Signal and Image Processsing

MSIAM first year

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Contents

Chapter 1

Introduction

The principle of signal acquisition is to convert an analogical signal into a digital signal. Numerical signal processing is the set of operations that are done on the signal among which the most popular ones are spectral analysis, linear and non linear filtering, modulation, detection, parameters extraction. The digitalization of the signal is made of two steps: sampling and coding are essential and need to be clearly analyzed.

To start with we will recall, in chapter 1, basics on Hilbert spaces that will be used throughout the course. For signal sampling the theory of distribution is essential, and will be introduced briefly in chapter 2. Then, we will recall some properties on the Fourier transform of distributions in the same chapter. In chapter 3, we finally introduce filtering in that context.

The advent of digital signal processing is essentially related to the development of fast algorithms like the short time Fourier transform which will be studied in chapter 4.

Chapter 2

Hilbert Spaces

Let H be a vector space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

2.1 Definitions

Definition 1 *An inner product on H is an application $H \times H \to \mathbb{K}$ (denoted by $(x, y) \mapsto \langle x, y\rangle$) satisfying:*

(i) $\forall\, \lambda, \mu \in \mathbb{K},\ \forall\, x, y, z \in H,\ \langle \lambda x + \mu y, z\rangle = \lambda\langle x, z\rangle + \mu\langle y, z\rangle$

(ii) $\forall\, x, y \in H,\ \langle x, y\rangle = \overline{\langle y, x\rangle}$

(iii) $\forall\, x \in H,\ \langle x, x\rangle \geq 0$

(iv) $\forall\, x \in H,\ \langle x, x\rangle = 0 \Longleftrightarrow x = 0$

Proposition 1

One has:

1) Cauchy-Schwarz inequality:

$$\forall\, x, y \in H,\ |\langle x, y\rangle|^2 \leq \langle x, x\rangle \cdot \langle y, y\rangle$$

(equality when x and y are colinear)

2) Parallelogram identity: If one denotes $\|x\| = \sqrt{\langle x, x\rangle}$, one has:

$$2\left(\|x\|^2 + \|y\|^2\right) = \|x + y\|^2 + \|x - y\|^2$$

Proof If $\langle x, y\rangle = 0$, it is clear; otherwise let $t \in \mathbb{C}$.

$$\begin{aligned} 0 \leq \langle x + ty, x + ty\rangle &= \langle x, x\rangle + t\,\langle y, x\rangle + \bar{t}\,\langle x, y\rangle + |t|^2\,\langle y, y\rangle \\ &= \langle x, x\rangle + 2\,Re\,(t\,\langle y, x\rangle) + |t|^2\,\langle y, y\rangle \end{aligned}$$

$\exists \theta \in [0, 2\pi[$ and $r > 0$ such that $\langle y, x \rangle = r \cdot e^{\imath\theta}$. Let us take $t = s \cdot e^{-\imath\theta}$ with $s \in \mathbb{R}$. $\langle x, x \rangle + 2rs + s^2 \langle y, y \rangle \geq 0$. The discriminant of this polynomial with respect to the variable s has to be negative.

Proposition 2

The application $x \longmapsto \sqrt{\langle x, x \rangle} = \|x\|$ is a norm on H.

Proof

$$\begin{aligned} \|x+y\|^2 &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + 2\,Re\langle x, y \rangle + \|y\|^2 \\ &\leq \|x\|^2 + \|y\|^2 + 2\,\|x\|\,\|y\| \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

Definition 2 *A prehilbertian space is a vector space H equipped with an inner product. If this space is complete for the norm associated with the inner product, one says that H is an Hilbert space.*

Examples 1

1) $H = \mathbb{R}^N$, $\mathbb{K} = \mathbb{R}$, $\langle x, y \rangle = \sum_{j=1}^{n} x_j y_j$ is an Hilbert space.

2) $H = \mathbb{C}^N$, $\mathbb{K} = \mathbb{C}$, $\langle x, y \rangle = \sum_{j=1}^{n} x_j \overline{y_j}$ is an Hilbert space.

3) Let $\Omega \subset \mathbb{R}^N$ be a bounded open set.
$H = L^2(\Omega)$, $\mathbb{K} = \mathbb{C}$, $\langle x, y \rangle = \int_\Omega x(t)\,\overline{y(t)}\,dt$ is an Hilbert space.

4) $H = l^2_{\mathbb{C}}(\mathbb{N}) = \{x = (x_n)_{n\in\mathbb{N}} \mid \forall\, n,\ x_n \in \mathbb{C}, \sum_{n\geq 0} |x_n|^2 < +\infty\}$, $\langle x, y \rangle = \sum_{n\geq 0} x_n \overline{y_n}$ is an Hilbert space.

Definition 3 *Let H be a prehilbertian space.*

1) $x, y \in H$ are orthogonal if $\langle x, y \rangle = 0$.

2) Let $A \subset H, A \neq \emptyset$. One calls the orthogonal to the set A in H, the set denoted by $A^\perp$ such that:

$$A^\perp = \{x \in H \mid \forall\, y \in A,\ \langle x, y \rangle = 0\}$$

Proposition 3

Let H be a prehilbertian space

1) Let $A \subset H, A \neq \emptyset$. Then $A^{\perp}$ is a closed subspace of H.
2) Let $A, B \subset H, A \neq \emptyset$ and $A \subset B$, then $B^{\perp} \subset A^{\perp}$.
3) Let $A \subset H, A \neq \emptyset$, then $A \subset \left(A^{\perp}\right)^{\perp}$.
4) Let A be a subspace of H. If H is complete then $\overline{A} = \left(A^{\perp}\right)^{\perp}$.

2.2 Projection on a convex set

Theorem 1 Projection on a closed convex set

Let H be an Hilbert space, and $C \subset H$ a closed non empty convex set. Then for all $x \in H$, there exists a unique $a \in C$ such that:

$$\|x-a\| = d\left(x,C\right) = \inf_{y \in C} \|x-y\|$$

Proof

- Existence:

  Let $\left(y_n\right)_{n\in\mathbb{N}}$ a sequence of elements in C such that $\|x-y_n\| \longrightarrow d\left(x,C\right)$.

  $\forall\, n \neq m$, $\dfrac{1}{2}\left\|y_n - y_m\right\|^2 = \|x-y_n\|^2 + \|x-y_m\|^2 - 2\left\|x - \dfrac{y_n+y_m}{2}\right\|^2$

  As $\dfrac{y_n+y_m}{2} \in C$ because C is convex, one has $\left\|x - \dfrac{y_n+y_m}{2}\right\| \geq d\left(x,C\right)$. We may then deduce that:

  $$0 \leq \frac{1}{2}\left\|y_n - y_m\right\|^2 \leq \underbrace{\|x-y_n\|^2 + \|x-y_m\|^2 - 2d\left(x,C\right)^2}_{\xrightarrow[n,m\to+\infty]{} 0}$$

  and thus $\left(y_n\right)_{n\in\mathbb{N}}$ is a Cauchy sequence. As H is complete, there exists $a \in H$ such that $\lim y_n = a$, and finally as C is closed, $a \in C$. $d\left(x,C\right) = \lim_{n\to+\infty} \|x-y_n\| = \|x-a\|$

- Unicity :

  Let us suppose that there exist a and $b \in C$ such that $\|x-a\| = \|x-b\| = d\left(x,C\right)$. Then we define the sequence $\left(y_n\right)_{n\in\mathbb{N}}$ of elements of C by: $y_{2p} = a$ and $y_{2p+1} = b$. We have that $\|x-y_n\| \longrightarrow d\left(x,C\right)$, meaning $\left(y_n\right)$ is a Cauchy sequence in H (complete), and thus $a = b$ by unicity of the limit.

Theorem 2 Characterization of the projection

Let H be an Hilbert space, and $C \subset H$ a closed non empty convex set. We have the following equivalence:

(i) $a \in C$ is the projection of x on C

(ii) $a \in C$ such that $\forall\, y \in C,\ Re\,\langle x-a, y-a\rangle \leq 0$

Proof

- $i) \Rightarrow ii)$, Let $y \in C$ and let $t \in\,]0,1]$. $(1-t)\,a + ty \in C$ as C is convex. We first remark that $\|x - ((1-t)\,a + ty)\|^2 \geq \|x-a\|^2$. But,
$$\begin{aligned}\|x - ((1-t)\,a + ty)\|^2 &= \|x - a - t\,(y-a)\|^2 \\ &= \|x-a\|^2 + t^2\,\|y-a\|^2 - 2\,Re\,(t\langle x-a, y-a\rangle)\,.\end{aligned}$$
So $\forall t \in]0,1]$, $2\,Re\,(t\langle x-a, y-a\rangle) \leq t^2\,\|y-a\|^2$, from this we deduce that $\forall t \in]0,1]$, $2\,Re\,\langle x-a, y-a\rangle \leq t\,\|y-a\|^2$. Then making $t \longrightarrow 0$, we get $Re\,\langle x-a, y-a\rangle \leq 0$.
- $ii) \Rightarrow i)$. Let $y \in C$, $\|x-y\|^2 = \|x-a+a-y\|^2 = \|x-a\|^2 + \|y-a\|^2 - 2\,Re\langle x-a, y-a\rangle$. Using ii) we get $\|x-y\|^2 \geq \|x-a\|^2 + \|y-a\|^2 \geq \|x-a\|^2$, and thus: $\|x-a\| = \inf\limits_{y\in C} \|x-y\|$

Proposition 4

Let H be an Hilbert space. and $C \in H$ a closed non empty convex set. For $x \in H$, we denote by $P_c x$ the projection of x on C. One has:

$$\forall\, x_1, x_2 \in H,\ \|P_c x_1 - P_c x_2\| \leq \|x_1 - x_2\|$$

Proof Let us put $a_j = P_c x_j$ pour $j = 1, 2$. $\|a_1 - a_2\|^2 = \langle a_1 - a_2, a_1 - a_2\rangle = Re\langle a_1 - a_2, a_1 - a_2\rangle$ $= \underbrace{Re\langle a_1 - x_1, a_1 - a_2\rangle}_{\leq 0} + Re\langle x_1 - x_2, a_1 - a_2\rangle + \underbrace{Re\langle x_2 - a_2, a_1 - a_2\rangle}_{\leq 0}$ using theorem 2 (page 12)
$\leq Re\langle x_1 - x_2, a_1 - a_2\rangle \leq \|x_1 - x_2\|\,\|a_1 - a_2\|$ from Cauchy-Schwarz theorem.

Corollary 1

Let H be an Hilbert space and F a closed subspace of H (F is thus convex).

1) $a = P_F\,x$ is characterized by: $\begin{cases} a \in F \\ \forall\, y \in F,\ \langle x-a, y\rangle = 0 \end{cases}$

2) P_F is linear and continuous

3) $H = F \oplus F^{\perp}$

Proof

1) $a = P_F\, x$ is characterized by:

$$(1) \begin{cases} \forall\, y \in F,\ Re\langle x-a, y-a\rangle \leq 0 \text{ from the theorem 2 (page 12)} \\ a \in F \end{cases}$$

This is equivalent to:

$$(2) \begin{cases} a \in F \\ \forall\, y \in F,\ \langle x-a, y\rangle = 0 \end{cases}$$

Indeed:

- (2) $\Rightarrow$ (1). $\forall\, y \in F,\ \langle x-a, y-a\rangle = 0$ as $y-a \in F$, we have $Re\langle x-a, y-a\rangle \leq 0$
- (1) $\Rightarrow$ (2). One has $\forall\, y \in F,\ Re\langle x-a, y+a-a\rangle \leq 0$ as $y+a \in F$ i.e. $\forall\, y \in F,\ Re\langle x-a, y\rangle \leq 0$. But if $y \in F$, then $-y \in F$, and thus $Re\langle x-a, -y\rangle \leq 0$ Finally, $\forall\, y \in F,\ Re\langle x-a, y\rangle = 0$.

  If $y \in F, iy \in F,\ Re\langle x-a, iy\rangle = 0$ and thus $Im\langle x-a, y\rangle = 0$

2) Let $x, x' \in H, \lambda, \mu \in \mathbb{C}$. We would like to show that: $P_F\left(\lambda x + \mu x'\right) = \lambda P_F x + \mu P_F x'$. Using 1) we get $\forall y \in F,\ \langle \lambda x + \mu x' - P_F\left(\lambda x + \mu x'\right), y\rangle = 0$ and $\forall y \in F, \langle \lambda x + \mu x' - \lambda P_F x - \mu P_F x', y\rangle = \lambda\langle x - P_F x, y\rangle + \mu\langle x' - P_F x, y\rangle = 0$

3) $F \cap F^{\perp} = \{0\}$, $\forall\, x \in H,\ x = \underbrace{P_F x}_{\in F} + \underbrace{x - P_F x}_{\in F^{\perp}}$

Remark 1 *In Theorem 1 (page 11), one can alternatively assume H is prehilbertian and $C \subset H$ a non empty convex complete set.*

2.3 Riesz Representation

Theorem 3 Riesz Theorem

> Let $L \in \mathcal{L}\left(H, \mathbb{K}\right) = H'$, H Hilbert. There exists a unique $a \in H$ such that:
>
> $$\forall\, x \in H,\ L\left(x\right) = \langle x, a\rangle$$
>
> Furthermore: $|\, \|L\|\, | = \|a\|$

Proof

Let us denote $F = L^{-1}\left(0\right)$, which is closed since L is continuous.

1. If $F = H$, then $L \equiv 0$ and one can consider $a = 0$.
2. Otherwise, let $z \in F^{\perp} \backslash \{0\}$ (a basis of $F^{\perp}$ since its dimension is 1)

- Existence:

  $$\forall\, x \in H,\ \exists \lambda \in \mathbb{K}, y \in F, \text{such that } x = y + \lambda z \quad \left(H = F \oplus \overbrace{Vect(z)}^{=F^{\perp}} \right)$$

  $$L\,(x) = L\,(y) + \lambda \cdot L\,(z) = \lambda \cdot L\,(z) \Longrightarrow \lambda = \frac{L(x)}{L(z)}$$

  $0 = \langle y, z\rangle = \langle x - \frac{L(x)}{L(z)} \cdot z, z\rangle = \langle x, z\rangle - \frac{L(x)}{L(z)} \cdot \|z\|^2$

  $L(x) = \frac{L(z)}{\|z\|^2} \cdot \langle x, z\rangle = \langle x, \overline{\frac{L(z)}{\|z\|^2}} \cdot z\rangle$

  One takes $a = \dfrac{\overline{L(z)}}{\|z\|^2} \cdot z$

- Unicity :

  $\forall\, x \in H,\ \langle x, a\rangle = \langle x, a'\rangle \Longleftrightarrow \forall\, x \in H,\ \langle x, a - a'\rangle = 0 \Longleftrightarrow a = a'$

From Cauchy-Schwarz inequality:

$|L(x)| = |\langle x, a\rangle| \leq \|x\| \cdot \|a\| \Rightarrow |\,\|L\|\,| \leq \|a\|$

If $L \equiv 0$, then $|\,\|L\|\,| = \|a\| = 0$. Otherwise, $a \neq 0$ et $L\left(\dfrac{a}{\|a\|}\right) = \|a\| \Rightarrow |\,\|L\|\,| = \|a\|$

2.4 Hilbertian Basis

Definition 4 *Let H be an Hilbert space, and $(H_n)_{n\in\mathbb{N}}$ a sequence of closed subspaces of H. One says that H is an Hilbertian sum of the H_n denoted by:* $H = \bigoplus_{n\geq 0} H_n$ *if:*

1) $\forall\, n \neq m,\ \forall\, x \in H_n,\ \forall\, y \in H_m,\ \langle x, y\rangle = 0$

2) The space F generated by the H_n is dense in H

(F is the set of all the finite linear combinations of elements in $H_n, n \in \mathbb{N}$).

Theorem 4

One supposes that $H = \bigoplus_{n\geq 0} H_n$. If $x \in H$, we denote $x_n = P_{H_n}x$.

Then, one has:

1) $x = \sum_{n\geq 0} x_n \qquad (i.e.\ x = \lim_{N\to+\infty} \sum_{n=0}^{N} x_n)$

2) $\|x\|^2 = \sum_{n\geq 0} \|x_n\|^2$ (Bessel-Parseval identity)

Conversely, let $(x_n)_{n\in\mathbb{N}}$ a sequence of elements in H such that:

$$\forall\, n \in \mathbb{N},\ x_n \in H_n$$

and satisfying :

$$\sum_{n\geq 0} \|x_n\|^2 < +\infty,$$

then the series $\sum_{n\geq 0} x_n$ converges in H and if $x = \sum_{n\geq 0} x_n$, then $\forall\, n \geq 0,\ x_n = P_{H_n}x$

Proof

1) Let us define for $N \in \mathbb{N}$, $S_N = \sum_{n=0}^{N} P_{H_n}$. If $x \in H$, $S_N x = \sum_{n=0}^{N} x_n$, and we would like to show:

$$\lim_{N\to+\infty} \|x - S_N x\| \to 0$$

$$\|S_N x\|^2 = \langle S_N x, S_N x\rangle = \langle \sum_{n=0}^{N} x_n, \sum_{n=0}^{N} x_n\rangle = \sum_{n=0}^{N} \langle x_n, x_n\rangle = \sum_{n=0}^{N} \|x_n\|^2 \ (4)$$

which we could have directly proven using Pythagore theorem.

Let $\varepsilon > 0$, as F is dense is H, there exists $x^* \in F$ such that $\|x - x^*\| \leq \varepsilon/2$, and then $N_0 \in \mathbb{N}, \quad \forall\, N \geq N_0,\ S_N x^* = x^*$. Let $N \geq N_0$,

$$\begin{aligned} \|S_N x - x\| &\leq \|S_N x - S_N x^*\| + \|S_N x^* - x\| \\ &= \|S_N x - S_N x^*\| + \|x - x^*\| \\ &\leq \|x - x^*\| + \|x - x^*\| \ \text{ as } S_N \text{ is a projection operator} \\ &\leq \varepsilon \end{aligned}$$

2) $(4) \Rightarrow \|x\|^2 = \sum_{n\geq 0} \|x_n\|^2$

Conversely Let us put $u_N = \sum_{n=0}^{N} x_n$, and consider $p > q$ in $\mathbb{N}$:

$$\|u_p - u_q\|^2 = \left\| \sum_{n=q+1}^{p} x_n \right\|^2 = \sum_{n=q+1}^{p} \|x_n\|^2 \xrightarrow[p,q\to+\infty]{} 0$$

So, u is a Cauchy sequence, and thus converges. and thus, one can put:

$$x = \sum_{n \geq 0} x_n$$

Let $z_n \in H_n$, $\langle x - x_n, z_n \rangle = \langle x, z_n \rangle - \langle x_n, z_n \rangle$ and $\langle x, z_n \rangle = \lim_{N \to +\infty} \sum_{p=0}^{N} \langle x_p, z_n \rangle = \langle x_n, z_n \rangle$ (when $N \geq n$).
Thus,

$$\forall\, z_n \in H_n,\ \langle x - x_n, z_n \rangle = 0$$

From corollary 1 (page 12) (with $x_n \in H_n$), one has: $x_n = P_{H_n} x$

Definition 5 *One calls hilbertian basis of an Hilbert space H, a sequence $(e_n)_{n \geq 0}$ of elements in H such that:*

1) $\forall\, n \neq m$, $\langle e_n, e_m \rangle = 0$ and $\forall\, n$, $\langle e_n, e_n \rangle = 1$

2) The space generated by the $(e_n)_{n \in \mathbb{N}}$ is dense in H.

Remark 2 *Let H be an Hilbert space and $(e_n)_{n \in \mathbb{N}}$ an hilbertian basis of H. From Theorem 4 with $H_n = \mathbb{K}\, e_n$, any $x \in H$ can be written:*

$$x = \sum_{n \geq 0} \langle x, e_n \rangle e_n$$

and

$$\|x\|^2 = \sum_{n \geq 0} |\langle x, e_n \rangle|^2$$

The $\langle x, e_n \rangle$ are called the Fourier coefficients of x with respect to the hilbertian basis $(e_n)_{n \geq 0}$.

