Minimal zeros of copositive matrices

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Outline

Copositive cone

Definition

A real symmetric $n \times n$ matrix A such that $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n_+$ is called copositive.

the set of all such matrices is a regular convex cone, the copositive cone $\mathcal{C}_n$

- many applications in optimization
- difficult to describe

related cones

- completely positive cone $\mathcal{C}_n^*$
- sum $\mathcal{N}_n + \mathcal{S}_n^+$ of nonnegative and positive semi-definite cone
- doubly nonnegative cone $\mathcal{N}_n \cap \mathcal{S}_n^+$

$$\mathcal{C}_n^* \subset \mathcal{N}_n \cap \mathcal{S}_n^+ \subset \mathcal{N}_n + \mathcal{S}_n^+ \subset \mathcal{C}_n$$

Extreme rays

Definition

Let $K \subset \mathbb{R}^n$ be a regular convex cone. An nonzero element $u \in K$ is called extreme if it cannot be decomposed into a sum of other elements of K in a non-trivial manner. In other words, $u = v + w$ with $v, w \in K$ imply $v = \alpha u$, $w = \beta u$ for some $\alpha, \beta \geq 0$.

in [Hall, Newman 63] the extreme rays of $\mathcal{C}_n$ belonging to $\mathcal{N}_n + \mathcal{S}_n^+$ have been described:

- the extreme rays of $\mathcal{N}_n$: E_{ii} and $E_{ij} + E_{ji}$
- rank 1 matrices $A = xx^T$ with x having both positive and negative elements

the other extreme rays of $\mathcal{C}_n$ are called exceptional

Perspective goal: **Describe the exceptional extreme rays of $\mathcal{C}_n$.**

Reduced rays

Definition (Diananda 62, Baumert 65)

A copositive matrix $A \in \mathcal{C}_n$ is called reduced if it cannot be represented as a sum of a copositive and a nonnegative matrix in a non-trivial manner. In other words, $A = B + C$ with $B \in \mathcal{C}_n$ and $C \in \mathcal{N}_n$ imply $B = A$ and $C = 0$.

Lemma

Let $A \in \mathcal{C}_n$ be an extreme matrix. Then A is either reduced or nonnegative.

exceptional extreme rays have to be reduced

[Hall, Newman 1963] reduced matrices satisfy $A_{ij} \leq \sqrt{A_{ii}A_{jj}}$

Zero patterns

zero patterns helpful in the description of reducedness

Definition (Baumert 65)

Let $A \in \mathcal{C}_n$ be a copositive matrix. A nonzero nonnegative vector $u \in \mathbb{R}_+^n$ is called zero of A if $u^T A u = 0$. The index set $\operatorname{supp} u = \{i \mid u_i > 0\}$ is called the support of u.
The set of supports of all zeros of A is called the zero pattern of A.

the zero pattern is a set of subsets of $\{1, \dots, n\}$

Theorem (Diananda 62)

Let $A \in \mathcal{C}_n$ be a copositive matrix, u a zero and $I = \operatorname{supp} u$ its support.
Then the principal submatrix $A_I = (A_{ij})_{i,j \in I}$ is positive semi-definite.

Size of supports

Lemma (Baumert 65)

Let $A \in \mathcal{C}_n$ be irreducible with respect to the cone of nonnegative matrices. If there exists a zero u of A with $|\operatorname{supp} u| \geq n-1$, then $A \in \mathcal{S}_+^n$.

zeros u of exceptional extreme copositive matrices with nonzero diagonal satisfy $2 \leq |\operatorname{supp} u| \leq n-2$

General reduced rays

Definition (Dür et al, 2013)

A copositive matrix $A \in \mathcal{C}_n$ is called reduced with respect to a subset $\mathcal{M} \subset \mathcal{S}_n$ if it cannot be in a non-trivial manner represented as a sum $A = B + C$ with B copositive and $C \in \mathcal{M}$.

