we do not symmetrize the whole set of maxima (resp. minima) points, which would result in a 9 times larger problem, but only the maxima (resp. minima) points that are ”close” to the boundary of the support of $\tilde{f}$. That is, we define the subset of the maxima points to be symmetrized as follows:

$$\mathcal{S}_{\max} = \left\{ A_i \in \mathcal{D}_{\max}, \quad \min_{B_i \in B} \|A_i - B_i\|_2 \leq \sqrt{\frac{1}{|\mathcal{D}_{\max}|}} \right\} \tag{1}$$

where $\|.\|_2$ is the Euclidean norm and where B is the boundary of the support of $\tilde{f}$. We similarly define a set $\mathcal{S}_{\min}$. Note that the upper bound in (1) defines a band of length equal to the inverse of the density of the maxima since it can also be written as $\frac{1}{N}\sqrt{\frac{N^2}{|\mathcal{D}_{\max}|}}$. In other words, the higher the density of maxima (resp. minima) is, the smaller the bandwidth is. Then, one remarks that the convexhull of the set of points:

$$\mathcal{D}_{\max} \cup T(\mathcal{S}_{\max}), \tag{2}$$

where T is the symmetry operator with respect to the boundary of the support of $\tilde{f}$ may still not contain $\mathcal{D}$ (the same conclusion holds for the set of points obtained with the minima). To ensure this condition, we add four arbitrary points which are the corners of $\tilde{f}$ extended by a band of $\sqrt{\frac{1}{|\mathcal{D}_{\max}|}}$ (resp. $\sqrt{\frac{1}{|\mathcal{D}_{\min}|}}$) in each direction. We call $\mathcal{D}^{ext}_{\max}$ (resp. $\mathcal{D}^{ext}_{\min}$) the set of points thus defined, the triangulation built on $\mathcal{D}^{ext}_{\max}$ (resp. $\mathcal{D}^{ext}_{\min}$) covers $\mathcal{D}$. When symmetrizing the maxima (resp. minima) points, we also symmetrize the corresponding values of f. We also assign at the four added corner points the value of f at the closest point (in terms of the Euclidean norm) of the union defined by (2) (or the corresponding one for minima). Such a choice prevents the propagation of numerical artifacts since the values at the added corner points only depend on the local behavior of the decomposition.

The envelop of the maxima (resp. minima) used in the SP is then defined by the piecewise cubic interpolation of f on the Delaunay triangulation of $\mathcal{D}^{ext}_{\max}$ (resp. $\mathcal{D}^{ext}_{\min}$), which we then restrict to the support of $\tilde{f}$. The envelop of the maxima (resp. minima) is denoted in the following by $Env_{\max}$ (resp. $Env_{\min}$).

On the Number of Iterations in the SP

Once we have defined the interpolation procedure we use in the SP, we determine the appropriate number of iterations in the SP to build IMFs. This number appears to be independent from the kind of image under consideration with the criterion we choose. To define our criterion, we investigate what is on average the impact on the computation of the first IMF of the number of iterations in the SP. We compute the median of $|Env_{mean}(x, y)| = \frac{1}{2}|Env_{\max}(x, y)) + Env_{\min}(x, y)|$ as a function of the number of iterations in the SP, for the first IMF and for either a Gaussian white noise, the image of Lenna or the image of Lenna corrupted by a Gaussian white noise. We display in Figure 1 (A)-(C) and (F)-(G) the average median of $|Env_{mean}(x, y)|$ as a function of the number of iterations for the first IMF and either for Gaussian white noises (image size 64×64) or for the image of Lenna corrupted by a Gaussian white noise (image size 512×512). For Gaussian white noises, we take $\sigma \in \{0.1, 1, 10\}$ and for the image of Lenna corrupted by a Gaussian white noise we take $\sigma \in \{1, 10, 100\}$. In Figure 1 (D), we display the median of $|Env_{mean}(x, y)|$ as a function of the number of iterations for the first IMF corresponding to the image of Lenna free of noise. Note that in the last four cases, the median of

$|Env_{mean}(x,y)|$ after one iteration is not displayed because it is too large.