Conversely *If $\lambda = (\lambda_n) \in l^2_{\mathbb{K}}(\mathbb{N})$, then the series $\sum_{n \geq 0} \lambda_n e_n$ is converging to an element $x \in H$ and $\forall\, n$, $\lambda_n = \langle x, e_n \rangle$ and $\|x\|^2 = \sum_{n \geq 0} |\lambda_n|^2$ (example: $\lambda_n = \dfrac{1}{n+1}$ $(n \geq 0)$).*

Remark 3 *Every separable Hilbert space admits an hilbertian basis.*

H is separable if there exists a countable subset $D \subset H$ dense in H. $D = \{v_0, v_1, \ldots, v_n, \ldots\}$ with $v_j \neq 0$ for all j. $F_{n+1} = Vect(v_0, v_1, \ldots, v_n)$, $\bigcup_{n \geq 1} F_n$ is dense in H.
In F_1, one takes a vector e_1 with norm 1. Then, one defines F_2 such that $\dim F_2 = \dim F_1 + 1 = 2$ by considering the vector e_2 with unit norm and orthogonal to e_1, etc.

2.5 The space $L^2(I)$

2.5.1 Definitions

Definition 6 *$L^2(I)$ is the space of functions $f : I \to \mathbb{C}$ such that $|f|^2$ is integrable.*

Definition 7 *Inner product on $L^2(I)$*
$\forall f, g \in L^2(I), \langle f, g\rangle = \int_I f(x)\overline{g(x)}dx$ is an inner product on $L^2(I)$ and the corresponding norm satisfies: $\|f\|_2^2 = \int_I |f(x)|^2 \, dx$

2.5.2 Properties

Theorem 5

$\boxed{(L^2(I), \|\cdot\|_2) \text{ is an Hilbert space}}$

2.6 Hilbertian basis in $L^2(I)$

2.6.1 Definition

Definition 8 *Let $\mathcal{B} = (\varphi_n)_{n\in\mathbb{N}}, \varphi_n \in L^2(I)$. $\mathcal{B}$ is an hilbertian basis $L^2(I)$ if and only if:*

1. $\forall n, m \in \mathbb{N}, \langle \varphi_n, \varphi_m\rangle = \delta_{n,m}$
2. *The finite linear combinations of functions φ_n are dense in $L^2(I)$*

$$\iff \mathcal{B}^{\perp} = \left\{g \in L^2(I) \mid \forall n \in \mathbb{N}, \langle g, \varphi_n\rangle = 0\right\} = \{0\}$$

2.6.2 Parseval Theorem

Theorem 6

$\mathcal{B} = (\varphi_n)_{n\in\mathbb{N}}$ is an hilbertian basis if it satisfies one of the following two equivalent properties:

1. Convergence of the series in $L^2(I)$:
$$\forall f \in L^2(I), f = \sum_{n\in\mathbb{N}} \langle f, \varphi_n\rangle \varphi_n$$
2. $$\forall f \in L^2(I), \|f\|_2^2 = \sum_{n\in\mathbb{N}} |\langle f, \varphi_n\rangle|^2$$

Remark 4 *If the series was converging pointwise then we could have written that:*

$$\forall x \in I, f(x) = \sum_{n\in\mathbb{N}} \langle f, \varphi_n\rangle \varphi_n(x)$$

When considering hilbertian basis, the convergence has to be understood in the L^2 sense, namely:

$$f = \sum_{n \in \mathbb{N}} c_n \varphi_n \; dans \; L^2(I) \iff \lim_{N \to +\infty} \left\| f - \sum_{n=0}^{N} c_n \varphi_n \right\|_2 = 0 \iff \lim_{N \to +\infty} \int_I \left| f(x) - \sum_{n=0}^{N} c_n \varphi_n(x) \right|^2 dx = 0$$

2.7 Fourier Series

One considers the space $L^2(]0,T[)$ equipped with the inner product $\langle f, g \rangle = \int_0^T f(x)\overline{g(x)}dx$

Theorem 7

$\mathcal{B} = (e_n)_{n \in \mathbb{Z}}$ with $e_n(x) = \frac{e^{in\frac{2\pi x}{T}}}{\sqrt{T}}$ is an hilbertian basis of $L^2(]0,T[)$ thus

$$\forall f \in L^2(]0,T[), c_n(f) = \langle f, e_n \rangle = \int_0^T f(x) \frac{e^{-in\frac{2\pi x}{T}}}{\sqrt{T}} dx$$

1. $$f = \sum_{n \in \mathbb{Z}} c_n(f) e_n$$

2. $$\int_0^T |f(x)|^2 \, dx = \sum_{n \in \mathbb{Z}} |c_n(f)|^2 .$$

For the proof, we are going to use the two following lemma:

Lemma 1 $\mathcal{B}$ *est orthonormal*

Proof

$$\langle e_n, e_m \rangle = \int_0^T \frac{e^{i(n-m)\frac{2\pi x}{T}}}{T} dx = \delta_{m,n}$$

Theorem 8

Let $\varphi \in C_c^\infty(]0,T[)$, and then define :

$$\forall n \in \mathbb{Z}, c_n(\varphi) = \int_0^T \varphi(x) \frac{e^{-in\frac{2\pi x}{T}}}{\sqrt{T}} dx$$

Then:

$$\varphi(x) = \sum_{n \in \mathbb{Z}} c_n(\varphi) \frac{e^{in\frac{2\pi x}{T}}}{\sqrt{T}}$$

and the series <u>uniformly</u> converges on $[0,T]$

Proof

- Uniform convergence

$$\begin{aligned}\left|c_n(\varphi)\frac{e^{in\frac{2\pi x}{T}}}{\sqrt{T}}\right| &= \frac{1}{\sqrt{T}}\left|c_n(\varphi)\right| = \frac{1}{T}\left|\int_0^T \varphi(x)e^{-in\frac{2\pi x}{T}}dx\right| \\ &= \frac{1}{T}\left|\frac{1}{(-i\frac{2\pi n}{T})^2}\int_0^T \varphi''(x)e^{-inx}dx\right| \leq \frac{T}{4\pi^2 n^2}\underbrace{\int_0^T \left|\varphi''(x)\right| dx}_{c}\end{aligned}$$

Then the series $\sum c_n(\varphi)\frac{e^{in\frac{2\pi x}{T}}}{\sqrt{T}}$ converges normaly, and thus uniformely.

- Let us then compute $\sum\limits_{n=-N}^{N} e^{in\theta}$

$$\begin{aligned}\sum_{n=-N}^{N} e^{in\theta} &= e^{-N\theta}\sum_{n=0}^{2N} e^{in\theta} = e^{-N\theta}\frac{1-e^{i(2N+1)\theta}}{1-e^{i\theta}} \\ &= e^{-N\theta}\frac{e^{i\left(\frac{2N+1}{2}\theta\right)}\left(e^{-i\left(\frac{2N+1}{2}\theta\right)}-e^{+i\left(\frac{2N+1}{2}\theta\right)}\right)}{e^{i\theta/2}\left(e^{-i\theta/2}-e^{i\theta/2}\right)} = \frac{\sin\left(N+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}\end{aligned}$$

This last expression is called the Féjer kernel. From this we deduce that

$$\frac{1}{T}\int_0^T\left(\sum_{n=-N}^{N} e^{in\frac{2\pi\theta}{T}}\right)d\theta = \sum_{n=-N}^{N}\left(\int_0^T \frac{e^{in\frac{2\pi\theta}{T}}}{T}d\theta\right) = 1$$

and thus

$$\frac{1}{T}\int_0^T \frac{\sin\left(N+\frac{1}{2}\right)\frac{2\pi\theta}{T}}{\sin\frac{\pi\theta}{T}}d\theta = 1.$$

With this in mind, let us write:

$$\begin{aligned}\varphi(t)-\sum_{n=-N}^{N} c_n(\varphi)\frac{e^{in\frac{2\pi t}{T}}}{\sqrt{T}} &= \varphi(t)-\sum_{n=-N}^{N}\left(\int_0^T \varphi(x)\frac{e^{-in\frac{2\pi x}{T}}}{\sqrt{T}}dx\right)\frac{e^{in\frac{2\pi t}{T}}}{\sqrt{T}} \\ &= \varphi(t)-\int_0^T \frac{\varphi(x)}{T}\left(\sum_{n=-N}^{N} e^{in\frac{2\pi(t-x)}{T}}\right)dx \\ &= \varphi(t)-\int_0^T \frac{\varphi(x)}{T}\frac{\sin\left((N+\frac{1}{2})\frac{2\pi}{T}(x-t)\right)}{\sin\left(\frac{\pi}{T}(x-t)\right)}dx \\ &= \frac{1}{T}\int_0^T \frac{\sin\left(\left(N+\frac{1}{2}\right)\frac{2\pi}{T}(x-t)\right)}{\sin\left(\frac{\pi}{T}(x-t)\right)}\varphi(t)dx - \int_0^T \frac{\varphi(x)}{T}\frac{\sin\left((N+\frac{1}{2})\frac{2\pi}{T}(x-t)\right)}{\sin\left(\frac{\pi}{T}(x-t)\right)}dx \\ &= \frac{1}{T}\int_0^T \frac{\varphi(x)-\varphi(t)}{\sin\left(\frac{\pi}{T}(x-t)\right)}\sin\left(\left(N+\frac{1}{2}\right)\frac{2\pi}{T}(x-t)\right)dx \\ &= \frac{1}{T}\int_{-t}^{T-t}(\varphi(u+t)-\varphi(t))\frac{\sin\left(\left(N+\frac{1}{2}\right)\frac{2\pi}{T}u\right)}{\sin\left(\frac{\pi}{T}u\right)}du\end{aligned}$$

As $t\in]0,T[$ and since $\frac{\varphi(u+t)-\varphi(t)}{\sin(\frac{\pi}{T}u)} \sim_0 \frac{T}{\pi}\varphi'(t)$, the function is continuous and Riemann-Lebesgue theorem enables us to conclude.

Proof of the theorem

1. $C_c^\infty(]0,T[)$ is dense in $L^2(]0,T[)$, that is to say for all $f \in L^2(]0,T[)$, one has the property:

$$\forall \varepsilon > 0, \exists \varphi \in C_c^\infty(]0,T[)\ \|f-\varphi\|_{L^2} < \varepsilon$$

2. (Fjer-Dirichlet theorem)

$$\varphi \in C_c^\infty(]0,T[), \varepsilon > 0, \exists N_0, \forall N \geq N_0, \|\varphi - S_N\varphi\|_\infty \leq \varepsilon,$$

where S_N is the projection operator on the H_n defined by the e_n. Then

$$\|\varphi - S_N\varphi\|_2 = \int_0^T |\varphi(x) - S_N\varphi(x)|^2\,dx \leq T\,\|\varphi - S_N\varphi\|_\infty^2 < 2\pi\varepsilon^2$$

Conclusion:

$$\forall f \in L^2(]0,T[), \forall \varepsilon > 0,\ \forall N \geq N_0,$$

$$\begin{aligned}
\|f - S_N f\|_2 &= \|f - \varphi + \varphi - S_N\varphi + S_N\varphi - S_N f\|_2 \\
&\leq \|f-\varphi\|_2 + \|\varphi - S_N\varphi\|_2 + \|S_N\varphi - S_N f\|_2 \\
&\leq 2\varepsilon + \sqrt{T}\varepsilon,
\end{aligned}$$

using the fact that S_N being a projection operator on a convex set, it is 1-Lipschitz.

Another important theorem related to Fourier series is when function is periodic and piecewise differentiable, for which we have the following:

Theorem 9 Dirichlet Theorem

Let f be a function piecewise C^1 then the Fourier series converges at x_0 to $\frac{f(x_0^+)+f(x_0^-)}{2}$

In particular, at each point x where f is C^1, we may write:

$$f(x) = \sum c_n(f)\frac{e^{in\frac{2\pi}{T}x}}{\sqrt{T}}$$

In the next chapter we are going to see how to compute approximations of Fourier coefficients using the so called *discrete Fourier transform.*

2.8 exercices

Exercise 1

Let H be a prehilbertian space

1. $A \subset H$ such that $A \neq \emptyset$, then $A^\perp$ is closed

2. $A \subset B$ such that $A \neq \emptyset$, then $B^\perp \subset A^\perp$

3. $A \subset (A^\perp)^\perp \quad \forall A \subset H$

4. $\overline{A}^\perp = A^\perp$

5. If A is a subspace of H and if H is complete then $\overline{A} = (A^{\perp})^{\perp}$

Exercise 2
Let H be ans Hilbert space on $\mathbb{R}$. We denote by $\langle .,. \rangle$ the inner product on H and $\|.\|$ the corresponding norm. Let $\alpha_1, \alpha_2, \cdots, \alpha_m$, be m real strictly positive numbers, we denote by H_i the space H equipped with inner product

$$\langle x, y \rangle_i = \alpha_i \langle x, y \rangle$$

and $\mathcal{H}= H_1 \times H_2 \times \cdots \times H_m$ the product space. For $X = (x_1, x_2, \cdots, x_m)$ and $Y = (y_1, y_2, \cdots, y_m)$ in $\mathcal{H}$, let us put

$$\langle\langle X, Y \rangle\rangle = \sum_{i=1}^{m} \langle x_i, y_i \rangle_i$$

1. Show that the application $(X, Y) \to \langle\langle X, Y \rangle\rangle$ is an inner product on $\mathcal{H}$. Check that $\mathcal{H}$ is an Hilbert space for this inner product.

2. Let $\mathcal{A}$ be the subspace of $\mathcal{H}$ defined by:

$$\mathcal{A} = \{X = (x_1, x_2, \cdots, x_m) | x_1 = x_2 = \cdots = x_m\}$$

   - Show that $\mathcal{A}$ is closed in $\mathcal{H}$
   - Let $\mathcal{A}^{\perp}$ be the orthogonal of A in $\mathcal{H}$. Show that:
   $$\mathcal{A}^{\perp} = \left\{ Y = (y_1, y_2, \cdots, y_m) | \sum_{i=1}^{m} \alpha_i y_i = 0 \right\}$$
   - Let $S \in \mathcal{H}$. Show that there exist $X \in \mathcal{A}$ and $Y \in \mathcal{A}^{\perp}$ unique such that $S = X + Y$.

3. Compute X et Y.

Exercise 3
Let H be an Hilbert space, and $T \in \mathcal{L}(H)$ satisfying the following property:

$$\exists M > 0 \text{ such that } \|Tx\| \geq M\|x\| \quad \forall x \in H$$

1. Show that T is injective

2. Let us denote $Im(T) = \{y \in H; \exists \quad x \in H \text{ such that } Tx = y\}$. Show that $Im(T)$ is complete in H.

3. Show that for all $z \in H$, there exists $b \in Im(T)$ unique such that

$$\|z - b\| = \inf_{y \in Im(T)} \|z - y\|$$

4. We assume that T is self-adjoint, meaning that it satisfies the following property:

$$\langle Tx, y\rangle = \langle x, Ty\rangle \quad \forall x, y \in H$$

Show that T is bijective.

5. Let $A \in \mathcal{L}(H)$, self-adjoint, satisfying the following property:

$$\exists C > 0 \text{ such that } \inf_{\|x\|=1} | < Ax, x > | = C$$

Show that A est bijective.

Exercise 4
Define the orthogonal projection onto a closed affine subspace.

Exercise 5
Let us define

$$l^2_{\mathbb{C}}(\mathbb{N}) = \left\{ (x_n)_{n\in\mathbb{N}}; \sum_{n\geq 0} |x_n|^2 < \infty \right\}$$

1. Show that $l^2_{\mathbb{C}}(\mathbb{N})$ is a vector space.

2. For $x = (x_n) \in l^2$, we put $\|x\| = \left(\sum\limits_{n\geq 0} |x_n|^2\right)^{\frac{1}{2}}$. Show that $\|.\|$ is a norm and that $l^2_{\mathbb{C}}(\mathbb{N})$ is complete for this norm. Deduce that $l^2_{\mathbb{C}}(\mathbb{N})$ is an Hilbert space for a specific inner product.

Exercise 6
Let us introduce the space of functions

$$H = \{f : [-\pi, \pi] \to \mathbb{R} \ ; \ f \in L^2([-\pi, \pi]) \text{ } and \text{ } f \text{ } odd\}$$

1. Show that H, equipped with the inner product:

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$$

is an Hilbert space.

2. Show that the family $\mathcal{B} = (\frac{1}{\sqrt{\pi}} \sin nx)_{n\geq 1}$ is an Hilbertian basis of H.

3. Let f be a function defined on $[-\pi, \pi]$ by

$$f(x) = \begin{cases} -\frac{1}{2}(\pi + x) & si \quad x < 0 \\ \frac{1}{2}(\pi - x) & si \quad x \geq 0 \end{cases}$$

Compute the coefficients c_n of f on the basis $\mathcal{B}$: $f = \sum_{n=1}^{+\infty} c_n \frac{\sin nx}{\sqrt{\pi}}$.

4. What can be said about the convergence of the series $\sum_{n=1}^{+\infty} c_n \frac{\sin nx}{\sqrt{\pi}}$ in terms of :
a) point-wise convergence,

b) normal convergence,
c) convergence in the space H ?

Exercise 7

Haar basis

Let us consider the space $L^2([0,1])$ equipped with the usual inner product. Letφ be the indicator function of $[0,1]$ and ψ the function defined by:

$$\psi(x) = \varphi(2x) - \varphi(2x-1) \tag{2.1}$$

1. Plot ψ and check that φ and ψ are orthogonal.

We define, for $j \geq 0$ and $0 \leq k \leq 2^j - 1$:

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \tag{2.2}$$

2. Show that the the family $(\varphi, \psi_{j,k})$ is an Hilbertian basis of $L^2([0,1])$.

3. Show that the function $\psi_{j,k}$ is the interval $I_{j,k} = [k2^{-j}, (k+1)2^{-j}]$. Deduce that if f is constant on the intervall $[a,b] \subset [0,1]$, then $\langle f, \psi_{j,k} \rangle = 0$ if $I_{j,k} \subset [a,b]$.

4. Write an algorithm of any function f on the finite dimensional basis $(\varphi, \psi_{j,k})_{0 \leq j \leq J}$.

Exercise 8

Legendre polynomial

For $n \in \mathbb{N}$ and $x \in [-1,1]$, one defines:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x^2-1)^n\right) \tag{2.3}$$

1. Show that P_n is a degree n polynomial.

2. Show that (P_n) is orthogonal in $L^2([-1,1])$ for the usual inner product.

3. Show that

$$\int_{-1}^{1} P_n(x)^2 dx = \frac{2}{2n+1} \tag{2.4}$$

4. Deduce that $(\sqrt{\frac{2n+1}{2}}P_n)$ is an Hilbertian basis of $L^2([-1,1])$.

5. Show that (P_n) satisfies the following recurring relation:

$$\begin{cases} P_0 = 1 \ ; \ P_1 = x \\ P_n = \frac{2n-1}{n} x P_{n-1} - \frac{n-1}{n} P_{n-2} \quad n \geq 2 \end{cases}$$

6. Show that the P_n are solution of the differential equation:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Exercise 9

Chebyshev polynomials

We define for f and g in the space $L^2(]-1,1[)$:

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\frac{dx}{\sqrt{1-x^2}} \tag{2.5}$$

1. Show that the application $(f,g) \to \langle f,g \rangle$ is an inner product on $L^2(]-1,1[)$.
One defines for $n \geq 0$ the following sequence:

$$T_n(x) = \cos(narccosx) \tag{2.6}$$

2. Show that the family $(\sqrt{\frac{1}{\pi}}T_0, \sqrt{\frac{2}{\pi}}T_n)$ is an Hilbertian basis of $L^2(]-1,1[)$.
3. Show that T_n satisfies the following recurring relation:

$$\begin{cases} T_0 = 1 \ ; \ T_1 = x \\ T_n = 2xT_{n-1} - T_{n-2} \quad n \geq 2 \end{cases}$$

4. Show that $T_n(1) = 1$, $T_n(-1) = (-1)^n$, $T_{2n}(0) = (-1)^n$, $T_{2n+1}(0) = 0$.
5. Show that the T_n are solutions to the following differential equation:

$$(1-x^2)y'' - xy' + n^2 y = 0$$

Chapter 3

Discrete Fourier Transform

3.1 DFT definition and properties

3.1.1 Definition

Definition 1 *Let* $X = (x_0, x_1, \cdots, x_{N-1})$ *a vector of* $\mathbb{C}^{\mathbb{N}}$*. We call discrete Fourier transform (DFT) of* X *the vector* $\hat{X} = (\hat{x}_0, \hat{x}_2, \cdots, \hat{x}_{N-1})$ *defined by:*

$$\hat{x}_k = \sum_{n=0}^{N-1} x_n e^{-2i\pi\frac{kn}{N}} \;\; k = 0, 1 \cdots, N-1 \tag{3.1}$$

which admits as inverse transform:

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} \hat{x}_k e^{2i\pi\frac{kn}{N}} \;\; n = 0, 1 \cdots, N-1 \tag{3.2}$$

The proof is left as an exercise.