Lemma

Let $A \in \mathcal{C}_n$ be an extreme matrix. Then A is either reduced with respect to $\mathcal{S}_+^n + \mathcal{N}_n$ or in $\mathcal{S}_+^n + \mathcal{N}_n$.

exceptional extreme rays are reduced with respect to $\mathcal{S}_+^n + \mathcal{N}_n$

Description of general irreducibility

Theorem (Dickinson, H. 2014)

Let $A \in \mathcal{C}_n$. Then for a matrix $B \in \mathcal{S}_n$ there exists $\delta > 0$ such that $A + \delta B \in \mathcal{C}_n$ if and only if $u^T B u \geq 0$ for all zeros u of A, and $(Bu)_i \geq 0$ for all zeros u of A such that $u^T B u = 0$ and all i such that $(Au)_i = 0$.

[Dickinson, H.: Considering copositivity locally (submitted)]

Automorphism group

the group $\mathbb{R}^n_{++}$ acts on $\mathcal{C}_n$ by $d : A \mapsto \operatorname{diag}(d) A \operatorname{diag}(d)$

for every $A \in \mathcal{C}_n$, there exists a normalized A' in the orbit of A such that

$$\operatorname{diag} A' \in \{0,1\}^n$$

if $\operatorname{diag} A' \not> 0$, then $\operatorname{diag} A \not> 0$ and $A \in \mathcal{C}_{n-1} + \mathcal{N}_n$
we may assume $\operatorname{diag} A = \mathbf{1}$ w.l.o.g.

the permutation group S_n acts on $\mathcal{C}_n$ by $P : A \mapsto PAP^T$
this action respects the property of being normalized with respect to the action of $\mathbb{R}^n_{++}$

these groups leave also $\mathcal{N}_n$ and $\mathcal{S}_n^+$ invariant $\Rightarrow$ they respect the property of being reduced with respect to $\mathcal{N}_n + \mathcal{S}_n^+$

Extreme rays in low dimensions

Theorem (Diananda 1962)

For $n \leq 4$ the relation $\mathcal{C}_n = \mathcal{S}_+^n + \mathcal{N}_n$ holds.

no exceptional extreme rays for $n \leq 4$

Theorem (H., 2011)

Let $A \in \mathcal{C}_5$ be an exceptional extreme ray. Then A is in the orbit of a T-matrix with $\psi = (\psi_1, \dots, \psi_5) \in \mathbb{R}^5_{++}$ and $\psi_1 + \dots + \psi_5 < \pi$ or in the orbit of the Horm matrix with respect to the action of $\mathrm{Aut}(\mathcal{C}_5)$.

T-matrices

a T-matrix is a matrix of the form

$$T(\psi) = \begin{pmatrix} 1 & -\cos\psi_4 & \cos(\psi_4+\psi_5) & \cos(\psi_2+\psi_3) & -\cos\psi_3 \\ -\cos\psi_4 & 1 & -\cos\psi_5 & \cos(\psi_5+\psi_1) & \cos(\psi_3+\psi_4) \\ \cos(\psi_4+\psi_5) & -\cos\psi_5 & 1 & -\cos\psi_1 & \cos(\psi_1+\psi_2) \\ \cos(\psi_2+\psi_3) & \cos(\psi_5+\psi_1) & -\cos\psi_1 & 1 & -\cos\psi_2 \\ -\cos\psi_3 & \cos(\psi_3+\psi_4) & \cos(\psi_1+\psi_2) & -\cos\psi_2 & 1 \end{pmatrix}$$

with $\psi_1, \dots, \psi_5 \geq 0$ and $\sum_{k=1}^{5} \psi \leq \pi$

the Horn matrix is of the form $T(\psi)$ with $\psi = 0$

Approach:

Find necessary conditions on the minimal zero pattern of matrices which are reduced with respect to $\mathcal{S}_+^n$.

For every pattern found, find the extremal matrices corresponding to it.

Minimal zeros

Definition

A zero u of a copositive matrix A is called minimal if there exists no zero v of A such that the inclusion $\operatorname{supp} v \subset \operatorname{supp} u$ holds strictly.