The average number of iterations necessary to obtain the first IMF is then given by the minimal curvature of the curve defined by $(j, g(j))$ where $g(j) = \frac{1}{P}\sum_{i=1}^{P} median(|Env^{i}_{mean,j}(x,y)|)$ where P is the number of realizations and $Env^{i}_{mean,j}$ is the mean envelop after j iterations in the SP and for the ith realization. As illustrated in Figure 1, the average number of iterations given by our criterion is independent from the kind of images: the appropriate number of iterations always equals 3. The criterion we use to compute the number of iterations leads to a mean envelop with low amplitude and reduces numerical artifacts that appear for large numbers of iterations. As the number of extrema diminishes considerably when we substract the first IMF to the signal (see Table I for an illustration), the obtaining of the successive IMFs will not require more than 3 iterations.

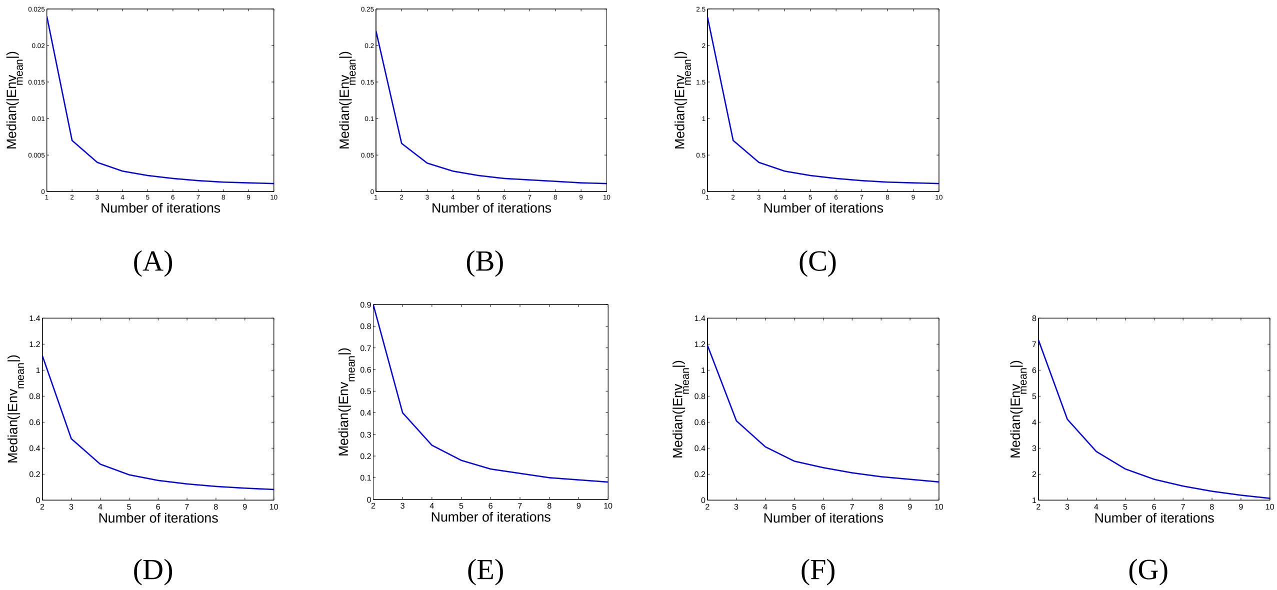

Fig. 1. (A): Evolution of the average $median(|Env_{mean}(x,y)|)$ for the first 10 iterations for a 64×64 Gaussian white noise with $\sigma = 0.1$, (B): idem for $\sigma = 1$, (C): idem for $\sigma = 10$,(D): $median(|Env_{mean}(x,y)|)$ for the first 10 iterations for the image of Lenna (512×512) free of noise,(E): idem as (A) but for the image of Lenna with Gaussian additive noise $\sigma = 1$,(F): idem as (E) but for $\sigma = 10$,(G): idem as (E) but for $\sigma = 100$

On the Number of IMFs

The criterion to stop the decomposition (i.e. how many IMFs to consider) is then based on the property that the number of extrema in the residual signal must decrease throughout the decomposition. When the algorithm stops there are very few remaining extrema (see Table I) ; the increase in the number of extrema is due to artifacts of low energy. We visualize the behavior of the algorithm on a bidimensional

64×64 Gaussian white noise with $\sigma = 1$. The number of iterations in the SP equals 3, for each IMF. This illustration also shows that there are very little boundary effects even for such small images: this is due to the symmetrization procedure we use in our interpolation method.