3.2 Properties of DFT

We have the property of norm conservation:

Theorem 1 (Parseval formula)

One has the relation:

$$\sum_{n=0}^{N-1} x_n \bar{y}_n = \frac{1}{N} \sum_{k=0}^{N-1} \hat{x}_k \overline{\hat{y}}_k$$

and consequently

$$\sum_{n=0}^{N-1} |x_n|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |\hat{x}_k|^2$$

Proposition 1

Periodising (x_n) with period N over all $\mathbb{Z}$, one also periodizes $(\hat{x}_k)$ with period N. The sequences (x_n) and $(\hat{x}_k)$ are thus extended to $\mathbb{Z}$ by periodization of length N. Let $(x_n) \stackrel{DFT}{\rightarrow} (\hat{x}_k)$ then:

1. $(x_{-n}) \stackrel{DFT}{\rightarrow} (\hat{x}_{-k})$
2. $(\overline{x_n}) \stackrel{DFT}{\rightarrow} (\overline{\hat{x}_{-k}})$
3. $(\overline{x_{-n}}) \stackrel{DFT}{\rightarrow} (\overline{\hat{x}_k})$

Proposition 2

Using the same notation one has:

i) (x_n) is even (resp. odd) $\Leftrightarrow$ $(\hat{x}_k)$ is even (resp. odd)

ii) (x_n) is real $\Leftrightarrow \forall k \in \mathbb{Z}\ \hat{x}_{-k} = \overline{\hat{x}_k}$

iii) (x_n) is even and real $\Leftrightarrow$ $(\hat{x}_k)$ is real and even

iv) (x_n) is real and odd $\Leftrightarrow$ $(\hat{x}_k)$ is odd and imaginary.

3.3 Applications

3.3.1 Approximation of Fourier coefficients using DFT

On seek to compute an approximation of the Fourier coefficients of f periodic with period T, Let (we use the normalization used by the physicists) $c_k = \frac{1}{T}\int_0^T f(t)e^{-2i\pi k\frac{t}{T}}dt$, for $-\frac{N}{2} \leq k < \frac{N}{2}$. Using the left rectangle method to approximate the integral we get:

$$\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} f(\frac{nT}{N})e^{-2i\pi\frac{k}{T}\frac{nT}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} y_n\omega_N^{-nk}, \quad -\frac{N}{2} \leq k < \frac{N}{2}.$$

Putting $y_n = f(\frac{nT}{N})$ and $\omega_N = e^{\frac{2i\pi}{N}}$. Then calling Y_k the DFT of y_n/N, we obtain that $c_k \approx \tilde{c}_k = Y_k$ if $0 \leq k \leq \frac{N}{2}$ and $c_k \approx \tilde{c}_k = Y_{k+N}$ si $-\frac{N}{2} \leq k < 0$. So the approximation of Fourier coefficients using the left rectangle method corresponds to the DFT of the sampled signal (modulo the transformation of Y into $\tilde{c}$, fftshift command in Matlab).

3.3.2 Relation between Fourier coefficients and DFT

Let us consider the Fourier series of a signal f assumed to be periodic with period T:

$$f(t) = \sum_{n=-\infty}^{n=+\infty} c_n e^{2i\pi n\frac{t}{T}}$$

where we assume, for the sake of simplicity that the series converges normaly:

$$\sum_{n=-\infty}^{n=+\infty} |c_n| < +\infty,$$

which is the case if f is continuous and piecewise C^1. One can group the indices as follows:

$$\frac{1}{N} f(k\frac{n}{N}) = \frac{y_n}{N} = \frac{1}{N} \sum_{m=-\infty}^{m=+\infty} c_m \omega_N^{mk} = \frac{1}{N} \sum_{n=-N/2}^{N/2-1} \left(\sum_{q=-\infty}^{q=+\infty} c_{n+qN} \right) \omega_N^{nk}$$

from which we deduce using the inverse DFT:

$$\tilde{c}_n = \sum_{q=-\infty}^{q=+\infty} c_{n+qN}$$

leading to:

$$\tilde{c}_n - c_n = \sum_{q \neq 0} c_{n+qN}.$$

We deduce from this that for a fixed N the approximation $c_n \approx \tilde{c}_n$ for $-\frac{N}{2} \leq n < \frac{N}{2}$ is all the better that the Fourier coefficients tend faster to 0 when n tends to $+\infty$.

3.3.3 Computation of the Fourier transform of finitely supported signals

Let f be a function compactly supported on $[0, T]$ and integrable on that interval. Then, we may write:

$$\hat{f}(\xi) = \int_0^T f(t) e^{-2i\pi t\xi} dt \approx \frac{1}{T} \sum_{k=0}^{N-1} f(\frac{kT}{N}) e^{-2i\pi \frac{kT}{N}\xi}$$

From this we deduce that

$$\hat{f}(\frac{n}{T}) \approx \frac{1}{T} \sum_{k=0}^{N-1} f(\frac{kT}{N}) e^{-2i\pi \frac{kn}{N}}$$

So $(\hat{f}(\frac{n}{T}))_{n=0,\cdots,N-1}$ can be approximated by the DFT of $(\frac{1}{T} f(\frac{kT}{N}))_{k=0,\cdots,N-1}$. However, note that due to periodicity of the obtained approximation, the approximation is valid only for the $\frac{N}{2}$ first coefficients. The last $N/2$ coefficients are the conjugate of the Fourier transform obtained for negative frequencies, due to symmetry properties of the DFT.

3.4 FFT algorithm

This algorithm was initially proposed by Cooley and Tuckey (1965). Let us suppose that $N = 2m$ and then set $\omega_N = e^{\frac{2i\pi}{N}}$. In $\hat{x}_k$ Let us group the terms with even indices and those with odd indices.

$$\hat{x}_k = P_k + \omega_N^{-k} I_k$$

with

$$\begin{cases} P_k &= x_0 + x_2 \omega_N^{-2k} + \cdots + x_{N-2} \omega_N^{-(N-2)k} \\ I_k &= x_1 + x_3 \omega_N^{-2k} + \cdots + x_{N-1} \omega_N^{-(N-2)k} \end{cases}$$

We remark that $P_{k+m} = P_k$ and also that $I_{k+m} = I_k$. Furthermore as $\omega_N^{-(k+m)} = -\omega_N^{-k}$, we can save some computational time. For $k = 0, \cdots, m-1$, one successively computes:

1. One computes P_k and $\omega_N^{-k} I_k$
2. Then one writes $\hat{x}_k = P_k + \omega_N^{-k} I_k$
3. And then one deduces $\hat{x}_{k+m} = P_k - \omega_N^{-k} I_k$

The computational cost of this first step is $2(m-1)^2 + m - 1$ so approximative $\frac{1}{2}N^2$ multiplications while N^2 multiplications are needed for a direct computation.
Then, we remark that P_k and I_k are Fourier transforms, independent one from another. So we propose to do the carry out the previous decomposition assuming m is still even. To go from stage (vector of length m) to another (vecteur of length $2m$) is carried out using the following formulae

$$\left\{ \begin{array}{rcl} \hat{x}_k & = & P_k + \omega_N^{-k} I_k \\ \hat{x}_{k+m} & = & P_k - \omega_N^{-k} I_k \end{array} \right.$$

Computational cost of the algorithm:
For $N = 2^p$, let us denote M_p the number of multiplications used by this algorithm, and let denote by by A_p the number of additions:

- Computational cost for P_k : $[M_{p-1}, A_{p-1}]$
- Computational cost for I_k : $[M_{p-1}, A_{p-1}]$
- Multiplications by ω_N^{-k}, $k \geq 1$: $[2^{p-1} - 1, 0]$
- Additions : $[0, 2^p]$.

From this we deduce the following relations:

$$\left\{ \begin{array}{rcl} M_1 & = & 0 \\ M_p & = & 2M_{p-1} + 2^{p-1} - 1 \end{array} \right. \qquad \left\{ \begin{array}{rcl} A_1 & = & 2 \\ A_p & = & 2A_{p-1} + 2^p \end{array} \right.$$

Finally we get:

$$\left\{ \begin{array}{rcl} M_p & = & (p-2)2^{p-1} + 1 \\ A_p & = & p2^p \end{array} \right.$$

or with respect to N: $[N/2(log_2(N) - 2) + 1, Nlog_2(N)]$.

Chapter 4

Continuous time Fourier transform

4.1 Fourier transform in $L^1(\mathbb{R})$

4.1.1 Density theorems

Definition 1 *Let f be a continuons function on a open set Ω of $\mathbb{R}^N$. The support of the function f which we denote by Supp(f) is the complement set in Ω of the largest open set on which f is null.*

$$\text{Supp}\,(f) = \overline{\{x \in \Omega \quad | \quad f(x) = 0\}}$$

Definition 2 *$C_c(\Omega)$ stands for the vector space of the functions continuous on Ω and compactly supported, i.e. :*

$$C_c(\Omega) = \{f \in C(\Omega) \quad | \quad \exists K \, compact\ set\ , K \subset \Omega s.t.\ x \in \Omega \backslash K \ \ f(x) = 0\}$$

Theorem 1 Density theorem

Let $\Omega \subset \mathbb{R}^N$ be an open set. $C_c(\Omega)$ is dense in $L^p(\Omega)$ for $p \in \{1, 2\}$ i.e. :

$$\forall\, p \in \{1, 2\},\ \forall\, \varepsilon > 0,\ \forall\, f \in L^p(\Omega),\ \exists g \in C_c(\Omega)\, \|f - g\|_p \leq \varepsilon$$

Remark: This theorem remains true for functions in $C_c^k(\Omega)$, $k \leq \infty$.

4.1.2 Definition of the Fourier transform in $L^1(\mathbb{R})$

Definition 3 (Fourier Transform) *Let $f \in L^1(\mathbb{R})$, we define the Fourier transform $\hat{f}$ of f as:*

$$\forall \nu \in \mathbb{R},\ \hat{f}(\nu) \stackrel{def}{=} \int_{-\infty}^{+\infty} f(x) e^{-2i\pi\nu x} dx$$

ν is called the frequency (Hz) The application: $\mathcal{F} : f \mapsto \hat{f}$ is called Fourier transform.

4.1.3 Riemann-Lebesgue Theorem

Theorem 2 **(Riemann-Lebesgue)**

1. $\mathcal{F} : f \mapsto \hat{f}$ is a linear application, continuous from $L^1(\mathbb{R})$ onto $L^\infty(\mathbb{R})$.
2. if $f \in L^1(\mathbb{R})$, then $\hat{f}$ is continuous on $\mathbb{R}$ and $\lim_{\nu \to \pm\infty} \hat{f}(\nu) = 0$.

Proof

1. - $\mathcal{F}$ is linear (linearity of $\int$).
   - To show the continuity of $\mathcal{F}$ it suffices to prove the result at 0:

$$\forall f \in L^1(\mathbb{R}),\ \|\mathcal{F}(f)\|_{L^\infty(\mathbb{R})} \leq C\|f\|_{L^1(\mathbb{R})}$$

$$\forall \nu,\ \left|\hat{f}(\nu)\right| \ = \ \left|\int_{-\infty}^{+\infty} f(x)e^{-2i\pi\nu x}\right| \leq \int_{-\infty}^{+\infty} |f(x)|\, dx = \|f\|_{L^1(\mathbb{R})}$$

   so $\hat{f} \in L^\infty(\mathbb{R})$ et $\|\hat{f}\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}$.

2. Let $g \in C_c^1(\mathbb{R})$, then

$$\begin{aligned}
\hat{g}(\nu) &= \int_{\mathbb{R}} g(x)e^{-2i\pi\nu x}dx = \left[g(x)\frac{e^{-2i\pi\nu x}}{-2i\pi\nu}\right]_{-\infty}^{+\infty} - \int_{\mathbb{R}} g'(x)\frac{e^{-2i\pi\nu x}}{-2i\pi\nu}dx \\
|\hat{g}(\nu)| &\leq \frac{1}{2\pi|\nu|}\int_{\mathbb{R}} \left|g'(x)\right| dx \xrightarrow[\nu \to \pm\infty]{} 0 \text{ since } \left\|g'\right\|_{L^1(\mathbb{R})} < +\infty.
\end{aligned}$$

   But $C_c^1(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, so : $\forall f \in L^1(\mathbb{R}),\ \forall \varepsilon > 0,\ \exists g \in C_c^1(\mathbb{R}),\ \|f - g\|_{L^1(\mathbb{R})} < \varepsilon$. Then, as

$$|\hat{f}(\nu)| \leq \|f - g\|_{L^1(\mathbb{R})} + |\hat{g}(\nu)|,$$

   we get $\lim_{\nu \to \pm\infty} \hat{f}(\nu) = 0$.

4.1.4 Example

$\Pi = \mathbb{1}_{]-\frac{1}{2};\frac{1}{2}[}$, $\hat{\Pi}(\nu) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2i\pi\nu x}dx = \frac{\sin(\pi\nu)}{\pi\nu}$: cardinal sine function.

4.1.5 Other Properties

Proposition 1 (delay)

$f \in L^1(\mathbb{R}),\ \tau \in \mathbb{R}$
$\forall x \in \mathbb{R},\ g(x) = f(x-\tau)$ for $g \in L^1(\mathbb{R})$, then $\forall \nu \in \mathbb{R},\ F(g)(\nu) = \hat{g}(\nu) = e^{-2i\pi\nu\tau}\hat{f}(\nu)$

Proposition 2

$f \in L^1(\mathbb{R}),\ a > 0.$
$\forall x \in \mathbb{R},\ g(x) = f(ax)$ with $g \in L^1(\mathbb{R})$. $\forall \nu \in \mathbb{R},\ \mathcal{F}(g(\nu)) = \frac{1}{a}f(\frac{\nu}{a})$.

Theorem 3

1. if $x \to x^k f(x)$ is in $L^1(\mathbb{R})$ for $k \in \{0, \cdots, n\}$ then $\hat{f}$ is n times differentiable, and one has:
$$\hat{f}^{(k)}(\nu) = \widehat{g_k}(\nu) \quad \forall \nu \in \mathbb{R}$$
where $g_k(x) = (-2i\pi x)^k f(x)$
2. If $f \in L^1(\mathbb{R}) \bigcap C^n(\mathbb{R})$ and if $f^{(k)} \in L^1(\mathbb{R})$ then for all $k \in \{1, \cdots, n\}$ one has:
$$\widehat{f^{(k)}}(\nu) = (2i\pi\nu)^k \hat{f}(\nu) \quad \forall \nu \in \mathbb{R}$$
3. If $f \in L^1(\mathbb{R})$ and if $supp(f)$ is bounded, then $\hat{f} \in C^\infty(\mathbb{R})$.

Proof

1. For all $k \leq n$, $\frac{\partial^k f(x)e^{-2i\pi\nu x}}{\partial^k \nu}$ is continuous for all ν and almost all x. Furthermore, $|\frac{\partial^k f(x)e^{-2i\pi\nu x}}{\partial^k \nu}| = |(-2i\pi x)^k f(x)|$ belongs to $L^1(\mathbb{R})$ and $\hat{f}$ belongs to C^k and then one applies the theorem on the differentiation of an integral dependent on a parameter.

2. Let us compute $\widehat{f'}$. By integrating by parts, we get that:
$$\widehat{f'}(\nu) = [f(x)e^{-2i\pi\nu x}]_{-\infty}^{\infty} + \int_{\mathbb{R}} f(x)(2i\pi\nu)e^{-2i\pi\nu x}dx.$$
Here we need to remark that if f is integrable and belongs to C^1, and is such that f' is also integrable then
$$f(x) = f(a) + \int_a^x f'(t)dt.$$
As f' is integrable, the integral has a limit when x tends to $\pm\infty$, so $f(x)$ has a limit when x tends to infinity. Moreover, this limit is necessarily null since f is integrable. We thus get $\widehat{f'}(\nu) = (2i\pi\nu)\hat{f}(\nu)$. Reasoning by induction, we get the expected result.

Compute the following Fourier transform

- Let $-\infty < a < b < +\infty$ et $f = \chi_{[a,b]}$.
- We denote $u(t)$ the Heavydide function (equal to 1 if $t > 0$ et 0 otherwise). $sign(t)$ is the sign function. Let α be a complex number with positive real part. Compute the following Fourier transforms:

$$\begin{array}{ll} i) f(t) = e^{-\alpha t}u(t) & ii) f(t) = e^{\alpha t}u(-t) \\ iii) f(t) = e^{-\alpha|t|} & iv) f(t) = \frac{t^k}{k!}e^{-\alpha t}u(t) \\ v) f(t) = \frac{t^k}{k!}e^{\alpha t}u(-t) & vi) f(t) = sign(t)e^{-\alpha|t|} \end{array}$$

- Compute $\mathcal{F}(f)$ with $f : x \mapsto e^{-\pi x^2}$

Proposition 3

Let $f, g \in L^1(\mathbb{R})$, then $f\hat{g}$ et $\hat{f}g$ both belong to $L^1(\mathbb{R})$ and one has:

$$\int_{\mathbb{R}} f\hat{g} = \int_{\mathbb{R}} \hat{f}g$$

4.1.6 Inversion of the Fourier transform in $L^1(\mathbb{R})$

Definition 4 *For any function f belonging to $L^1(\mathbb{R})$ let us write:*

$$\overline{\mathcal{F}}(f)(\nu) = \int_{\mathbb{R}} f(x)e^{2i\pi\nu x}dx.$$

One then have the following inversion theorem:

Theorem 4

1. Let $f \in L^1(\mathbb{R})$. Let us assume f is continuous at $x \in \mathbb{R}$ and that $\hat{f} \in L^1(\mathbb{R})$. Then,

$$\overline{\mathcal{F}}\hat{f}(x) = f(x)$$

2. Let $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$ then

$$\overline{\mathcal{F}}\hat{f}(x) = f(x) \text{ for almost all } x$$

Proof 1) Let us first prove the first point. For $n \in \mathbb{N}^*$, let us define $g_n(x) = e^{-\frac{2\pi}{n}|x|}$, for which we get $\widehat{g_n}(\nu) = \frac{1}{\pi}\frac{n}{1+n^2\nu^2}$. Since g_n is in $L^1(\mathbb{R})$, we can write:

$$\int_{\mathbb{R}} \hat{f}(\nu)g_n(\nu)e^{2i\pi x\nu}d\nu = \int_{\mathbb{R}} f(\nu)\widehat{g_n}(\nu - x)d\nu$$

The term on the left hand side tends to $\overline{\mathcal{F}}\hat{f}(x)$ using the dominated convergence theorem. Let us show that the term on the right hand side tends to $f(x)$. As $\int_{\mathbb{R}} \widehat{g_n}(\nu)d\nu = 1$, one may write

$$\int_{\mathbb{R}} f(\nu)\widehat{g_n}(\nu - x)d\nu - f(x) = \int_{\mathbb{R}} (f(\nu + x) - f(x))\widehat{g_n}(\nu)d\nu.$$

Let $\epsilon > 0$, there exists $\eta = \eta(\epsilon, x)$ such that $|y - x| \leq \eta \Rightarrow |f(y) - f(x)| \leq \epsilon$ (f continuous at x). One can then write:

$$\int_{\mathbb{R}} (f(x+\nu) - f(x))\widehat{g_n}(\nu)d\nu = \int_{|\nu|\leq\eta} (f(x+\nu) - f(x))\widehat{g_n}(\nu) + \int_{|\nu|\geq\eta} (f(x+\nu) - f(x))\widehat{g_n}(\nu).$$

For all $n \in \mathbb{N}^*$, one has:

$$\int_{|\nu|\leq\eta} |f(x+\nu) - f(x)|\widehat{g_n}(\nu)d\nu \leq \epsilon \int_{\mathbb{R}} \widehat{g_n}(\nu)d\nu = \epsilon.$$