Lemma

Let $A \in \mathcal{C}_n$ and let $I \subset \{1, \dots, n\}$ be a nonempty index set. Then the following are equivalent:

- *A has a minimal zero with support I,*
- *the principal submatrix A_I is positive semi-definite with corank 1, and the generator of the kernel of A_I can be chosen such that all its elements are positive.*

Consequences of a minimal zero

let $A \in \mathcal{C}_n$ and let u be a minimal zero of A with support I

- for every index subset such that $J \subset I$ strictly, $A_J \succ 0$
- the subvector u_I of the zero generates the kernel of A_I
- the minimal zero with support I is unique up to scaling
- I not comparable by inclusion to the support of any other minimal zero

Corollary

The number of minimal zeros of a copositive matrix is finite up to scaling.

convenient for treatment with combinatorial methods

Decomposition of zeros

Lemma

Let $A \in \mathcal{C}_n$ and let u be a zero of A with support I. Then the set of zeros v of A with support $\operatorname{supp} v \subset I$ is a polyhedral cone, namely v_I is in the intersection of $\ker A_I$ with the nonnegative orthant $\mathbb{R}_+^{|I|}$. The extreme rays of this cone are generated exactly by the minimal zeros v of A with $\operatorname{supp} v \subset I$.

Corollary

Every zero of A can be represented as a convex combination of minimal zeros.

Sufficient condition for minimality

Lemma

Let $A \in \mathcal{C}_n$ and u be a zero of A with support I. Suppose that A_I has a principal submatrix of size $|I| - 1$ which is positive definite. Then u is a minimal zero.

sufficient if there exists a minimal zero v with support $J \not\subset I$ such that $|I \setminus J| = 1$

Overlapping zeros

Theorem

Let $A \in \mathcal{C}_n$ and $I \subset \{1, \dots, n\}$ an index set such that $A_I \succ 0$. Let $u^1, \dots, u^m$ be zeros of A such that $(\operatorname{supp} u^l) \setminus I = \{k^l\}$ consists of exactly one element, $u^1, \dots, u^m$ are mutually different modulo scaling, and $\operatorname{supp} u^1 \cap I \subset \dots \subset \operatorname{supp} u^m \cap I$ for all $r = 1, \dots, m-1$.
Then $k^1, \dots, k^m$ are mutually different, and $u^1, \dots, u^m$ are minimal. If v is a zero of A with $\operatorname{supp} v \subset I \cup \{k^1, \dots, k^m\}$, then $v = \sum_{i=1}^m \alpha_i u^i$ for some nonnegative scalars α^i. If in addition v is minimal, then v is proportional to one of the u^k.

note: condition $A_I \succ 0$ guaranteed by existence of a minimal zero u such that $I \subset \operatorname{supp} u$ strictly

Two overlapping zeros

Corollary

Let $A \in \mathcal{C}_n$ *and* u, v *minimal zeros of* A *with supports* $\operatorname{supp} u = I$, $\operatorname{supp} v = J$. *Suppose* $|J \setminus I| = 1$ *consists of one element. Then every zero* w *of* A *with support* $\operatorname{supp} w \subset I \cup J$ *can be represented as a convex conic combination* $w = \alpha u + \beta v$ *with* $\alpha, \beta \geq 0$.

no minimal zeros w with $\operatorname{supp} w \subset I \cup J$ other than u and v

Irreducibility with respect to $\mathcal{S}_+^n$

Theorem

A copositive matrix $A \in \mathcal{C}_n$ is irreducible with respect to the cone $\mathcal{S}_+^n$ if and only if the linear span of the minimal zeros of A equals $\mathbb{R}^n$. Equivalently, the number of linearly independent minimal zeros is at least n.

in particular, the number of minimal zeros is at least n

Supports of size 2

Lemma

Let $A \in \mathcal{C}_n$ with $\operatorname{diag} A = \mathbf{1}$*. Let u be a zero of A with* $\operatorname{supp} u = \{i, j\}$*.*
Then u is minimal and $u_i = u_j$.

without loss of generality we may assume $u_i = u_j = 1$

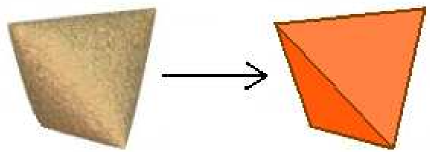
Consequence: $A_{ij} = -1$ if and only if $\{i, j\}$ is a minimal zero support
then $A_{ik} + A_{jk} \geq 0$ for all k

define $\alpha_{ij} = \frac{1}{\pi} \arccos(-A_{ij})$
for reduced matrices $\alpha_{ij} \in [0, 1]$ and above conditions become
$\alpha_{ij} = 0$ ($\alpha_{ij} > 0$) and $\alpha_{ik} + \alpha_{jk} \geq 1$