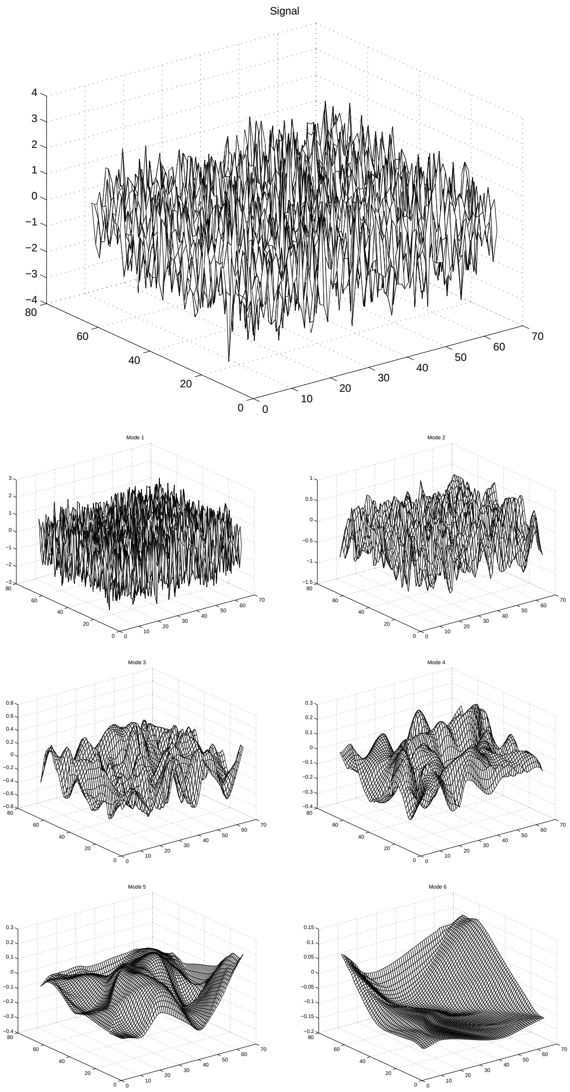

Fig. 2. From top to bottom and left to right: the signal to be analyzed and the first 6 IMFs that correspond to this signal

Bidimensional IMF Definition

The algorithm we detailed above give a possible construction for IMFs which are not given any definition yet. The definition that corresponds the most to IMFs we obtain is the following:

Definition 2: an image is a bidimensional IMF, if it has a zero mean, if the maxima are positive and the minima are negative and if the number of maxima equals the number of minima.

The second property replaces the condition in one dimension that the number of zero-crossings equals the number of extrema (modulo 1). To show the relevance of this definition, we display in Table I the number of maxima, of positive maxima, of minima, of negative minima as well as the mean and the standard deviation of IMFs either for a Gaussian white noise, for the image of Lenna or for the image of Lenna corrupted by a Gaussian white noise (in both cases $\sigma = 1$). We notice that the first two conditions of Definition 2 are satisfied while the number of minima are not exactly equal to the number of maxima but they are quite close.

Now that we have defined our SP, we compare our method with existing methods in terms of computational cost and we also investigate the modes decorrelation as well as the frequency responses of IMFs.

IV. Results and Comparison with Existing Methods

A. Computational Cost

We show the gain of computational cost with our bidimensional EMD compared with that proposed in [3] that uses thin-plate splines, on a bidimensional Gaussian white noise. Note that the conclusions we draw could be extended to any image that contain many extrema. We give an illustration of the computational gain in Table II, where we display the computational cost of both algorithms as a function of N and when IMFs are obtained after 3 iterations in the SP.

B. Bidimensional EMD as a Filter Bank and Modes Decorrelation

In this section, we put forward that the bidimensional EMD we propose creates a selective filter bank when applied to bidimensional Gaussian white noises similarly as in the one dimensional case [7]. The analysis of more complex noises such as Brownian or fractional Brownian noises will be the subject of future developments. We consider the average Fourier transform of IMFs over 1000 realizations of a bidimensional 64×64 Gaussian white noise ($\sigma = 1$). The results are displayed in Figure 3 for the first 6 modes. As expected the first modes contain the highest frequencies while the others contain lower frequencies. To stress this point, we compute the maximum amplitude of the Fourier Transform for the first 5 modes taking into account the isotropy of the Fourier representation ; these are located at $\|\nu\| = 0.39, 0.15, 0.0075, 0.006, 0.003$ respectively. We also notice that contrary to the one-dimensional case the spectra of IMFs do not vanish in the vicinity of zero.