Furthermore,

$$|\int_{|\nu|\geq\eta} f(x)\widehat{g_n}(\nu)|d\nu \leq |f(x)|(1 - \frac{2}{\pi}atan(\eta n)),$$

which tends to 0 when n tends to infinity. Furthermore, as $\widehat{g_n}$ is even and decreasing over $\mathbb{R}^+$

$$|\int_{|\nu|\geq\eta} f(x+\nu)\widehat{g_n}(\nu)| \leq \widehat{g_n}(\eta)\|f\|_1,$$

this expression tends to 0 when n tends to infinity. This proves the theorem.
2) Let us now show point 2. We multiply the function to be integrated by $\hat{h}_\epsilon(\nu) = e^{-\pi\epsilon^2\nu^2}$:

$$I_\epsilon = \int_{\mathbb{R}} (\int_{\mathbb{R}} f(u)e^{-\pi\epsilon^2\nu^2}e^{2i\pi\nu(x-u)}du)d\nu.$$

Then, we have $(u,\nu) \to \phi(u,\nu) = f(u)e^{-\pi\epsilon^2\nu^2}e^{2i\pi\nu(x-u)} \quad \in L^1(\mathbb{R}^2)$. By applying Fubini theorem, we get two different expressions of I_ϵ:

i) By integrating with respect to u, one gets $I_\epsilon = \int_{\mathbb{R}} \hat{f}(\nu)e^{-\pi\epsilon^2\nu^2}e^{2i\pi\nu x}d\nu$.
But since $|\hat{f}(\nu)e^{-\pi\epsilon^2\nu^2}e^{2i\pi\nu x}| \leq |\hat{f}(\nu)|$ which belongs to $L^1(\mathbb{R})$, and since $\lim_{\epsilon\to 0} e^{-\pi\epsilon^2\nu^2} = 1$, by applying the dominated convergence theorem, we get that $\lim_{\epsilon\to 0} I_\epsilon = \int_{\mathbb{R}} \hat{f}(\nu)e^{2i\pi\nu x}dx$.

ii) Integrating with respect to v:

$$I_\epsilon = \int_{\mathbb{R}} f(u)\left(\int_{\mathbb{R}} e^{-\pi\epsilon^2\nu^2}e^{2i\nu(x-u)}d\nu\right)du = \int_{\mathbb{R}} f(u)\frac{1}{\epsilon}e^{-\pi(\frac{x-u}{\epsilon})^2}du,$$

using the properties of the Fourier transforms of Gaussian functions and the dilation formula. Furthermore, we know that the function $h_\epsilon(x) = \frac{1}{\epsilon}e^{-\pi(\frac{x}{\epsilon})^2}$ has its integral equal to 1. One then deduce that:

$$\begin{aligned}
\int_{\mathbb{R}} |I_\epsilon(x) - f(x)| &= \int_{\mathbb{R}}\int_{\mathbb{R}} |(f(x-u) - f(x))h_\epsilon(u)|du \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x-\epsilon u) - f(x)|h_\epsilon(u)du \\
&\leq \int_{\mathbb{R}} \|f(x-\epsilon u) - f(x)\|_1 h(u)du.
\end{aligned}$$

In $L^1(\mathbb{R})$, one has the following property:

Proposition 4

Let $f \in L^1(\mathbb{R})$, $h \in \mathbb{R}$, and define $\tau_h f(x) = f(x-h)$. Then $\tau_h h \in L^1(\mathbb{R})$ et $\lim_{h\to 0} \|\tau_h f - f\|_1 = 0$.

Proof The theorem uses the density of $C_0(\mathbb{R})$ in $L^1(\mathbb{R})$. Indeed, let g_n be a sequence in $C_0(\mathbb{R})$ tending to f in $L^1(\mathbb{R})$, i.e.:

$$\forall \epsilon > 0 \; \exists N \; \forall n \geq N \; \|f - g_n\|_1 \leq \epsilon$$

One may then write:

$$\int_{\mathbb{R}} |f(x+\eta) - f(x)| \leq \int_{\mathbb{R}} |f(x+\eta) - g_n(x+\eta)| + \int_{\mathbb{R}} |g_n(x+\eta) - g_n(x)| + \int_{\mathbb{R}} |g_n(x) - f(x)|$$

Let N be such that $|f - g_N| \leq \frac{\epsilon}{3}$ and choose η such that $\|g_N(x+\eta) - g_N(x)\|_1 \leq \frac{\epsilon}{3}$ (dominated convergence theorem), hence the result.

Since $\|f(x-\epsilon u) - f(x)\|_1 |h(u)| \leq 2\|f\|_1 |h(u)|$ which belongs to $L^1(\mathbb{R})$, applying the dominated convergence theorem, we deduce that: $\lim_{\epsilon\to 0} \|I_\epsilon - f\|_1 = 0$.

So I_ϵ tends to f in $L^1(\mathbb{R})$ so there exists a sub-sequence $I_{\phi(\epsilon)}$ converging to f almost everywhere, hence the result.

4.1.7 Convolution product in $L^1(\mathbb{R})$

Theorem 5 (and definition)

$f \in L^1(\mathbb{R})$, $g \in L^1(\mathbb{R})$. Let us define : $\forall x \in \mathbb{R}$, $(f \star g)(x) = \int_{\mathbb{R}} f(y)g(x-y)dy$.
Then $(f \star g)$ is defined almost everywhere, integrable and $\|f \star g\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}$.

Proof Using Fubini theorem:

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(x-y)| \, dy \right) dx = \int_{\mathbb{R}} |f(y)| \left(\int_{\mathbb{R}} |g(x-y)| \, dx \right) dy \tag{4.1}$$

and by changing variables $u = x - y$, we obtain:

$$(4.1) \quad = \quad \int_{\mathbb{R}} |f(y)| \left(\int_{\mathbb{R}} |g(u)| \, du \right) dy = \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})} < +\infty,$$

since $\int_{\mathbb{R}} |g(u)| \, du = \|g\|_{L^1(\mathbb{R})}$. So $x \mapsto \int_{\mathbb{R}} |f(y)g(x-y)| \, dy$ is integrable and thus finite almost everywhere. Consequently $(f \star g)$ is defined almost everywhere, integrable and:

$$\int_{\mathbb{R}} |(f \star g)(x)| \, dx \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}$$

Proposition 5

Let $f, g, h \in L^1(\mathbb{R})$.

- $f \star g = g \star f$
- $(f \star g) \star h = f \star (g \star h)$
- $(f + g) \star h = f \star h + g \star h$

4.1.8 Illustration: moving average

At each point $x \in \mathbb{R}$, one replaces $f(x)$ by its average $\bar{f}(x)$ over an interval of length τ :

$$\bar{f}(x) = \frac{1}{\tau}\int_{x-\frac{\tau}{2}}^{x+\frac{\tau}{2}} f(t)dt = \frac{1}{\tau}\int_{\mathbb{R}} \chi_{[x-\frac{\tau}{2};x+\frac{\tau}{2}]}(t)f(t)dt = \int_{\mathbb{R}} h(x-t)f(t)dt$$

where $h : u \mapsto \frac{1}{\tau}\mathbb{1}_{[-\frac{\tau}{2};\frac{\tau}{2}]}$.

<u>In pratice</u>

- choice of a more regular window.
- choice for τ depends on the scale of the phenomena one wants to highlight.

4.1.9 Convolution and Fourier transform

Theorem 6 (Convolution and Fourier transform)

i) Let $f \in L^1(\mathbb{R})$, $h \in L^1(\mathbb{R})$. Then $\forall \nu \in \mathbb{R}$, $\mathcal{F}(f \star h)(\nu) = \hat{f}(\nu)\hat{h}(\nu)$.

ii) Let $f \in L^1(\mathbb{R})$, $h \in L^1(\mathbb{R})$ such that $\hat{f}$ and $\hat{h}$ are also in $L^1(\mathbb{R})$, then for almost all ν, one has: $\hat{f} \star \hat{h} = \mathcal{F}(fh)$.

Example 1 *$\mathcal{F}(\bar{f})(\nu) = \hat{h}(\nu)\hat{f}(\nu) = \frac{\sin(\pi\nu\tau)}{\pi\nu\tau}\hat{f}(\nu)$. $\hat{h}$ is called transfer function. One can then adapt $\frac{1}{\tau}$ to the frequencies of interest in signal f.*

Proof i) Applying Tonelli's theorem: $\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |f(y)g(x-y)|\, dy\right) \left|e^{-2i\pi\nu x}\right| dx = \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})} < +\infty$

since $\left|e^{-2i\pi\nu x}\right| = 1$. Then, from Fubini's theorem:

$$\begin{aligned}
\mathcal{F}(f \star g)(\nu) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(y) g(x-y) dy \right) e^{-2i\pi\nu x} dx \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(y) g(x-y) dy \right) e^{-2i\pi\nu(x-y+y)} dx \\
&= \int_{\mathbb{R}} f(y) e^{-2i\pi\nu y} \left(\int_{\mathbb{R}} g(x-y) e^{-2i\pi\nu(x-y)} dx \right) dy \\
&= \int_{\mathbb{R}} f(y) e^{-2i\pi\nu y} \left(\int_{\mathbb{R}} g(u) e^{-2i\pi\nu u} du \right) dy = \hat{f}(\nu)\hat{g}(\nu)
\end{aligned}$$

ii) Since $\hat{f}$ and $\hat{g}$ are both in $L^1(\mathbb{R})$, we get, remarking that $\overline{\mathcal{F}}$ has the same properties as $\mathcal{F}$:

$$\begin{aligned}
\overline{\mathcal{F}}(\hat{f} \star \hat{g}) &= \overline{\mathcal{F}}(\hat{f})\overline{\mathcal{F}}(\hat{g}) \\
&= fg \text{ almost everywhere}
\end{aligned}$$

Finally, since $f = \overline{\mathcal{F}}(\hat{f})$, f is bounded one can compute the Fourier transform of fg to obtain: $\hat{f} \star \hat{g} = \mathcal{F}(fg)$.

4.2 Fourier transform on $L^2(\mathbb{R})$

One of the main drawback with considering the Fourier transform in $L^1(\mathbb{R})$, is its non invertibility in general. In what follows, we are going to see how to define the Fourier transform on $L^2(\mathbb{R})$ as a bijective application from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.

4.2.1 The space $L^2(\mathbb{R})$

Let $f, g \in L^2(\mathbb{R})$, we recall that $L^2(\mathbb{R})$ is equipped with the inner product $\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)} dx$ and that the norm on $L^2(\mathbb{R})$ is defined by:$\|f\|_2 = \sqrt{\langle f, f \rangle}$. $L^2(\mathbb{R})$ is an Hilbert space for which one has the Cauchy-Schwarz theorem:

Theorem 7 (Cauchy-Schwarz)

Let f and g belong to $L^2(\mathbb{R})$, we then have the following property:

$$\left| \int_{\mathbb{R}} f(t)\bar{g}(t) \right| \leq \sqrt{\int_{\mathbb{R}} |f|^2(t) dt} \sqrt{\int_{\mathbb{R}} |g|^2(t) dt}$$

4.2.2 Convolution in $L^2(\mathbb{R})$

Convolution in $L^2(\mathbb{R})$ is defined for f and g in $L^2(\mathbb{R})$ by $F(x) = \int_{\mathbb{R}} f(x-y) g(y) dy$, satisfying :

Theorem 8

$F \in C^0(\mathbb{R}) \bigcap L^\infty(\mathbb{R})$

Proof $F \in L^\infty(\mathbb{R})$ by direct application of Cauchy-Schwarz theorem. Then, we have

$$\begin{aligned}|F(x+\eta)-F(x)| &= |\int_{\mathbb{R}} (f(x+\eta-y)-f(x-y))\, g(y)|dy \\ &\leq \int_{\mathbb{R}} |f(x+\eta-y)-f(x-y)|^2 dy\ \|g\|_2.\end{aligned}$$

The term depending on η tends to 0 with η (to prove it we use the density of $C_c(\mathbb{R})$ in $L^2(\mathbb{R})$). So, f is continuous at x.

Example : the correlation in $L^2(\mathbb{R})$ is defined by:

$$G(x) = \int_{\mathbb{R}} f(x+t)\bar{f}(t)dt = \check{f} \star \bar{f}(-x),$$

which is continuous.

4.2.3 Property of the Fourier Transform in $L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$

Theorem 9 (Plancherel-Parseval)

Let f and h belonging to $L^2(\mathbb{R}) \bigcap L^1(\mathbb{R})$, then one has:

$$\int_{\mathbb{R}} f(t)\overline{h(t)}dt = \int_{\mathbb{R}} \hat{f}(\nu)\overline{\hat{h}(\nu)}d\nu$$

If $f = h$, one has the following property : $\int_{\mathbb{R}} |f|^2 = \int_{\mathbb{R}} |\hat{f}|^2$

Proof We first prove that the Fourier transform of a function in $L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$ is in $L^2(\mathbb{R})$ showing that $\|f\|_2^2 = \|\hat{f}\|_2^2$.

Let us first consider $g_\alpha(x) = e^{-\alpha x^2}$, whose Fourier transform is $\hat{g}_\alpha(x) = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\pi^2 x^2}{\alpha}}$. Applying the monotone convergence theorem, one gets:

$$\int_{\mathbb{R}} g_\alpha(x)|\hat{f}(x)|^2 \underset{\alpha \to 0}{\to} \int_{\mathbb{R}} |\hat{f}|^2 \leq +\infty$$

since $g_\alpha(x)|\hat{f}(x)|^2$ is positive, belongs to $L^1(\mathbb{R})$ and is increasing when α decreases.
Moreover, as the function $(x,u,y) \to f(y)\overline{f(u)}e^{i2\pi x(u-y)}g_\alpha(x)$ is in $L^1(\mathbb{R}^3)$ (applying Tonnelli's theorem).

$$\begin{aligned}\int_{\mathbb{R}} g_\alpha(x)|\hat{f}(x)|^2 &= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} \overline{f(u)} \int_{\mathbb{R}} e^{-i2\pi x(y-u)} g_\alpha(x) dx dy du \\ = \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} \overline{f(u)}\hat{g}_\alpha(y-u) dy du &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(y+u)\overline{f(u)} du\ \hat{g}_\alpha(y) dy \\ &= \int_{\mathbb{R}} G(y)\hat{g}_\alpha(y)dy = \int_{\mathbb{R}} G(\sqrt{\frac{\alpha}{\pi}}y)e^{-\pi y^2}dy \\ &\underset{\alpha \to 0}{\to} G(0) = \|f\|_2^2,\end{aligned}$$

the limit being obtained applying the dominated convergence theorem, this means that the Fourier transform of a function in $L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$ is in $L^2(\mathbb{R})$.
We are now going to show Plancherel formula. Let f and h be in $L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$. $\hat{f}\overline{\hat{h}}$ belongs to $L^1(\mathbb{R})$ as a product of functions in $L^2(\mathbb{R})$. Moreover, defining $\check{h}(t)=\bar{h}(-t)$, one has $\mathcal{F}(f*\check{h})=\hat{f}\overline{\hat{h}}$ which belongs to $L^1(\mathbb{R})$. So, from the inversion theorem of the Fourier transform, and as $f*\check{h}$ is continuous being, the convolution of functions in $L^2(\mathbb{R})$, one has $f*\check{h}(x)=\overline{\mathcal{F}}(\hat{f}\overline{\hat{h}})(x)$, for all x. Considering its value at $x=0$, one gets Plancherel inequality.

4.2.4 Fourier transform in $L^2(\mathbb{R})$

Proposition 6

$L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.

Proof Let us define $f_N(x)=\chi_{[-N,N]}(x)f(x)$ which belongs to $L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$, one checks that f_N tends to f in $L^2(\mathbb{R})$.

Let f_N be a sequence of functions in $L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$ converging to f in $L^2(\mathbb{R})$. We have seen that $\hat{f}_N$ belongs to $L^2(\mathbb{R})$, furthermore $\hat{f}_N$ is a Cauchy sequence in $L^2(\mathbb{R})$ since

$$\|\hat{f}_N-\hat{f}_P\|_2=\|f_N-f_P\|_2\to 0 \text{ when P and N tend to infinity.}$$

We then define $\hat{f}_\infty$ the limit in $L^2(\mathbb{R})$ of $\hat{f}_N$.
It remains to show that this limit is independent of the choice of sequence f_N tending to f. It is easing to see that this arises from Parseval equality. Indeed, let f_N and $\tilde{f}_N$ in $L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$ tending to f in $L^2(\mathbb{R})$, then:

$$\|f_N-\tilde{f}_N\|_2=\|\hat{f}_N-\widehat{\tilde{f}_N}\|_2\to 0,$$

meaning the Fourier transforms have the same limit in $L^2(\mathbb{R})$.
We then have the following definition:

Definition 5 *The Fourier transform of a function $f\in L^2(\mathbb{R})$ is defined as the limit in $L^2(\mathbb{R})$ of the Fourier transform of any $f_N\in L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$ tending to f in $L^2(\mathbb{R})$.*

In the sequel, we will note $\mathcal{F}(f)$ the Fourier transform of f when the latter is in $L^2(\mathbb{R})$.
Remark: for the sake of simplicity, one takes $f_N=\chi_{[-N,N]}f$.

4.2.5 Property of the Fourier transform in $L^2(\mathbb{R})$

Theorem 10

The Fourier transform (resp $\overline{\mathcal{F}}$) can be extended into an isometry from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$. Let us denote $\mathcal{F}$ et $\overline{\mathcal{F}}$, these extensions, one then gets:

- $\forall f\in L^2(\mathbb{R})$ $\mathcal{F}\overline{\mathcal{F}}(f)=\overline{\mathcal{F}}\mathcal{F}(f)=f$ almost everywhere.
- $\forall f,g\in L^2(\mathbb{R})$ $\int_{\mathbb{R}} f(x)\overline{g(x)}dx=\int_{\mathbb{R}}\mathcal{F}(f)\overline{\mathcal{F}}(g)d\xi$
- $\forall f\in L^2(\mathbb{R})$ $\|f\|_2=\|\mathcal{F}(f)\|_2$

Proof The proof stems from the density theorem of functions of $L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$ in $L^2(\mathbb{R})$ (the equalities being true in $L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$, they are also true in $L^2(\mathbb{R})$).

4.3 Exercises

Exercise 1 *Properties of the Fourier transform F*
Show the following properties:

1. $F(f+\lambda g) = \hat{f} + \lambda\hat{g}$, $\forall f, g \in L^1(\mathbb{R})$, $\forall \lambda \in \mathbb{R}$.
2. $F[f(ax)](\nu) = \frac{1}{|a|}\hat{f}(\frac{\nu}{a})$, $\forall f \in L^1(\mathbb{R})$, $\forall a \in \mathbb{R}^*$.
3. $F[f(x-\tau)](\nu) = e^{-2i\pi\nu\tau}\hat{f}(\nu)$, $\forall f \in L^1(\mathbb{R})$, $\forall \tau \in \mathbb{R}$.
4. $F[f'](\nu) = 2i\pi\nu\hat{f}(\nu)$, $\quad \forall f \in L^1(\mathbb{R})\bigcap C^1(\mathbb{R})$, such that $f' \in L^1(\mathbb{R})$.
5. Compute $F[xf(x)](\nu)$ as a function of $\hat{f}(\nu)$ (hypotheses on f?).

Exercise 2 *Computation of simple Fourier transforms*

1. Compute Fourier transform of $f(x) = e^{-|x|}$, that of $g(x) = U(x)f(x)$, U Heavyside function.
2. Compute Fourier transform of $\rho_n(x) = n\Pi(nx)$ (Π indicator function of $[-1/2, 1/2]$).
3. Plot ρ_n and $\hat{\rho}_n$. What happens when $n \to +\infty$?
4. *Modulation* :Compute $F[\cos(2\pi\nu_0 x)f(x)]$. Example : $f(x) = \chi_{[-a,a]}(x)$.

Exercise 3 *Computation of the Fourier transform of $f(x) = e^{-\pi x^2}$.*

1. Check that $f \in L^1(\mathbb{R})$
2. Show that f is solution to the following differential equation

$$y' + 2\pi xy = 0 \tag{4.2}$$

3. Compute Fourier transform of (4.2) and deduce differential equation satified by $\hat{f}$.
4. Deduce the computation of $\hat{f}$.