A last point we investigated involves modes decorrelation which we studied on a Gaussian white noise. Our experiments show very little correlation between modes except for IMF_1 and IMF_2 where

		nb maxima	nb minima	nb maxima > 0	nb minima < 0	mean	standard deviation
IMF_1	Noise	30645	30692	30645	30692	9.10^{-4}	0.98
	Lenna	21776	21294	21776	21296	-4.10^{-2}	11
	Lenna+Noise	22846	22357	22846	22357	-0.07	10.59
IMF_2	Noise	8889	8910	8889	8910	-6.10^{-4}	0.36
	Lenna	4236	4206	4236	4206	0.28	11.6
	Lenna+Noise	4529	4600	4529	4600	0.18	11.4
IMF_3	Noise	3153	3198	3153	3198	-3.10^{-4}	0.20
	Lenna	1420	1424	1420	1424	-0.26	12.2
	Lenna+Noise	1533	1517	1533	1517	0.27	12.1
IMF_4	Noise	1486	1480	1486	1480	10^{-3}	0.12
	Lenna	928	980	928	980	0.25	11.4
	Lenna+Noise	961	931	961	931	0.22	11.9
IMF_5	Noise	836	896	836	896	-2.10^{-5}	0.07
	Lenna	873	812	873	812	0.35	10.3
	Lenna+Noise	741	765	741	765	0.55	11.4
IMF_6	Noise	656	652	656	652	10^{-4}	0.05
	Lenna	676	712	676	712	-1.6	12
	Lenna+Noise	706	680	706	680	1.02	10.4
IMF_7	Noise	576	562	576	562	5.10^{-3}	0.03
	Lenna	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
	Lenna+Noise	706	718	706	718	-0.6	11.7
residual signal	Noise	177	211	$\times$	$\times$	$\times$	$\times$
	Lenna	169	161	$\times$	$\times$	$\times$	$\times$
	Lenna+Noise	119	131	$\times$	$\times$	$\times$	$\times$

TABLE I

NUMBER OF MAXIMA, NUMBER OF POSITIVE MAXIMA, NUMBER OF MINIMA, NUMBER OF NEGATIVE MINIMA, MEAN AND STANDARD DEVIATION OF IMFS AND OF THE RESIDUAL SIGNAL. THE NUMBER OF ITERATIONS IN THE SP EQUALS 3. THE SYMBOL $\times$ MEANS WE DO NOT MAKE THE COMPUTATION. THE IMAGE SIZE IS 512×512

the correlation equals 0.2.

V. CONCLUSION

The scope of this paper was to introduce a new algorithm for bidimensional EMD that is efficient to analyze images that contain many extrema. The fast method we propose is based on the definition of a

	PC	TPS
$N = 16$	1	2
$N = 32$	2	6
$N = 64$	4	57
$N = 128$	10	$\times$
$N = 256$	43	$\times$
$N = 512$	185	$\times$

TABLE II

COMPUTATIONAL COST OF THE BIDIMENSIONAL EMD WITH RESPECT TO N FOR EITHER PIECEWISE CUBIC INTERPOLATION (PC) OR THIN-PLATE SPLINE (TPS) INTERPOLATION, FOR 3 ITERATIONS OF THE SP FOR EACH MODE. THE SYMBOL $\times$ MEANS WE DO NOT MAKE THE COMPUTATION

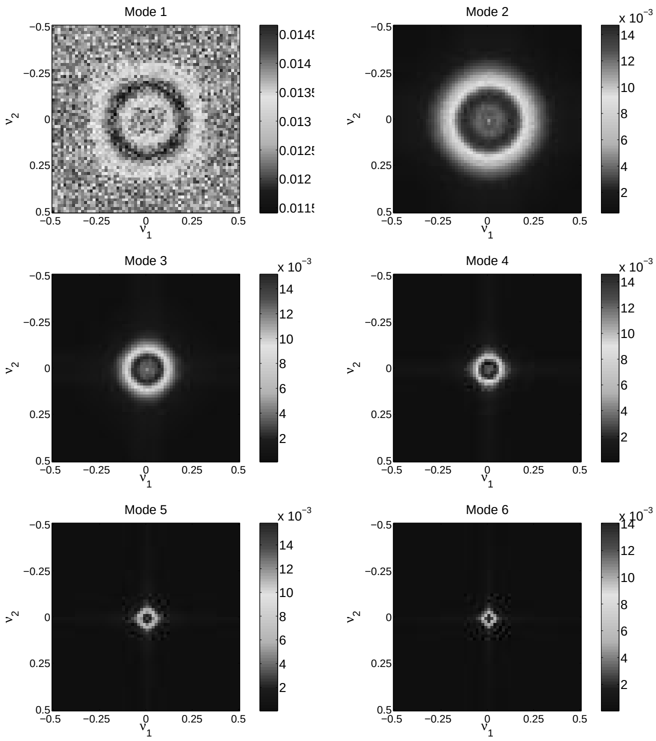

Fig. 3. From top to bottom and left to right, the average Fourier Transform of IMFs obtained over 1000 realizations of a 64×64 bidimensional Gaussian white noise $\sigma^2 = 1$

specific ”sifting” process (SP) that uses Delaunay triangulation and piecewise cubic interpolation, IMFs being then obtained by a fixed number of iterations in the SP. We showed that the decomposition of a

Gaussian white noise leads to the creation of a selective filter bank. Future work will involve the study of Brownian and fractional Brownian noises with this bidimensional EMD but also will deal with the analysis of textures that are images containing many extrema and for which previous studies [5] have shown promising results by use of EMD.

VI. ACKNOWLEDGMENTS

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REFERENCES

[1] F.L . Bookstein, *Principal Warps : Thin-Plate Splines and the Decompositions of Deformations,* IEEE Transactions on Pattern Analysis and Machine Intelligence, vol. 11,no. 6,pp. 567-585, 1989.

[2] N.E. Huang, Z. Shen, S. R. Long, M.C. Wu, H.H. Shih, Q. Zheng, N.C. Yen, C.C. Tung and H.H. Liu, *The empirical mode Decomposition and the Hilbert Spectrum for Nonlinear and nonstationary time series analysis*, Proceedings of the Royal Society London A, pp. 903-1005, 1998.

[3] A. Linderhed, *2D Empirical Mode Decompositions in the Spirit of Image Compression*, Wavelet and Independent components analysis applications IX,SPIE proceedings, vol. 4738, pp. 1-8, 2002.

[4] A. linderhed, *Adaptive Image Compression with Wavelet Packets and Empirical Mode Decomposition*, Phd thesis, Linköping Studies in Science and Technology, 2004.

[5] J.C. Nunes, Y. Bouaoune, E. Delechelle, O. Niang, and Ph Bunel, *Image Analysis by Bidimensional Empirical Mode Decomposition*, Image Vis. Comput, vol. 21, no. 12, 2003.

[6] J.C Nunes, *Analyse Multiéchelle d'image. Application à l'Angiographie Rétinienne et à la DMLA*, PhD Thesis, Université Paris 12, décembre 2003.

[7] G. Rilling, P. Flandrin and P. Goncalves, *Empirical Mode Decomposition As a Filter Bank*, IEEE Signal Processing Letters, vol. 11, no. 2, pp. 112-114, 2004.

[8] G. Rilling, P. Flandrin and P. Goncalves, *Detrending and Denoising with Empirical Mode Decomposition* EUSIPCO-04, Wien.

[9] G. Rilling, P. Flandrin and P. Goncalves, *On empirical Mode Decomposition and Its Algorithm,* IEEE-EURASIP Worshop on NonLinear Signal and Image Processing NSIP-03, Grado I.

[10] T.Y. Yang, *Finite Element Structural Analysis*, Prentice-Hall, pp. 446-449, 1986.

[11] Z. Liu and S. Peng *Boundary Processing of Bidimensional EMD Using Texture Synthesis*, IEEE Signal Processing Letters, vol. 12, no. 1, pp. 33-36, 2005.