Exercise 4 *Door function*

1. Compute $\hat{\Pi}$. Check that $\lim_{\nu\to\infty}\hat{\Pi}(\nu) = 0$.
2. Deduce the value of the integral:

$$\int_{-\infty}^{+\infty} \frac{\sin \pi\nu}{\pi\nu} e^{2i\pi\nu x} dx$$

3. Deduce that (difficult question):

$$\int_{-\infty}^{+\infty} \frac{\sin t}{t} dt = \pi$$

4. Compute $\int_{-\infty}^{+\infty} \Pi^2(x)dx$
5. Deduce that :

$$\int_{-\infty}^{+\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi$$

Exercise 5 *Hat function*
Let Λ be the piecewise affine function, equal to 0 on $]-\infty,-1]$ and $[1,+\infty[$, with value 1 at $x=0$.

1. Give the expression of $\Lambda(x)$.
2. Show that $\Lambda'(x)=\Pi(x+1/2)-\Pi(x-1/2)$.
3. Compute the Fourier transform of Λ'. Deduce that of Λ.

Exercise 6 *On the relation between Fourier transform and Fourier coefficients*
f_0 a function of $L^1(\mathbb{R})$, null outside the interval $[0,T]$. f T-periodic extension of f:

$$f(x)=\sum_{n\in\mathbb{Z}} f_0(x+nT)$$

1. Since f periodic function integrable on $[0,T]$, show that Fourier coefficients of f, $c_n(f)$ satisfy:

$$c_n(f)=\frac{1}{T}\hat{f}_0(\frac{n}{T})$$

where $\hat{f}_0$ is the Fourier transform of function f_0.

Exercise 7 Let us define

$$f(x)=\int_0^\infty \frac{1}{\sqrt{t}}e^{-\frac{x^2}{2t}-\frac{t}{2}}dt$$

1. Show that $f\in L^1(\mathbb{R}^+)$.
2. Compute $\hat{f}$. Deduce that $f(x)=\sqrt{2\pi}e^{-|x|}$.

Exercise 8 Let $f(x)=\frac{\sin x}{|x|}$ and $f_\lambda(x)=e^{-\lambda|x|}\frac{\sin x}{|x|}$ $(\lambda>0)$.

1. Show that f and f_λ belong to $L^2(\mathbb{R})$, and that if λ tends to 0, f_λ converges to f in $L^2(\mathbb{R})$.
2. Compute of a fixed ξ, $\frac{\partial}{\partial\lambda}\hat{f}_\lambda(\xi)$. Deduce $\hat{f}_\lambda(\xi)$ and then $\hat{f}(\xi)$.

Exercise 9 Let a and b two real numbers such that $a,b>0$ et $a\neq b$.

1. Compute the Fourier transform of $e^{-a|x|}$.
2. Deduce the values of the following convolution products: $\frac{1}{a^2+x^2}*\frac{1}{b^2+x^2}$ and $e^{-a|x|}*e^{-b|x|}$.

Exercise 10 *Heat equation de la chaleur*
Let us consider the following partial derivatives equation:

$$\left\{\begin{array}{l}\frac{\partial^2 f}{\partial x^2}=\frac{\partial f}{\partial t}\\ f(x,0)=\varphi(x)\end{array}\right. \tag{4.3}$$

where φ belongs to $C_c^\infty(\mathbb{R})$. Let us define:

$$F(\nu,t)=\int_{-\infty}^{+\infty} f(x,t)\ e^{-2i\pi\nu x}dx$$

1. Let us assume $f\in L^1(\mathbb{R})$. Check that F satisfies:

$$\frac{\partial F}{\partial t}+4\pi^2\nu^2F=0$$

2. Deduce F, and then f.

Chapter 5

Fourier transform of discrete sequences

The Fourier transform of Discrete sequences makes use of the theory of distributions of which we give a brief introduction.

5.1 Motivations for the introduction of distributions

The theory of distributions has been introduced to extend the notions of functions and that of derivation. It is the basis to the unification of discrete and continuous phenomena, and are widely used in mechanical physics, electronic, and probabilities.
To model impulses, the physician P. Dirac had the idea, around 1920 to use a pseudo-fonction, already introduced by par O. Heaviside, now known as the Dirac distribution and assumed to satisfy:

$$\delta_a(x) \quad = \quad \begin{cases} +\infty \text{ if } & x = a \\ 0 \text{ otherwise} & \end{cases}$$

and, for any continuous function ϕ

$$\int_{-\infty}^{+\infty} \delta_a(x)\varphi(x)dx = \varphi(a).$$

δ_a is definitely not a function but one had to wait until the years 1945-1950, and the work by L. Schwartz, for a proper mathematical definition of this object. This is the main motivation to the introduction of distribution theory.

5.1.1 The space of test functions

The distributions are going to be defined as applications on a function space which is called the *space of test functions.*

Definition 1 *One defines $\mathcal{D}(\Omega)$ (also denoted $C_0^\infty(\Omega)$) the set of smooth functions (admitting derivatives of any orders) defined on Ω, with values in $\mathbb{C}$, and compactly supported in Ω.*

$\mathcal{D}(\Omega)$ is a vector space.

Remark 1 *Let* $\varphi \in \mathcal{D}(\Omega)$. $\mathrm{Supp}\,(\varphi)$ *is a compact set and* $\mathrm{Supp}\,(\varphi) \subset \Omega$. *If* $\tilde{\varphi}(x) = \begin{cases} 0 & si\ x \in \mathbb{R} \setminus \mathrm{Supp}\,(\varphi) \\ \varphi(x) & si\ x \in \mathrm{Supp}\,(\varphi) \end{cases}$ *then* $\tilde{\varphi} \in \mathcal{D}(\mathbb{R})$.

Example 1 $\varphi(x) = \begin{cases} \exp \frac{-1}{1-\|x\|^2} & si\ \|x\| < 1 \\ 0 & si\ \|x\| \geq 1 \end{cases}$

$\varphi \in \mathcal{D}(\mathbb{R})$, $\mathrm{Supp}\,(\varphi) = \overline{B(0,1)}$

Definition 2 *Convergence in* $\mathcal{D}(\Omega)$ *Let* φ_n *and* $\varphi \in \mathcal{D}(\Omega)$. $\underline{\varphi_n \text{ converges to } \varphi \text{ in } \mathcal{D}(\Omega)}$ *if:*

- $\exists K$ *a compact set* , $K \subset \Omega$ *such that* $\forall\, n$, $\mathrm{Supp}\,(\varphi_n) \subset K$
- $\forall\, \alpha \in \mathbb{N}^N$, $\partial^\alpha \varphi_n \longrightarrow \partial^\alpha \varphi$ *uniformly.*

Theorem 1

$$\boxed{\mathcal{D}(\Omega) \text{ is dense in } L^p(\Omega),\ 1 \leq p < +\infty.}$$

5.1.2 Definitions of the distribution space

Definition 3 *A* <u>*distribution*</u> T *on* Ω *is a linear form continuous on* $\mathcal{D}(\Omega)$, *i.e.*

(i) $\forall\, \varphi_1, \varphi_2 \in \mathcal{D}(\Omega)$, $\forall\, \lambda \in \mathbb{C}$, $T(\varphi_1 + \lambda\varphi_2) = T(\varphi_1) + \lambda T(\varphi_2)$

(ii) If $\varphi_n \longrightarrow \varphi$ *in* $\mathcal{D}(\Omega)$, *then* $T(\varphi_n) \longrightarrow T(\varphi)$ *in* $\mathbb{C}$

One notes $\langle T, \varphi \rangle$ or $T(\varphi)$.

Remark 2 *Point ii) is equivalent to showing: for any compact set* $K \subset \Omega$, *there exists* $C_k > 0$ *and* $k \in \mathbb{N}$ *such that for all* $\varphi \in \mathcal{D}(\Omega)$ *with* $\mathrm{Supp}\,(\varphi) \subset K$, *one has* $\langle T, \varphi \rangle \leq C_k \|\varphi\|_{C^k(K)}$, *with* $\|\varphi\|_{C^k(K)} = \max\limits_{\alpha \leq k} \|\varphi^{(\alpha)}\|_{\infty,K}$.

One denotes $\mathcal{D}'(\Omega)$ the set of distributions on Ω, which is a vector space.
Examples

1. $L^1_{loc}(\Omega)$: set of mesurable functions on Ω, integrable on any compact set of Ω (for instance, $\frac{1}{\sqrt{|x|}} \in L^1_{loc}(\mathbb{R})$).

   Let $f \in L^1_{loc}(\Omega)$. For $\varphi \in \mathcal{D}(\Omega)$ one puts: $\langle T_f, \varphi \rangle = \int_\Omega f(x)\varphi(x)\,dx$

   $T_f \in \mathcal{D}'(\Omega)$:

   - T_f is well defined since
     $|f(x)\varphi(x)| \leq \|\varphi\|_\infty\, \mathbb{1}_{\mathrm{Supp}(\varphi)}(x)|f(x)| \in\ L^1(\Omega)$
   - T_f is linear (linearity of the integral).
   - T_f is continuous on $\mathcal{D}(\Omega)$:
     Let $\varphi_n \in \mathcal{D}(\Omega)$ be such that $\varphi_n \to 0$ dans $\mathcal{D}(\Omega)$.
     $|\langle T_f, \varphi_n \rangle| = \left| \int_\Omega f(x)\varphi_n(x)dx \right| \leq \int |f(x)|\, |\varphi_n(x)|\, dx \leq \|\varphi_n\|_\infty \int_K |f(x)|\, dx$

2. Let $a \in \mathbb{R}$. For all $\varphi \in \mathcal{D}(\mathbb{R})$ one puts: $\langle \delta_a, \varphi \rangle = \varphi(a)$

   $\delta_a \in \mathcal{D}'(\mathbb{R})$:

   - Linearity :
     $\langle \delta_a, \varphi_1 + \lambda\varphi_2 \rangle = (\varphi_1 + \lambda\varphi_2)(a) = \varphi_1(a) + \lambda\varphi_2(a) = \langle \delta_a, \varphi_1 \rangle + \lambda \langle \delta_a, \varphi_2 \rangle$
   - Continuity :
     If $\varphi_n \longrightarrow 0$ in $\mathcal{D}(\mathbb{R})$: $|\langle \delta_a, \varphi_n \rangle| = |\varphi_n(a)| \leq \|\varphi_n\|_\infty \longrightarrow 0$

   When $a = 0$, we put $\delta = \delta_0$.

Proposition 1

The application from $L^1_{loc}(\Omega)$ on $\mathcal{D}'(\Omega)$ which maps f to T_f is linear and injective.

Proof

- $\forall\, f_1, f_2 \in L^1_{loc}(\Omega)$, $\forall\, \lambda \in \mathbb{C}$, $T_{f_1+\lambda f_2} = T_{f_1} + \lambda T_{f_2}$
  Indeed, let $\varphi \in \mathcal{D}(\Omega)$,
  $$\langle T_{f_1+\lambda f_2}, \varphi \rangle = \int_\Omega (f_1 + \lambda f_2)\, \varphi = \int_\Omega f_1\varphi + \lambda \int_\Omega f_2\varphi = \langle T_{f_1}, \varphi \rangle + \lambda \langle T_{f_2}, \varphi \rangle$$
- If $\forall \varphi \in \mathcal{D}(\Omega), \langle T_f, \varphi \rangle = \displaystyle\int_\Omega f(x)\varphi(x)dx = 0$, alors $f = 0$. Indeed, since $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$, letting φ_n a sequence coverging to f in L^2, then $\int_\Omega f(x)\varphi_n(x) = 0$ tends to $\int_\mathbb{R} |f(x)|^2 = 0$ and so f is null almost everywhere.

Remark 3 *The application defined by proposition 1 is not surjective, but enables to identify $L^1_{loc}(\Omega)$ to a subspace $\mathcal{D}'(\Omega)$ called* regular distributions*.*

5.1.3 Convergence in the distribution space

Definition 4 *A sequence of distribution $T_n \in \mathcal{D}'(\Omega)$ converges to the distribution $T \in \mathcal{D}'(\Omega)$ if for all $\varphi \in \mathcal{D}(\Omega)$, $\langle T_n, \varphi \rangle \ \longrightarrow\ \langle T, \varphi \rangle$.*

$\displaystyle\sum_{n\geq 0} T_n$ is said to converge and sums to T if the sequence $S_p = \displaystyle\sum_{n=0}^{p} T_n$ converges to T.

Examples 2

1. *Let $f_n(x) = \cos(nx)$, $f_n \in L^1_{loc}(\mathbb{R})$. $T_{f_n} \in \mathcal{D}'(\mathbb{R})$.*
   $\lim\limits_{n\to+\infty} T_{f_n} = 0$ *(because $\forall\, \varphi \in \mathcal{D}(\mathbb{R})$, $\langle T_{f_n}, \varphi \rangle = \int \cos(nx)\varphi(x)dx \xrightarrow[n\to\infty]{} 0$).*
2. *Let $n \in \mathbb{N}$, $\delta_n \longrightarrow 0$ in $\mathcal{D}'(\mathbb{R})$.*
   Let $\varphi \in \mathcal{D}(\mathbb{R})$, $\langle \delta_n, \varphi \rangle = \varphi(n) = 0$ for a large enough n (since φ is compactly).

Theorem 2

Let $T_n \in \mathcal{D}'(\Omega)$, $n \in \mathbb{N}$.
If for all $\varphi \in \mathcal{D}(\Omega)$, $\langle T_n, \varphi\rangle$ has a limit in $\mathbb{C}$, then T_n has a limit in $\mathcal{D}'(\Omega)$.

Proof

- $\varphi \in \mathcal{D}(\Omega)$ implies $\lim_{n\to+\infty}\langle T_n, \varphi\rangle$ exists.
  $\varphi_1, \varphi_2 \in \mathcal{D}(\Omega)$ et $\lambda \in \mathbb{C}$.
  $\lim\limits_{n\to+\infty}\langle T_n, \varphi_1 + \lambda\varphi_2\rangle = \lim\limits_{n\to+\infty}\langle T_n, \varphi_1\rangle + \lambda\langle T_n, \varphi_2\rangle = \lim\limits_{n\to+\infty}\langle T_n, \varphi_1\rangle + \lambda \lim\limits_{n\to+\infty}\langle T_n, \varphi_2\rangle$, hence the linearity.
- The continuity of $\lim T_n$ is a consequence of Banach-Steinhaus theorem (admitted)

Example 3 $\forall\, n,\ T_n = \sum\limits_{p=0}^{n} \delta_p \in \mathcal{D}'(\mathbb{R})$. *Indeed, let* $\varphi \in \mathcal{D}(\mathbb{R})$, **Exercise** $n_o \in \mathbb{N}\mathrm{Supp}\,(\varphi) \subset$

$[-n_0, n_0]$. $\langle T_n, \varphi\rangle = \sum\limits_{p=0}^{n_0}\varphi(p) \xrightarrow[n\to\infty]{} \sum\limits_{p=0}^{n_0}\varphi(p)$, *so there exists* $T \in \mathcal{D}(\mathbb{R})$ *such that* $T_n \longrightarrow T$ *dans* $\mathcal{D}'(\mathbb{R})$ *:* $T = \sum\limits_{p\geq 0}\delta_p$.

5.1.4 Derivation in the distribution space

Let $f \in C^1(\Omega)$ (so $\in L^1_{loc}(\Omega)$). For $\varphi \in \mathcal{D}(\Omega)$:

$$\int_\Omega f'(x)\varphi(x)dx = -\int_\Omega f(x)\varphi'(x)dx \Leftrightarrow \langle T_{f'}, \varphi\rangle = -\langle T_f, \varphi'\rangle$$

Extending this to more general distributions, we get:

Definition 5 *Let* $T \in \mathcal{D}'(\Omega)$, *one defines* T' *as :* $\langle T', \varphi\rangle \overset{def}{=} -\langle T, \varphi'\rangle$

Proposition 2

T is indefinitely differentiable, and one has: $\forall\, \varphi \in \mathcal{D}(\Omega)$, $\langle T^{(\alpha)}, \varphi\rangle = (-1)^\alpha\langle T, \varphi^{(\alpha)}\rangle$

Proof (du 1.)

- Let φ and $\psi \in \mathcal{D}(\Omega)$, $\lambda \in \mathbb{C}$.
  $$\begin{aligned}\langle T', \varphi + \lambda\psi\rangle &= -\langle T, (\varphi + \lambda\psi)'\rangle = -\langle T, \varphi' + \lambda\psi'\rangle\\ &= -\langle T, \varphi'\rangle - \lambda\langle T, \psi'\rangle = \langle T', \varphi\rangle + \lambda\langle T', \psi\rangle\end{aligned}$$
- Let $\varphi_n \longrightarrow 0$ in $\mathcal{D}(\Omega)$, then $\varphi'_n \longrightarrow 0$ in $\mathcal{D}(\Omega)$. One has: $\langle T', \varphi_n\rangle = -\langle T, \varphi'_n\rangle \longrightarrow 0$

Proposition 3

The derivation is a continuous operation on $\mathcal{D}'(\Omega)$.
If $T_n, T \in \mathcal{D}'(\Omega)$ and $T_n \longrightarrow T$ in $\mathcal{D}'(\Omega)$, then $\forall\, \alpha \in \mathbb{N}$, $T_n^{(\alpha)} \longrightarrow T^{(\alpha)}$ in $\mathcal{D}'(\Omega)$.

Proof
Let $\varphi \in \mathcal{D}(\Omega)$, $\langle T_n^{(\alpha)}, \varphi\rangle = (-1)^\alpha \langle T_n, \varphi^{(\alpha)}\rangle \longrightarrow (-1)^\alpha \langle T, \varphi^\alpha\rangle = \langle T^\alpha, \varphi\rangle$

Example 4 *If $f \in C^1(\Omega)$: $(T_f)' = T_{f'}$, the derivative is still a regular distribution.*

Examples 5 *Let us define T_Y, with Y the Heaviside function defined by:* $Y(x) = \begin{cases} 1 & si\ x > 0 \\ 0 & si\ x < 0 \end{cases}$

$Y \in L^1_{loc}(\mathbb{R})$, so Y is a distribution $T_Y \in \mathcal{D}'(\mathbb{R})$. Then, let $\varphi \in \mathcal{D}'(\mathbb{R})$, we get
$\langle T'_Y, \varphi\rangle = -\langle T_Y, \varphi'\rangle = -\displaystyle\int_0^{+\infty} \varphi'(x)dx = \varphi(0) = \langle \delta_0, \varphi\rangle$*, meaning that* $T'_Y = \delta_0$.

5.2 Fourier transform of distributions

The Fourier transform of distributions is going to be defined on a subset of $\mathcal{D}'(\mathbb{R})$, called *tempered distributions*. These are defined as continuous linear forms on the Schwartz space which we first introduce.

5.3 The Schwartz class

Definition 6 *$\mathcal{S}(\mathbb{R})$, called the* Schwartz class *is the set of functions $\phi : \mathbb{R} \to \mathbb{C}$ such that:*

- $\phi \in C^\infty(\mathbb{R})$
- $\forall \alpha \in \mathbb{N},\ n \in \mathbb{N}, \exists C$ *such that* $|\phi^{(\alpha)}(x)| \leq \frac{C}{(1+\|x\|)^n}$ *(fast decay)*

In other words, the functions in $\mathcal{S}(\mathbb{R})$ are $C^\infty(\mathbb{R})$ functions having all their derivatives with fast decay.
Example : $\phi(x) = e^{-x^2}$

Theorem 3

$\mathcal{S}(\mathbb{R})$ has the following properties:

1. $\forall \phi \in \mathcal{S}(\mathbb{R}), \quad \forall P \in \mathbb{C}[X] \quad P\phi \in \mathcal{S}(\mathbb{R})$
2. $\forall \phi \in \mathcal{S}(\mathbb{R}), \quad \phi' \in \mathcal{S}(\mathbb{R})$
3. $1 \leq p \leq \infty \quad \mathcal{S}(\mathbb{R}) \subset L^p(\mathbb{R})$

Theorem 4

$\mathcal{F}$ (i.e. the Fourier transform, $\mathcal{F}(\phi) = \hat{\phi}$) is a linear bijection from $\mathcal{S}(\mathbb{R})$ onto itself with inverse $\overline{\mathcal{F}}$.

Proof Let $f \in \mathcal{S}(\mathbb{R})$, since for all k, $x^k f(x)$ is in $L^1(\mathbb{R})$, $\hat{f}$ belongs to $C^\infty(\mathbb{R})$. Let $n \in \mathbb{N}$ and $p \in \mathbb{N}$, one has:

$$\begin{aligned} \xi^n \hat{f}^{(p)}(\xi) &= \xi^n \mathcal{F}((-2i\pi x)^p f(x)) \text{ by differentiation of an integral depending on a parameter} \\ &= \frac{1}{(2i\pi)^n} \mathcal{F}(((-2i\pi x)^p f(x))^{(n)}) \text{ using properties of the Fourier transform of derivatives.} \end{aligned}$$

Using the stability properties of $\mathcal{S}(\mathbb{R})$ by multiplication with a polynom and by derivation, we get that the function of which we compute the Fourier transform is in $\mathcal{S}(\mathbb{R})$, so that its Fourier transform is bounded, which proves that $\hat{f}$ belongs to $\mathcal{S}(\mathbb{R})$.
Furthermore, as f and $\hat{f}$ are in $L^1(\mathbb{R})$ and, as f is continuous, one gets $f(x) = \overline{\mathcal{F}}(\hat{f})(x)$.

The topology of $\mathcal{S}(\mathbb{R})$ is not defined by a norm but by a numerable family of norms:

$$\forall \phi \in \mathcal{S}(\mathbb{R}), \quad \mathcal{N}_p(\phi) = \max_{0 \le \alpha, \beta \le p} \sup_{x \in \mathbb{R}} |x^\alpha \phi^{(\beta)}((x)|, \quad p \in \mathbb{N}$$

One says that a sequence ϕ_n converges to ϕ in $\mathcal{S}(\mathbb{R})$ if:

$$\mathcal{N}_p(\phi_n - \phi) \to 0 \text{ for all } p \ge 0, \text{ when } n \to \infty.$$

One can alternatively define the convergence in $\mathcal{S}(\mathbb{R})$ as follows:

$$\forall \alpha, \beta \in \mathbb{N} \; x^\alpha \phi_n^{(\beta)}(x) \to x^\alpha \phi^{(\beta)}(x) \text{ uniformly on } \mathbb{R}$$

5.4 The space of tempered distributions $\mathcal{S}'(\mathbb{R})$

We need to define the Fourier transform in a more general framework than that of the functions so that the Fourier transform of sampled signals makes sense. The set of tempered distributions $\mathcal{S}'(\mathbb{R})$ is defined by:

$$\left\{ \begin{array}{llll} T: & \mathcal{S}(\mathbb{R}) & \to & \mathbb{C} \quad \textit{linear, continuous} \\ & \varphi & \mapsto & \langle T, \varphi \rangle \end{array} \right\}$$

Here, the continuity has to be understood in the following sense:

$$\exists m, C_m \text{ tel que } \forall \phi \in \mathcal{S}(\mathbb{R}), \quad |\langle T, \phi \rangle| \le C_m \mathcal{N}_m(\phi)$$

or, using sequences:

$$\phi_n \to \phi \text{ in } \mathcal{S}(\mathbb{R}) \Rightarrow \langle T, \phi_n \rangle \to \langle T, \phi \rangle \text{ in } \mathbb{C}.$$

Remark 4 *$\mathcal{D}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ and if T is continuous for the topology on $\mathcal{S}(\mathbb{R})$, it is also continuous for the topology on $\mathcal{D}(\mathbb{R})$, so $\mathcal{S}'(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$. Furthermore, one can show that $\mathcal{D}(\mathbb{R})$ is dense in $\mathcal{S}(\mathbb{R})$ for the topology of $\mathcal{S}(\mathbb{R})$. A consequence is that to prove the continuity in $\mathcal{S}'(\mathbb{R})$, one can restrain to functions in $\mathcal{D}(\mathbb{R})$, i.e.:*

$$\exists m, C_m \text{ tel que } \forall \phi \in \mathcal{D}(\mathbb{R}), \quad |\langle T, \phi \rangle| \leq C_m \mathcal{N}_m(\phi)$$

or, using sequences: $\phi_n \in \mathcal{D}(\mathbb{R})$, $\phi_n \to \phi$ in $\mathcal{S}(\mathbb{R}) \Rightarrow \langle T, \phi_n \rangle \to \langle T, \phi \rangle$ in $\mathbb{C}$

Examples 6

- *$f \in L^1(\mathbb{R})$, $L^2(\mathbb{R})$ ou $L^\infty(\mathbb{R}) \Rightarrow T_f \in \mathcal{S}'(\mathbb{R})$*
- *Dirac $\delta_a \in \mathcal{S}'(\mathbb{R})$*
- *Dirac comb $\sum_{n \in \mathbb{Z}} \delta_n \in \mathcal{S}'(\mathbb{R})$*
- *$\mathcal{E}'(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$*
- *Functions of slow increase are in $\mathcal{S}'(\mathbb{R})$. A function is of slow increase if:*

$$\exists c > 0 \quad \exists N \in \mathbb{N}, \quad \forall x \in \mathbb{R} \ |f(x)| \leq c(1 + |x|)^N$$

- *If the sequence $(y_n)_{n \in \mathbb{Z}}$ is of slow increase (i.e. $\exists m \in \mathbb{N}$, $c \in \mathbb{R}$ such that $|y_n| \leq C(1 + |n|)^m$, $n \in \mathbb{Z}$), the distribution*

$$T = \sum_{n \in \mathbb{Z}} y_n \delta_{na}$$

  is tempered.

5.5 Fourier transform in $\mathcal{S}'(\mathbb{R})$

Definition 7 *Let $T \in \mathcal{S}'(\mathbb{R})$, one defines its Fourier transform as follows*

$$\hat{T} : \begin{pmatrix} \varphi & \mapsto & \langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle \\ \mathcal{S}(\mathbb{R}) & \to & \mathbb{C} \end{pmatrix}$$

One has: $\hat{T} \in \mathcal{S}'(\mathbb{R})$

Remark 5

- *$\varphi \in \mathcal{S}(\mathbb{R}) \Rightarrow \hat{\varphi} \in \mathcal{S}(\mathbb{R})$*
- *So $\hat{T}$ is well defined, linear by linearity of T, continuous using the continuity of T (for that we use the fact that $\mathcal{S}(\mathbb{R})$ is stable through Fourier transform).*

Proposition 4

$T_n \in \mathcal{S}'(\mathbb{R})$ converges to T in $\mathcal{S}'(\mathbb{R})$, if $\forall \varphi \in \mathcal{S}(\mathbb{R})$, $\langle T_n, \varphi \rangle \to \langle T, \varphi \rangle$.

Proposition 5

The Fourier transform is a continuous application from $\mathcal{S}'(\mathbb{R})$ onto itself.

Proof Let T_n tending to T in $\mathcal{S}'(\mathbb{R})$ then for all φ in $\mathcal{S}(\mathbb{R})$, one has:

$$\langle \hat{T}_n, \varphi\rangle = \langle T_n, \hat{\varphi}\rangle \to \langle T, \hat{\varphi}\rangle = \langle \hat{T}, \varphi\rangle$$

Theorem 5

$$\begin{array}{cccc} \mathcal{F}: & T & \mapsto & \hat{T} \\ & \mathcal{S}'(\mathbb{R}) & \to & \mathcal{S}'(\mathbb{R}) \end{array}$$

is invertible, and its inverse is:

$$\begin{array}{cccc} \bar{\mathcal{F}}: & T & \mapsto & \bar{\mathcal{F}}(T) \\ & \mathcal{S}'(\mathbb{R}) & \to & \mathcal{S}'(\mathbb{R}) \end{array}$$

with $\langle \bar{\mathcal{F}}(T), \varphi\rangle = \langle T, \bar{\mathcal{F}}(\varphi)\rangle$

Proof Let $T \in \mathcal{S}'(\mathbb{R})$, $\forall \varphi \in \mathcal{S}(\mathbb{R}), \langle \bar{\mathcal{F}}\mathcal{F}(T), \varphi\rangle = \langle \mathcal{F}(T), \bar{\mathcal{F}}(\varphi)\rangle = \langle T, \mathcal{F}\bar{\mathcal{F}}(\varphi)\rangle = \langle T, \varphi\rangle$, because $\mathcal{F}\bar{\mathcal{F}} = Id$ in $\mathcal{S}(\mathbb{R})$.

Examples 7

(i) Let $f \in L^1(\mathbb{R})$ or $L^2(\mathbb{R})$, then $T_f \in \mathcal{S}'(\mathbb{R})$.

$$\langle \widehat{T_f}, \varphi\rangle = \langle T_f, \hat{\varphi}\rangle = \int_{\mathbb{R}} f(y)\hat{\varphi}(y)dy = \int_{\mathbb{R}} \hat{f}(y)\varphi(y)dy = \langle T_{\hat{f}}, \varphi\rangle$$

so $\widehat{T_f} = T_{\hat{f}}$.

Conclusion: if the Fourier transform exists in the functional sense, and is denoted by $\hat{f}$, its Fourier transform in the sense of distributions will be $T_{\hat{f}}$.

(ii) $\forall \varphi \in \mathcal{S}(\mathbb{R}), \langle \hat{\delta}, \varphi\rangle = \langle \delta, \hat{\varphi}\rangle = \hat{\varphi}(0) = \int_{-\infty}^{+\infty} \varphi(x)e^{-2i\pi 0x}dx = \int_{-\infty}^{+\infty} \varphi(x)dx = \langle T_1, \varphi\rangle$, so $\hat{\delta} = T_1$.

(iii) $\forall \varphi \in \mathcal{S}(\mathbb{R}), \langle \hat{\delta_a}, \varphi\rangle = \langle \delta_a, \hat{\varphi}\rangle = \hat{\varphi}(a) = \int_{-\infty}^{+\infty} \varphi(x)e^{-2i\pi ax}dx = \langle e^{-2i\pi ax}, \varphi\rangle$, so $\hat{\delta_a} = T_{e^{-2i\pi ax}}$.

(iv) $\forall \varphi \in \mathcal{S}(\mathbb{R}), \langle \widehat{T_{e^{2i\pi k_0 x}}}, \varphi\rangle = \langle T_{e^{2i\pi k_0 x}}, \hat{\varphi}\rangle = \int_{-\infty}^{+\infty} e^{2i\pi k_0 y}\hat{\varphi}(y)dy = \varphi(k_0) = \langle \delta_{k_0}, \varphi\rangle$, so $\widehat{T_{e^{2i\pi k_0 x}}} = \delta_{k_0}$ (in particular : $\widehat{T_1} = \delta_0$).

(v) Let T be a strictly positive real and $(y_n)_{n\in\mathbb{Z}}$ a sequence of slow increase, then

$$\mathcal{F}(\sum_{n\in\mathbb{Z}} y_n\delta_{nT}) = \sum_{n\in\mathbb{Z}} y_n e^{-2i\pi nTx},$$

which is a consequence of the continuity of the Fourier transform on $\mathcal{S}'(\mathbb{R})$:

$$\widehat{\sum_{n\in\mathbb{Z}} y_n\delta_{nT}} = \sum_{n\in\mathbb{Z}} y_n\hat{\delta}_{nT} = \sum_{n\in\mathbb{Z}} y_n e^{-2i\pi nTx}.$$

This last example is very important in signal processing, in which community one defines (often without proof of existence) the so-called discrete time Fourier transform (DTFT), as follows

Definition 8 *The discrete-time Fourier transform (DTFT) of a sequence (x_n) is defined by:*

$$X(e^{2i\pi\omega}) = \sum_{n\in\mathbb{Z}} x_n e^{-2i\pi n\omega} \tag{5.1}$$

which exists in $\mathcal{S}'(\mathbb{R})$ as soon as (x_n) is with slow increase.

In particular, when (x_n) is in $l_1(\mathbb{Z})$, one has normal convergence, and X is continuous. When (x_n) is in $l_2(\mathbb{Z})$, $X(e^{2i\pi\omega})$ can be viewed as a Fourier series of a 1-periodic function and the convergence takes place in $L^2([-1/2, 1/2[)$, so that we can write:

$$x_n = \int_{-1/2}^{1/2} X(e^{2i\pi\omega}) e^{2i\pi n\omega} d\omega, \ n \in \mathbb{Z} \tag{5.2}$$

Another consequence is the Parseval equality when (x_n) is in $l_2(\mathbb{Z})$:

$$\int_{-1/2}^{1/2} |X(e^{2i\pi\omega})|^2 d\omega = \sum_{n\in\mathbb{Z}} |x_n|^2.$$

5.5.1 Distributions with compact support

Definition 9 *Let $T \in \mathcal{D}'(\Omega)$ and $\omega \subset \Omega$ an open set. T is <u>null</u> on ω if for any $\varphi \in \mathcal{D}(\omega)$, $\langle T, \varphi\rangle = 0$.*

Example 8 *Let $a \in \mathbb{R}$, δ_a is null on $\mathbb{R} \setminus \{a\}$. If $\omega \subset \mathbb{R}$ is an open set and if $a \notin \omega$, then δ_a is null on ω.*

Definition 10 *Let $T \in \mathcal{D}'(\Omega)$, the <u>support</u> of T, denoted* $\operatorname{Supp}(T)$*, is the complement set (in Ω) of the largest open set ω on which T is null.*

One can show that ω exists.

Examples 9

- $\operatorname{Supp}(\delta_a) = \{a\}$
- *For all $a \in \mathbb{R}$ and all $\alpha \in \mathbb{N}$, one has* $\operatorname{Supp}(\partial^\alpha \delta_a) = \{a\}$
- *If $f \in C^0(\Omega)$ and* $\operatorname{Supp}(f)$ *is a compact set, then $T_f \in \mathcal{E}'(\Omega)$.*

Proposition 6

Let $T \in \mathcal{E}'(\mathbb{R})$, then $\widehat{T}$ belongs to $C^\infty(\mathbb{R})$ and is with slow increase.

5.5.2 Convolution $\mathcal{E}'(\mathbb{R}) * \mathcal{D}'(\mathbb{R})$

Let $u \in C_0(\mathbb{R})$ and $v \in L^1_{loc}(\mathbb{R})$ then for all $\varphi \in \mathcal{D}(\mathbb{R})$, since $(x,y) \mapsto u(y)v(x-y)\varphi(x) \in L^1(\mathbb{R}\times\mathbb{R})$, using Fubini theorem one may write:

$$\begin{aligned}\int_{\mathbb{R}} u * v(x)\varphi(x)dx &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} u(y)v(x-y)dy\right)\varphi(x)dx = \int_{\mathbb{R}} u(y)\left(\int_{\mathbb{R}} v(x)\varphi(x+y)dx\right)dy \\ &= \int_{\mathbb{R}} v(x)\left(\int_{\mathbb{R}} u(y)\varphi(x+y)dy\right)dx.\end{aligned}$$

which can be rewrite using distributions notations as:

$$\langle T_{u*v}, \varphi\rangle = \langle T_u, \langle T_v, \varphi(.+y)\rangle\rangle = \langle T_v, \langle T_u, \varphi(.+y)\rangle\rangle.$$

One can then generalize this remark through the following definition.

Definition 11 *Let $S \in \mathcal{E}'(\mathbb{R})$ and $T \in \mathcal{D}'(\mathbb{R})$.*

- *There exists a distribution, called convolution of S with T which we write $S*T$ and such that for all $\varphi \in \mathcal{D}(\mathbb{R})$, one has:*

$$\langle S*T, \varphi\rangle = \langle S_t, \langle T_x, \varphi(x+t)\rangle\rangle = \langle T_u, \langle S_x, \varphi(x+u)\rangle\rangle$$

- *The application $(S,T) \to S*T$ from $\mathcal{E}'(\mathbb{R}) \times \mathcal{D}'(\mathbb{R})$ onto $\mathcal{D}'(\mathbb{R})$ is continuous with respect to each variable.*

- *If $T \in \mathcal{S}'(\mathbb{R})$ then $S*T \in \mathcal{S}'(\mathbb{R})$.*

Examples: Let $T \in \mathcal{D}'(\mathbb{R})$, $\delta_a * T = T * \delta_a = \tau_a T$.
$\delta^{(k)} * T = T * \delta^{(k)} = T^{(k)}$

Proposition 7

Let S and T belonging to $\mathcal{D}'_+(\mathbb{R})$, the set of distribution with support limited to the left, then the convolution $S*T$ is still defined.

Proposition 8

Let $S \in \mathcal{E}'(\mathbb{R})$ and $T \in \mathcal{S}'(\mathbb{R})$, one has

$$\widehat{S*T} = \hat{S}\hat{T}$$

5.6 Exercises

Exercise 1
Are the following applications T, defined for $\varphi \in \mathcal{D}(\mathbb{R})$, distributions?
1. $\langle T, \varphi\rangle = \int_0^1 \varphi(x)dx$

2. $\langle T, \varphi \rangle = \int_0^1 |\varphi(x)| dx$

3. $\langle T, \varphi \rangle = \sum\limits_{n=0}^{+\infty} \varphi(n)$

4. $\langle T, \varphi \rangle = \sum\limits_{n=1}^{+\infty} \varphi(1/n)$

Exercise 2

Let $\varphi \in \mathcal{D}(\mathbb{R})$.

1. Show that there exist a constant $C(\varphi)$ such that:

$$\forall n \in \mathbb{Z}, \left| \int_{-\infty}^{+\infty} e^{inx} \varphi(x) dx \right| \leq \frac{C(\varphi)}{1+n^2}$$

2. Let $(a_n)_{n \in \mathbb{Z}}$ be a bounded sequence. Show that the series with general term:

$$a_n \int_{-\infty}^{+\infty} e^{inx} \varphi(x) dx$$

converges, and that the application which maps φ to the sum of this series is a distribution.

3. Show that when the sequence $n^2 a_n$ is bounded, the distribution is indeed a function.

Exercise 3

Let us consider the regular distributions e^{inx}.

1. Show that for all $n \neq 0$:

$$\forall \varphi \in \mathcal{D}(]-\pi, \pi[), \quad \left| \langle e^{inx}, \varphi \rangle \right| \leq \frac{C(\varphi)}{n^2}$$

where $C(\varphi)$ is a constant which does not depend on φ.

2. Let us define:

$$u_N(x) = \sum_{n=-N}^{N} e^{inx}$$

Show that the sequence of distributions T_N associated with functions u_N converges in the distributional sense on $]-\pi, \pi[$. Let T be its limit.

3. Show that:

$$u_N(x) = \frac{\sin(N + \frac{1}{2})x}{\sin(\frac{x}{2})}$$

4. Show that if $\varphi \in \mathcal{D}(]-\pi, \pi[)$ is such that $\varphi(0) = 0$, then $\frac{\varphi(x)}{\sin(\frac{x}{2})}$ belongs to $C_c^\infty(]-\pi, \pi[)$ (we recall that $\frac{\sin x}{x}$ is a smoot function).

5. Show that if $\varphi \in \mathcal{D}(]-\pi, \pi[)$ is such that $\varphi(0) = 0$, then $\langle T, \varphi \rangle = 0$.

6. Deduce that there exists a constant C such that:

$$T = \sum_{n=-\infty}^{+\infty} e^{inx} = C\delta.$$

We will admit that $C = 2\pi$.

Exercise 4
After having shown the following distributions are tempered distributions, compute the Fourier transform of the following distributions:

1. 1
2. x^n
3. $\delta^{(n)}$
4. $e^{2i\pi\nu_0 x}$

Exercise 5
Compute, using the definition of the Fourier transforms of distributions the following integral:

$$\int_{\mathbb{R}} e^{-\pi x^2} \cos(2\pi x) dx$$

Exercise 6
Show that if f belongs to $L^1(\mathbb{R})$ or $L^2(\mathbb{R})$ that $\widehat{T_f} = T_{\hat{f}}$.

Exercise 7
Fourier transform of $vp(1/x)$

1. Show that $vp(1/x)$ is a tempered distribution
2. We recall that $xvp(1/x) = 1$, deduce the Fourier transform of $vp(1/x)$

Exercise 8
Fourier transform of the Heavyside function
Remarking that $U(x) = \frac{1}{2}(sign(x) + 1)$, compute its Fourier transform.

Chapter 6

z-transform

6.1 Discrete signal definition

We study in this chapter, the signal x that are defined in the following manner:

$$x = \sum_{n=-\infty}^{n=+\infty} x_n \delta_{na}$$

with a fixed. We denote by X_a the set of these signals:

$$X_a = \{x \in \mathcal{D}'(\mathbb{R}), x = \sum_{n=-\infty}^{n=+\infty} x_n \delta_{na}\}$$

This is a vector space which is equipped with the same notion of convergence as the one on $\mathcal{D}'$:

$$(x^N \overset{\mathcal{D}'}{\to} x) \Leftrightarrow (\forall n \in \mathbb{Z} \quad x_n^N \to x_n)$$

6.2 z-transform

As we have noticed that the convergence of the Fourier transform of distribution is not automatic in a functional space, one replaces in the definition of DTFT $e^{2i\pi a\omega}$ by a complex z, to obtain the so-called Z-transform of the signal x :

$$X(z) = \sum_{n=-\infty}^{+\infty} x_n z^{-n} \quad z \in \mathbb{C}.$$

In what follows, $X(z)$ denotes the z-transform of (x_n).
To show the practical interest of such a transform, let us consider the Heaviside sequence $u_n = 1$ if $n \geq 0$ and 0 otherwise. We notice that $U(z) = \frac{1}{1-z^{-1}}$ if $|z| \in [0, 1[$, while the DTFT does not converge in that case.
In general, when it exists the z-transform converges on a ring $r < |z| < R$. To study discrete signal, one replaces the Fourier transform of discrete signals by the z-transform.

- Show that the z-transform of $x_n = \delta_{n,n_0}$ is $z^{-n_0} X(z)$
- Show that the z-transform of $x_n = \alpha^n u_n$ is $X(z) = \frac{1}{1-\alpha z^{-1}}$, for $|z| > \alpha$.
- Show that the z-transform of $x_n = -\alpha^n u_{-n-1}$ is $X(z) = \frac{1}{1-\alpha z^{-1}}$, for $|z| < \alpha$.

6.3 Rational z-transform

An important class of z-transforms consists of those that are rational functions, which could be written under the form

$$H(z) = \frac{B(z)}{A(z)}$$

where $A(z)$ and $B(z)$ are polynomials in z^{-1} with no common roots, of degree N and M respectively, the degree satisfies $M \leq N$, otherwise polynomial division would lead to sum of polynomial that and a rational function satisfying this constraint. The zeros of the numerator $B(z)$ and denominator $A(z)$ are called the zeros and poles of H.

Consider a finite sequence $h = (h_i)_{i=0,\cdots,M}$, then $H(z) = \sum\limits_{k=0}^{M} h_k z^{-k}$, which has M poles at $z = 0$ and M zeros at the roots $\{z_k\}_{k=1,\cdots,M}$. Therefore we may write:

$$H(z) = \frac{\sum\limits_{k=0}^{M} h_k z^{M-k}}{z^M} = \frac{h_0 \prod\limits_{k=1}^{M} (z - z_k)}{z^M} = h_0 \prod_{k=1}^{M} (1 - z_k z^{-1})$$

Consequently:

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 \prod\limits_{k=1}^{M} (1 - z_k z^{-1})}{a_0 \prod\limits_{k=1}^{N} (1 - p_k z^{-1})}$$

where $\{z_k\}_{k=1,\cdots,M}$ and $\{p_k\}_{k=1,\cdots,N}$ are the zeros and the poles of H respectively.

6.4 Inversion of the z-transform

Given a z-transform and its ring of convergence (ROC) how do we invert the z-transform? The general inversion formula for the z-transform involves contour integration which is a standard topic of complex analysis. However, most z-transforms encountered in practice can be inverted using simpler methods, which we now discuss.

6.4.1 Inversion by inspection

This method is just a way of recognizing certain z-transform pairs. For example, we see that the z-transform

$$H(z) = \frac{1}{1 - \frac{1}{4} z^{-1}}$$

has the form of $1/(1 - az^{-1})$ with $a = 1/4$. One recognizes that $H(z)$ is generated by:

$$\left(\frac{1}{4}\right)^n u_n \text{ if } |z| > 1/4$$

$$-\left(\frac{1}{4}\right)^n u_{-n-1} \text{ if } |z| < 1/4$$

6.4.2 Inversion using partial fraction expansion

When the z-transform is given as a rational function, partial fraction expansion results in a sum of terms, each of which can be inverted by inspection. Here we consider cases in which the numerator and denominator are polynomials in z^{-1}.

(i) $M < N$, simple poles: if all the N poles are of the first order, we can express $X(z)$ as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$

Each term has a simple inverse z-transform, which depends on the the ROC of $X(z)$. The ROC takes one of the following forms:

$$ROC = \begin{cases} \{z,\ |z| < |p_1|\} \\ \{z,\ |p_k| < |z| < |p_{k+1}|\} \\ \{z,\ |z| > |p_N|\}, \end{cases}$$

where we have assumed that $|p_1| \leq |p_2| \leq \cdots \leq |p_N|$, for simplicity. Each distinct ROC corresponds to a different sequence. Note that when the ROC is $\{z,\ |z| > |p_N|\}$, one has

$$x_n = \sum_{k=1}^{N} A_k (p_k)^n u_n$$

(ii) $M < N$, poles with multiplicity: suppose that $X(z)$ has pole p_i of order $s > 1$, in general the ith term is replaced by s terms:

$$\sum_{k=1}^{s} \frac{C_k}{(1 - p_i z^{-1})^k}.$$

The $k = 1$ term is inverted as before, and the terms for $k > 1$ are inverted using differentiation rules (see Properties of z-transform).

(iii) $M \geq N$. Assume that all the poles are of the first order; multiplicities can be interpreted as above. Using polynomial division, we can write $X(z)$ as

$$X(z) = \sum_{k=0}^{M-N} B_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}.$$

There are many possible ROCs, each determining a distinct sequence corresponding to the second summation. When the ROC is outside the largest pole, we get that:

$$x_n = \sum_{k=0}^{M-N} B_k \delta_{n-k} + \sum_{k=1}^{N} A_k (p_k)^n u_n$$

Example: (Inversion suing partial fraction expansion). Given

$$X(z) = \frac{1 - z^{-1}}{1 - 5z^{-1} + 6z^{-2}} = \frac{1 - z^{-1}}{(1 - 2z^{-1})(1 - 3z^{-1})} = \frac{-1}{1 - 2z^{-1}} + \frac{2}{1 - 3z^{-1}}$$

The original sequence is then

$$x_n = \begin{cases} (2^n - 2 \cdot 3^n) u_{-n-1}, \text{ if ROC} = \{z, |z| < 2\} \\ -2^n u_n - 2 \cdot 3^n u_{-n-1}, \text{ if ROC} = \{z, 2 < |z| < 3\} \\ (-2^n + 2 \cdot 3^n) u_n, \text{ if ROC} = \{z, |z| > 3\} \end{cases}$$

6.5 Properties of the z-transform

Here we list the most important properties of the z-transform:

i) Linearity: $\alpha x_n + \beta y_n \to \alpha X(z) + \beta Y(z)$, $\mathrm{ROC}_x \cap \mathrm{ROC}_y \subset \mathrm{ROC}_{\alpha x + \beta y}$

ii) Scaling in time:

- $x_{Nn} \to \frac{1}{N} \sum\limits_{k=0}^{N-1} X(W_N^k z^{1/N})$ on $(\mathrm{ROC}_x)^{1/N}$ with $W_N^k = e^{i\frac{2\pi k}{N}}$
- $\begin{cases} x_{n/N}, & n/N \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases} \to X(z^N)$ on $(\mathrm{ROC}_x)^N$

iii) Scaling in z: $\alpha^n x_n \to X(\alpha^{-1} z); |\alpha| \mathrm{ROC}_x$.

iv) Time reversal: $x_{-n} \to X(z^{-1}); \frac{1}{\mathrm{ROC}_x}$.

v) Differentiation: $n^k x_n \to (-1)^k z^k \frac{\partial^k X(z)}{\partial z^k}; \mathrm{ROC}_x$

vi) Moments: Computation of the kth moment using the z-transform results in

$$m_k = \sum_{n \in \mathbb{Z}} n^k x_n = \left(\sum_{n \in \mathbb{Z}} n^k x_n z^{-n} \right)_{|z=1} = \left((-1)^k \frac{\partial^k X(z)}{\partial z^k} \right)_{|z=1}$$

Chapter 7

Discrete-time filtering

7.1 Definition of discrete filters

One defines discrete filters in the following way:

Definition 1 *One calls discrete filter an application $D : X \to X_a$ linear, continuous and invariant through translations τ_{ka} $k \in \mathbb{Z}$, where X is a subspace of X_a containing δ_0, invariant through translations τ_{ka}, i.e. $D(\tau_{ka}x) = \tau_{ka}D(x)$ and equipped with the same notion of convergence as X_a.*

Proposition 1

Let: $D : X \to X_a$, a discrete filter and $h = D\delta_0$ (h is called the impulsional response to the filter h). Then D can be written in the convolution form:

$$\forall x \in X \; Dx = h * x$$

in the following two cases:

i) $X = X_a$ and h is finite

ii) $X = X_a \cap \mathcal{D}'_+$ and h belongs to this set.

In both cases, one can check that one has $y = Dx = h * x = \sum_{n \in \mathbb{Z}} y_n \delta_{na}$ with $y_n = \sum_{k=-\infty}^{+\infty} h_k x_{n-k}$ which is a finite sum.

Now the convolution can also be defined when the support of x and h extends from $-\infty$ to $+\infty$. in particular, we have the following proposition:

Proposition 2

> Consider the two discrete signals $h = \sum_{k=-\infty}^{k=+\infty} h_k \delta_{ka}$ and $x = \sum_{n=-\infty}^{n=+\infty} x_n \delta_{na}$, such that (h_k) is with fast decay and (x_n) is with slow increase. The convolution $h * x$ is defined, and we have:
> i) $h * x$ is a tempered distribution
> ii) $h * x = \sum_{n=-\infty}^{n=+\infty} y_n \delta_{na}$ with $y_n = \sum_{k=-\infty}^{k=+\infty} h_k x_{n-k}$, the series defining y_n converging absolutely
> iii) $\widehat{h * x} = \hat{h}\hat{x}$.

Other very important situations involve the spaces

$$l_a^p = \left\{ x = \sum_{n=-\infty}^{n=+\infty} x_n \delta_{na},\ \sum_{n=-\infty}^{n=+\infty} |x_n|^p < +\infty \right\}$$
$$l_a^\infty = \left\{ x = \sum_{n=-\infty}^{n=+\infty} x_n \delta_{na},\ \sup|x_n| < +\infty \right\}$$

One can show that we have $l_a^1 * l_a^\infty \subset l_a^\infty$, $l_a^2 * l_a^2 \subset l_a^\infty$ and $l_a^1 * l_a^2 \subset l_a^1 * l_a^\infty \subset l_a^\infty$.

7.2 Stability and causality of discrete filters

Let us start with the notion of causality:

Definition 2

$$(\text{ \textit{The filter} } D : X \to X_a \text{ \textit{is realizable} }) \Leftrightarrow ((\forall n < 0, \quad x_n = 0) \Rightarrow (\forall n < 0, \quad (Dx)_n = 0))$$

The stability of filters as follows:

Definition 3

$$(\text{ \textit{The filter} } D : X \to X_a \text{ \textit{is stable} }) \Leftrightarrow (\exists A > 0, \forall x \in X \cap l_a^\infty, \|Dx\|_\infty \leq A\|x\|_\infty)$$

To summarize the discrete filter:

$$D : X \to X_a$$
$$x \to D(x) = h * x$$

is defined in the following seven cases:

1. h is finite, $X = X_a$
2. h is causal and, $X = X_a \cap \mathcal{D}'_+$ (causal entries)
3. h is with fast decay, $X = X_a \cap \mathcal{S}'$ (slowly increasing entries)
4. $h \in l_a^1$, $X = l_a^\infty$.

5. $h \in l_a^2$, $X = l_a^2$.

6. $h \in l_a^\infty$, $X = l_a^2$.

7. $h \in X_a$, $X = X_a \cap \mathcal{E}'$ (finite entries)

Theorem 1

Let us consider a filter with impulse response h corresponding to one of the seven above cases, then one has:

i) D is stable $\Leftrightarrow \sum_{n\in\mathbb{Z}} |h_n| < +\infty$

ii) D is causal $\Leftrightarrow \forall\ n < 0 \quad h_n = 0$

Proof i) when the convolution exists, one can write:

$$|y_n| \leq \sum_{k\in\mathbb{Z}} |h_k||x_{n-k}| \leq \sup_n |x_n| \sum_{k\in\mathbb{Z}} |h_k|$$

so D is stable.
The reciprocal is trivial if D is finite. In the case 3 and 4, h is in l_a^1. In the case 2, h and x are with support bounded to the left (one can assume without any loss of generality that h is supported in $\mathbb{N}$).
Let p belonging to $\mathbb{N}$, and x^p defined by:

$$x_n^p = \begin{cases} \text{sign}(h_{p-n}) \text{ si } 0 \leq n \leq p \text{ et } h_{p-n} \neq 0 \\ 0 \text{ sinon} \end{cases}$$

These signals are finite and thus causal and $\|x^p\|_\infty \leq 1$. We can then write

$$y_n^p = \sum_{k=0}^{\infty} h_k \text{sign}(h_{p-n+k}).$$

Thus, $y_p^p = \sum_{k=0}^{p} |h_k| \leq A$, for all $p \geq 0$ and, $\sum_{k=0}^{\infty} |h_k| \leq +\infty$.
Note that this proof holds in the case 5, 6, 7 since the entries x^p are finite. ii) If D is realizable, $h = D\delta_0$ satisfies $h_n = 0$ for all n. Conversely, If this property holds we easily get that the filter is realizable.

7.3 Analyzing filter using z-transform

The distributions gives us a very nice setting for the existence of the convolution, Now to analyse the properties of the discrete filters the z-transform is the most commonly used tool since we have, as soon as the convolution of x et h exists, the very useful property:

Theorem 2

Let $h = \sum_{n\in\mathbb{Z}} h_n\delta_n$ and $x = \sum_{n\in\mathbb{Z}} x_n\delta_n$, such that the convolution exists. Then, denote y this convolution we have:

$$\forall z \in \mathrm{ROC}_x \bigcap \mathrm{ROC}_h, \quad Y(z) = H(z)X(z)$$

Remark 1 *Note that the definition domain is potentially empty.*

7.3.1 Filters governed by a linear difference equation

A very important class of filters are governed by a linear difference of the type:

$$y_n = \sum_{j=0}^{p} a_j x_{n-j} - \sum_{k=1}^{q} b_k y_{n-k} = \sum_{k\in\mathbb{Z}} h_k x_{n-k}$$

Let us compute the z-transform of the system, to get:

$$\left(\sum_{k=0}^{q} b_k z^{-k}\right) Y(z) = \left(\sum_{j=0}^{p} a_j z^{-j}\right) X(z)$$

from which we may write :

$$H(z) = \frac{\sum\limits_{j=0}^{p} a_j z^{-j}}{\sum\limits_{k=0}^{q} b_k z^{-k}}$$

which is a fractional rational z-transform encountered in the previous chapter. From $H(z)$ we can study the stability of filters, which leads to the following theorem:

Theorem 3

i) A filter is stable if and only if the ring of convergence contains the unit circle.

ii) If the filter is realizable, it is stable if and only the poles of $H(z)$ are located inside the unit circle.

To find the impulse response depending on the ROC we refer to the previous chapter. However, when the thought filter h is causal, one can determine the components of h through the following recurring principle:

$$\begin{cases} h_0 = a_0 \\ h_n = a_n - \sum\limits_{k=1}^{n} b_k h_{n-k} \quad n = 1, 2, \cdots \end{cases}$$

7.4 Convolution using finite impulse response filter

We have seen in the previous section that the filter corresponding to linear difference equation is associated with infinite impulse response h, and thus in practice the filtering process never uses the impulse response. But though one never uses the impulse response, one can characterize the nature of the filter, i.e. stability and realizability.

Another very important class of filtering process is when both h and x are finite with the length of h much smaller that N of x, in which case the convolution is carried out through circular convolution.

7.4.1 On the relation between convolution and circular convolution

Let us introduce:

Definition 4 *The circular convolution between sequence $h \in l_a^1$ and x a periodic sequence with period N:*

$$\begin{aligned}(h \circledast x)_n &= \sum_{0 \le k \le N-1} x_k h_{n-k \ mod \ N} \\ &= \sum_{0 \le k \le N-1} x_{(n-k) mod \ N} h_k \end{aligned} \tag{7.1}$$

Remark 2 *This sequence is itself periodic with period N.*

The relation between convolution and circular convolution is then the following. Let us define

$$h_{N,n} = \sum_{k \in \mathbb{Z}} h_{n-kN}, \tag{7.2}$$

then one has the following property:

Proposition 3

Let x be a discrete signal periodic with period N de taille N, then one has the following property:

$$(h * x)_n = (h_N \circledast x)_n \tag{7.3}$$

7.4.2 DFT and circular convolution

Theorem 4

Let (x_k) and (h_k) be two complex sequences, with period N, and let us denote by $(\hat{x}_n)$ et $(\hat{h}_n)$ their DFT.

i) The sequence (z_n) define through circular convolution

$$z_n = (x \circledast h)_n$$

has for DFT

$$\hat{z}_k = \hat{x}_k \hat{h}_k$$

ii) the sequence

$$p_n = x_n h_n$$

has for DFT

$$(\hat{x} \circledast \hat{h})_k$$

From this one can view the convolution as the product of two DFTs followed by an inverse DFT. If one carries out the direct computation, the convolution requires $O(N^2)$ operations, but using the FFT this can be carried using in $O(N\ log_2(N))$ operations.

7.5 Exercises

Exercise 1
Compute the impulse response corresponding to a causal version of the filter given by the following z-transform:

$$H(z) = \frac{1-z^{-1}}{1-5z^{-1}+6z^{-2}}$$

Is this filter stable?

Exercise 2
We sample the RC filter, $RCv' + v = f$ as $RC\frac{y_n - y_{n-1}}{a} + y_n = x_n$.

1. Write the filter as a linear difference equation
2. Deduce the transfert function of the filter
3. Compute the impulse response of the filter. Check that this filter is causal and stable.

Chapter 8

Shannon sampling theorem

One of the key issues we have not dealt with up to now is how to sample the signal to switch from the analogic representation to the digital one. The most important result is due to Shannon and tells us how to sample a signal given its frequency bandwidth. The Shannon theorem is based on Poisson formulae whose descriptions follow.

8.1 Poisson formula in $\mathcal{S}'(\mathbb{R})$

8.1.1 Dual formulation of Poisson formula

To sample a signal f every a seconds consists in considering a piecewise constant approximation for the signal at location (na) and then replace the value of the function by that of the integral of the latter, i.e. $af\Delta_a = a\sum\limits_{n\in\mathbb{Z}} f(na)\delta_{na}$.

Assume f is a tempered distribution such that its Fourier transform satisfies:

$$\mathrm{Supp}\left(\hat{f}\right) \subset [-\lambda_c, \lambda_c],$$

i.e. $\hat{f} \in \mathcal{E}'(\mathbb{R})$, so that f is a C^∞ slow increasing function belonging to $\mathcal{S}'(\mathbb{R})$:

$$af\Delta_a = a\sum_{n\in\mathbb{Z}} f(na)\delta_{na} \in \mathcal{S}'(\mathbb{R}), \tag{8.1}$$

so considering the Fourier transform in $\mathcal{S}'(\mathbb{R})$, one gets:

$$a\widehat{f\Delta_a}(\xi) = a\sum_{n\in\mathbb{Z}} f(na)e^{-2i\pi na\xi}. \tag{8.2}$$

Furtehrmore, considering the Fourier transform of distributions in $\mathcal{E}'(\mathbb{R}) * \mathcal{S}'(\mathbb{R})$, one obtains:

$$\hat{f} * \hat{\Delta}_a = \widehat{f\Delta_a}.$$

Thus:

$$\begin{aligned} a\widehat{f\Delta_a}(\xi) &= \hat{f} * \Delta_{\frac{1}{a}}(\xi)\text{Properties of the Fourier transform of a Dirac comb} \\ &= \sum_{n\in\mathbb{Z}} \tau_{\frac{n}{a}}\hat{f}(\xi). \end{aligned} \tag{8.3}$$

Using the expressions given by the expressions (8.1) and (8.3), one obtains the dual formulation of Poisson formula (true in $\mathcal{S}'(\mathbb{R})$):

$$\sum_{n\in\mathbb{Z}} \tau_{\frac{n}{a}} \hat{f}(\xi) \quad = \quad a \sum_{n\in\mathbb{Z}} f(na) e^{-2i\pi na\xi} \tag{8.4}$$

8.1.2 Direct formulation of Poisson formula

The direct formulation of Poisson formula corresponds to:

$$\sum_{n=-\infty}^{n=+\infty} \tau_{na} f(t) = \frac{1}{a} \sum_{n=-\infty}^{n=+\infty} \hat{f}(\frac{n}{a}) e^{2i\pi n\frac{t}{a}} \tag{8.5}$$

By analogy with the dual formulation, let us suppose that f is compactly supported ($f \in \mathcal{E}'(\mathbb{R})$), Then, one can write:

$$\begin{aligned}
f * \Delta_a &= \overline{\mathcal{F}}(\hat{f}\hat{\Delta}_a) \\
&= \overline{\mathcal{F}}\left(\frac{1}{a}\sum_{n\in\mathbb{Z}} \hat{f}(\frac{n}{a})\delta_{\frac{n}{a}}\right) \\
&= \frac{1}{a}\sum_{n\in\mathbb{Z}} \hat{f}(\frac{n}{a}) e^{2i\pi n\frac{t}{a}}
\end{aligned}$$

8.2 Poisson formula in $L^1(\mathbb{R})$

Theorem 1

Let $f \in L^1(\mathbb{R})$ and $F(t) = \sum\limits_{n=-\infty}^{+\infty} f(t - na)$, then

1. $F \in L^1_p(0, a)$ (periodic with period a) and the series converges in $L^1(0, a)$.
2. The formula (8.5) is true in $\mathcal{S}'(\mathbb{R})$.
3. If furthermore f' (the derivative being considered in the sense of distributions) is in $L^1(\mathbb{R})$, then F is continuous on $\mathbb{R}$ and $\sum\limits_{n=-\infty}^{+\infty} f(t - na)$ converges normally and (8.5) is true for all t (property of Fourier series).

8.3 Shannon Theorem

Shannon theorem expresses how to reconstruct a function in $L^2(\mathbb{R})$ whose Fourier transform is compactly supported by means of its samples.

Theorem 2 (Shannon)

Let f be a function in $L^2(\mathbb{R})$ such that its Fourier transform is supported in the interval $[-\lambda_c, \lambda_c]$. One can sample f with a sampling step a without any loss of information if $\frac{1}{a} \geq 2\,\lambda_c$.
In this case, one has:

$$f(x) = \sum_{n\in\mathbb{Z}} \underbrace{f(na)}_{\acute{e}chantillons} \cdot \underbrace{\sin_c(\frac{\pi}{a}(x-na))}_{Shannoninterpolationbasis}$$

Proof As f is slowly increasing it satisfies the dual formulation of Poisson formula:

$$\hat{f}(\xi) = a\sum_{n\in\mathbb{Z}} f(na)e^{-2i\pi na\xi} = \sum_{n\in\mathbb{Z}} \tau_{\frac{n}{a}}\hat{f}(\xi).$$

As $\hat{f}$ is in $L^2(\mathbb{R})$ and is compactly supported, one can easily show that the previous equality is also true in $L^2_p(-\frac{1}{2a}, \frac{1}{2a})$. With the hypothesis made on a, one has in $L^2_p(-\frac{1}{2a}, \frac{1}{2a})$ (periodic functions with period $1/a$ and square integrable over a period):

$$\hat{f}(\xi) = a\sum_{n\in\mathbb{Z}} f(na)e^{-2i\pi na\xi}\chi_{[-\frac{1}{2a},\frac{1}{2a}]}(\xi).$$

Finally, using the continuity of the Fourier transform on $L^2(\mathbb{R})$, one obtains:

$$\begin{aligned} f(x) &= a\sum_{n\in\mathbb{Z}} f(na)\mathcal{F}^{-1}\left(e^{-2i\pi na\xi}\chi_{[-\frac{1}{2a},\frac{1}{2a}]}\right) \\ &= \sum_{n\in\mathbb{Z}} f(na)\sin_c(\tfrac{\pi}{a}(x-na)) \end{aligned}$$

8.4 Exercises

In the following exercises, we will assume that Poisson formula is also valid for a function f in $L^1(\mathbb{R})$.

Exercise 1
Poisson formula in $L^1(\mathbb{R})$

1. On considers the series:

$$F(t) = \sum_{n=-\infty}^{n=\infty} \frac{1}{n^2+b^2} e^{2i\pi nt}.$$

Show that F is continuous
2. Show that $\frac{1}{n^2+b^2}$ corresponds to the sampling of the Fourier transform of a function f.

3. Apply Poisson formula to obtain another expression for F
4. Then compute F explicitly.

Exercise 2

Let $f \in L^2(\mathbb{R})$ the signal defined by $\hat{f}(\xi) = (1 - |\xi|)\mathbf{1}_{[-1,1]}(\xi)$.
1. Show that $f(x) = \frac{\sin^2(\pi x)}{\pi^2 x^2}$.
2. Using Shannon formula with $a = \frac{1}{2}$, show that

$$\tan(x) = x - \frac{8x^2}{\pi^2} \sum_{k=-\infty}^{k=+\infty} \frac{1}{(2k+1)^2(2x - (2k+1)\pi)}$$

for $x \in \mathbb{R}$ A where $A = \left\{\frac{2k+1}{2}\pi; k \in \mathbb{Z}\right\}$

Exercise 3

generalization of Shannon to trigonometric functions
We consider in this exercise the function $f(t) = e^{2i\pi\lambda t}$, $\lambda \in \mathbb{R}$.

1. Let g be a function periodic with period $\frac{1}{a}$ and equal to f on the interval $[-\frac{1}{2a}, \frac{1}{2a}[$. For λ a real and positive fixed number, show that the Fourier coefficients of f are:

$$c_n = \frac{a\sin(\frac{\pi}{a}(\lambda - na))}{\pi(\lambda - na)}$$

2. Applying Dirichlet theorem, show that:

$$e^{2i\pi\lambda t} = \sum_{n\in\mathbb{Z}} e^{2i\pi nat} \operatorname{sin}_c(\frac{\pi}{a}(\lambda - na)) \text{ for all } t \in]-\frac{1}{2a}, \frac{1}{2a}[,$$

which corresponds to Shannon theorem de permuting λ and t.

Exercise 4

Let f be a real functionsuch that $\hat{f} \in L^2(\mathbb{R})$ and $\mathrm{Supp}(\hat{f}(\xi)) = [f_1, f_2]$ for $\xi > 0$. We moreover assume that $2f_1 \geq f_2$.

1. What relation exists between $\hat{f}(\xi)$ and $\hat{f}(-\xi)$?

2. What constraint must satisfy the sampling frequency for Shannon theorem to apply ?

3. Explain how to periodize the signal by using the smallest sampling frequency as possible.

4. Deduce from that a technique for signal reconstruction.

Chapter 9

Linear Time-Frequency Analysis

Here we focus on linear time-frequency techniques, that is we are going to define linear transforms that map a function to its time-frequency representation. The focus is put on the short-time Fourier transform both in the continuous and discrete setting. In the latter case, the emphasis will be put on the relation with the Fourier transform of distributions.

9.1 Linear Time-Frequency analysis: the continuous time framework

Time-Frequency analysis is related to the definition of Short-Time Fourier Transform (STFT), the definition of which we now recall in different contexts.

9.1.1 Continuous Time Short Time Fourier Transform

Definition 1 *The STFT of a given signal $f \in L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$ and g a real window also in $L^2(\mathbb{R})$ is given by:*

$$V_f^g(t,\omega) \quad = \quad \int_{\mathbb{R}} f(u)g(u-t)e^{-i2\pi\omega(u-t)}du. \tag{9.1}$$

Remark 1 *The existence of the STFT is a direct consequence of Cauchy-Schwarz theorem.*

This transform is invertible under some assumptions:

Proposition 1

Assume $\int_{\mathbb{R}} g = 1$, and that $\hat{f}$ is in $L^1(\mathbb{R})$, the following reconstruction formula holds :

$$f(t) = \int\int_{\mathbb{R}^2} V_f^g(u,\omega)e^{i2\pi\omega(t-u)}dud\omega.$$

Proof Because of the hypothesis made on g, one has: $f(t) = \int_{\mathbb{R}} f(t)g(t-u)du$. Now, the Fourier transform of f reads:

$$\begin{aligned}\hat{f}(\omega) &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(t)g(t-u)e^{-2i\pi\omega(t-u)}dte^{-2i\pi\omega u}du \\ &= \int_{\mathbb{R}} V_f^g(u,\omega)e^{-2i\pi\omega u}du.\end{aligned}$$

So, since $\hat{f}$ is in $L^1(\mathbb{R})$, it is invertible and we can write: $f(t) = \int_{\mathbb{R}^2} V_f^g(u,\omega)e^{2i\pi\omega(t-u)}dud\omega$.

Proposition 2

Assume $\|g\|_2 = 1$, the following reconstruction formula holds (in $L^2(\mathbb{R})$) :

$$f(t) = \int\int_{\mathbb{R}^2} V_f^g(u,\omega)g(t-u)e^{i2\pi\omega(t-u)}dud\omega.$$

Proof The proof uses the fact that $\{g(t-u)e^{i2\pi\omega(t-u)}\}_u$ is a frame of $L^2(\mathbb{R})$, but it will not be detailed here.

Proposition 3

If one assumes that f is analytic (namely $\hat{f}(\omega) = 0$ if $\omega < 0$), g is continuous, iand both f and g are in $L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$, one may also write:

$$f(t) = \frac{1}{g(0)}\int_0^\infty V_f^g(t,\omega)d\omega.$$

Proof

$$\begin{aligned}\int_0^\infty V_f^g(t,\omega)d\omega &= \int_0^\infty\int_{\mathbb{R}} f(u)g(u-t)e^{-i2\pi\omega(u-t)} = \int_0^\infty\int_{\mathbb{R}} \hat{f}(\omega)\hat{g}(\omega-\nu)^* e^{i2\pi\omega t}d\omega d\nu \\ &= \int_0^\infty \hat{f}(\omega)e^{i2\pi\omega t}d\omega\int_{\mathbb{R}}\hat{g}(\nu-\omega)d\nu = f(t)g(0),\end{aligned}$$

so one has the following reconstruction formula: $f(t) = \frac{1}{g(0)}\int_0^\infty V_f^g(t,\omega)d\omega$.

9.1.2 Discrete-Time Short-Time Fourier Transform

Since this part of the course is more signal processing oriented, we replace the notation x_n for a sequence by $x[n]$. For a sequence $(f[n])_{n\in\mathbb{Z}}$ in $l_1(\mathbb{Z})$, and a discrete real window g also in $l_1(\mathbb{Z})$, the STFT is defined for each ω by:

$$V_{f,d}^g(m,\omega) = \sum_{n\in\mathbb{Z}} f[n]g[n-m]e^{-i2\pi\omega(n-m)}. \tag{9.2}$$

The STFT can be viewed as the Fourier transform of $\sum_{n\in\mathbb{Z}} f[n]g[n-m]\delta_n$ times the phase shift term $e^{2i\pi m\omega}$. For STFT, we have the following reconstruction formula:

Proposition 4

Assume $g(0) \neq 0$, then:

$$f[m] = \frac{1}{g(0)} \int_0^1 V_{f,d}^g(m,\omega)d\omega.$$

Proof Since $V_{f,d}^g(m,\omega)$ is 1-periodic with respect to ω, using Fourier series theory we get:

$$f[n]g[n-m] = \int_0^1 V_{f,d}^g(m,\omega)e^{i2\pi\omega(n-m)}d\omega,$$

and then considering $n = m$ and $g(0) \neq 0$, we obtain:

$$f[m] = \frac{1}{g(0)} \int_0^1 V_{f,d}^g(m,\omega)d\omega. \tag{9.3}$$

Proposition 5

Note that with the hypothesis put on g, $(V_{f,d}^g(m,\omega))_{m\in\mathbb{Z}}$ is also in $l_1(\mathbb{Z})$, and further assuming $\|g\|_2 = 1$, we get :

$$f[n] = \int_0^1 \sum_{m\in\mathbb{Z}} V_{f,d}^g(m,\omega)g[n-m]e^{2i\pi\omega(n-m)}d\omega.$$

Proof Indeed, we have:

$$\begin{aligned}\int_0^1 \sum_{m\in\mathbb{Z}} V_{f,d}^g(m,\omega)g[n-m]e^{2i\pi\omega(n-m)}d\omega &= \sum_{m,k\in\mathbb{Z}} f[k]g[k-m]g[n-m] \int_0^1 e^{i2\pi\omega(n-k)}d\omega \\ &= \sum_{m\in\mathbb{Z}} f[n]g[n-m]^2 = f[n] \sum_{m\in\mathbb{Z}} g[m]^2 = f[n]\end{aligned}$$

Proposition 6

Alternatively, if one considers a filter $g \in l_1(\mathbb{Z})$ such that $\sum\limits_m g[m] = 1$, the reconstruction of f is as follows:

$$f[n] = \int_0^1 \sum_{m\in\mathbb{Z}} V_{f,d}^g(m,\omega)e^{i2\pi\omega(n-m)}d\omega.$$

Proof The proof is similar to the previous one and is thus left as an exercice.

9.1.3 Short-Time Fourier Transform for finite length signal and filter

Now we assume the signal is of length L and that the filter g is supported on $\{-M, \cdots, M\}$ such that:

$$2^{\log_2 \lfloor 2M+1 \rfloor + 1} = N \leq L, \tag{9.4}$$

then we have the following reconstruction formula:

Proposition 7

Assume $g[0] \neq 0$, then we may write:

$$f[m] = \frac{1}{g[0]N} \sum_{k=0}^{N-1} V_{f,d}^{g}(m, \frac{k}{N}). \tag{9.5}$$

Proof Indeed,

$$\begin{aligned} V_{f,d}^{g}(m, \frac{k}{N}) &= \sum_{n \in \mathbb{Z}} f[n] g[n-m] e^{-i2\pi \frac{k(n-m)}{N}} = \sum_{n=-M}^{M} f[m+n] g[n] e^{-i2\pi \frac{kn}{N}}, \\ &= \sum_{n=0}^{2M} f[m+n-M] g[n-M] e^{-i2\pi \frac{k(n-M)}{N}}. \end{aligned}$$

Since g is null on $\{M+1, \cdots, N-1-M\}$, the STFT can be rewritten as:

$$V_{f,d}^{g}(m, \frac{k}{N}) e^{-i2\pi \frac{kM}{N}} = \sum_{n=0}^{N-1} f[m+n-M] g[n-M] e^{-i2\pi \frac{kn}{N}}.$$

Using the properties of the discrete Fourier transform, one obtains, for any $n \in \{0, \cdots, N-1\}$:

$$f[m+n-M] g[n-M] = \frac{1}{N} \sum_{k=0}^{N-1} V_{f,d}^{g}(m, \frac{k}{N}) e^{i2\pi \frac{k(n-M)}{N}}. \tag{9.6}$$

Finally, taking $n = M$ and assuming $g[0] \neq 0$:

$$f[m] = \frac{1}{g[0]N} \sum_{k=0}^{N-1} V_{f,d}^{g}(m, \frac{k}{N}).$$

Remark 2 *To reconstruct $f[m]$, one only needs the knowledge of $\left(V_{f,d}^{g}(m, \frac{k}{N})\right)_k$, while $\left(V_{f,d}^{g}(m, \frac{k}{N})\right)_k$ is non zero for $m \in \{-M, \cdots, L-1+M\}$, but the transform outside the support of f is not used in the reconstruction.*

Proposition 8

Now, if g is with unit energy, one has:

$$f[n] = \sum_{m=n-M}^{m=n+M} \frac{1}{N} \sum_{k=0}^{N-1} V_{f,d}^g(m, \frac{k}{N}) g[n-m] e^{i2\pi \frac{k(n-m)}{N}}.$$

Proof Indeed, we may write:

$$\begin{aligned}
\sum_{m=n-M}^{m=n+M} \frac{1}{N} \sum_{k=0}^{N-1} V_{f,d}^g(m, \frac{k}{N}) g[n-m] e^{2i\pi \frac{k(n-m)}{N}} &= \sum_{m=n-M}^{m=n+M} \sum_{p=0}^{p=m+M} f[p] g[p-m] g[n-m] \frac{1}{N} \sum_{k=0}^{N-1} e^{i2\pi \frac{k(n-p)}{N}} \\
&= \sum_{m=n-M}^{m=n+M} f[n] g[n-m]^2 = f[n] \sum_{m=-M}^{m=M} g[m]^2 = f[n].
\end{aligned}$$

This time, one needs the knowledge of $\left(V_{f,d}^g(m, \frac{k}{N})\right)_k$ for $m \in \{-M, \cdots, L-1+M\}$, while one would like to be able to reconstruct f using only $\left(V_{f,d}^g(m, \frac{k}{N})\right)_k$ for $m \in \{0, \cdots, L-1\}$.
To circumvent this difficulty, one can assume f is L-periodic instead of finite: the STFT is no longer in $l_1(\mathbb{Z})$ but is also L-periodic (in the sum defining the STFT p varies from $m-M$ to $m+M$). In this case, we may write:

Proposition 9

Assuming g is with unit energy, one has the following reconstruction formula assuming f is L-periodic:

$$f[n] = \sum_{m=n-M}^{n+M} \sum_{k=0}^{N-1} V_{f,d}^g(m \bmod L, \frac{k}{N}) g[n-m] \frac{e^{i2\pi \frac{k(n-m)}{N}}}{N}.$$

Note that the hypothesis that f is periodic could be avoided easily as well as on the unit energy for the filter. Indeed,

Proposition 10

assuming f is null outside its boundary, we have:

$$f[n] = \frac{\sum\limits_{m=\max(n-M,0)}^{m=n+M} \frac{1}{N} \sum\limits_{k=0}^{N-1} V_{f,d}^g(m, \frac{k}{N}) g[n-m] e^{2i\pi \frac{k(n-m)}{N}}}{\sum\limits_{m=\max(n-M,0)}^{m=n+M} g[m]^2}$$

Proof We may actually write

$$\begin{aligned}
&\sum_{m=\max(n-M,0)}^{m=n+M} \frac{1}{N} \sum_{k=0}^{N-1} V_{f,d}^{g}(m, \frac{k}{N}) g[n-m] e^{2i\pi \frac{k(n-m)}{N}} \\
= &\sum_{m=\max(n-M,0)}^{m=n+M} \sum_{p=0}^{p=m+M} f[p] g[p-m] g[n-m] \frac{1}{N} \sum_{k=0}^{N-1} e^{i2\pi \frac{k(n-p)}{N}} \\
= &\sum_{m=\max(n-M,0)}^{m=n+M} f[n] g[n-m]^2 = f[n] \sum_{m=\max(n-M,0)}^{m=n+M} g[m]^2 = f[n].
\end{aligned}$$

Similarly to what was done previously in the continuous time case, if we further assume that f is L-periodic, and that $\sum_{m=-M}^{M} g(m) = 1$, one has the following reconstruction formula:

$$f[p] = \sum_{m=p-M}^{p+M} \frac{1}{N} \sum_{k=0}^{N-1} V_{f,d}^{g}(m \bmod L, \frac{k}{N}) e^{i2\pi \frac{k(p-m)}{N}}.$$

Again, not assuming any periodicity hypothesis we also have.

$$f[p] = \frac{\sum_{m=max(p-M,0)}^{p+M} \frac{1}{N} \sum_{k=0}^{N-1} V_{f,d}^{g}(m \bmod L, \frac{k}{N}) e^{i2\pi \frac{k(p-m)}{N}}}{\sum_{m=max(p-M,0)}^{p+M} g[p]}.$$

To conclude on this part we have shown different reconstruction procedures for associated with the STFT and considering different hypothesis on the filter. We are going to use these developments in the study of reassignment technique in the following chapter.