Supports of size 3

the set $\{A \in \mathcal{S}_3^+ \mid \operatorname{diag} A = \mathbf{1}\}$ is bounded by the Cayley surface

the element-wise map $x \mapsto \frac{2}{\pi} \arcsin x$ transforms it into a tetrahedron with the same vertices



define $\alpha_{ij} = \frac{1}{\pi} \arccos(-A_{ij})$

Corollary

Let $A \in \mathcal{C}_n$ be reduced with $\operatorname{diag} A = \mathbf{1}$. Let u be a zero of A with $\operatorname{supp} u = \{i,j,k\}$. Then $\alpha_{ij} + \alpha_{jk} + \alpha_{ik} = 1$. If $\{i,j,k\}$ does not contain a minimal zero support, then $\alpha_{ij} + \alpha_{jk} + \alpha_{ik} > 1$.

MAXCUT polytope

Definition

The MAXCUT polytope $\mathcal{MC}_n \subset \mathcal{S}_+^n$ is the convex hull of all matrices $A \in \mathcal{S}_+^n$ such that $A_{ij} \in \{-1, +1\}$ for all $i, j = 1, \ldots, n$, i.e., all matrices of the form vv^T, $v \in \{-1, +1\}^n$.

Lemma (Hirschfeld 2004; Goemans, Williamson 1995)

Let $A \in \mathcal{S}_+^n$ be a positive semi-definite matrix with $A_{ii} = 1$, $i = 1, \ldots, n$. Let B be the real symmetric $n \times n$ matrix defined entry-wise by $B_{ij} = 2\alpha_{ij} - 1 = \frac{2}{\pi} \arcsin A_{ij}$, $i, j = 1, \ldots, n$. Then $B \in \mathcal{MC}_n$.

Linear relations

Corollary

Let $A \in \mathcal{C}_n$ *be reduced with* $\operatorname{diag} A = \mathbf{1}$*. Let* $I \subset \{1, \ldots, n\}$ *be the support of some minimal zero of* A*. Define*
$B_{ij} = 2\alpha_{ij} - 1 = \frac{2}{\pi} \arcsin A_{ij}$, $B = (B_{ij})$.
Then $B_I \in \mathcal{MC}_{|I|}$*. If* $J \subset I$ *strictly, then* $B_J \in \operatorname{relint} \mathcal{MC}_{|J|}$.

gives strict and nonstrict linear inequalities on α_{ij}

for every pairwise distinct indices $i_1, \ldots, i_5 \in \{1, \ldots, n\}$ we have $\sum_{1 \leq j < k \leq 5} \alpha_{i_j i_k} \geq 4$

Low dimensions

the number of equivalence classes (with respect to the action of S_n) of minimal zero patterns of matrices $A \in \mathcal{C}_n$ which satisfy all restrictions is

- 0 for $n \leq 4$
- 2 for $n = 5$
- 44 for $n = 6$
- 12378 for $n = 7$

hence $\mathcal{C}_n$ cannot have exceptional extreme rays for $n \leq 4$, proving quickly Dianandas theorem

Cone $\mathcal{C}_5$

the two equivalence classes of minimal zero patterns have representatives

$$\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,5\}\},$$

$$\{\{1,2,3\},\{2,3,4\},\{3,4,5\},\{1,4,5\},\{1,2,5\}\}$$

realized by the Horn form and the T-matrices, respectively

these are the exceptional extreme rays of $\mathcal{C}_5$

Cone $\mathcal{C}_6$

minimal zero patterns satisfying all necessary conditions

1	$\{1,2\},\{1,3\},\{1,4\},\{2,5\},\{3,6\},\{5,6\}$	23	$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,6\},\{2,4,6\},\{3,4,5,6\}$
2	$\{1,2\},\{1,3\},\{1,4\},\{2,5\},\{3,6\},\{4,5,6\}$	24	$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,6\},\{3,4,6\},\{3,5,6\}$
3	$\{1,2\},\{1,3\},\{1,4\},\{2,5\},\{3,5,6\},\{4,5,6\}$	25	$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,6\},\{3,4,6\},\{4,5,6\}$
4	$\{1,2\},\{1,3\},\{1,4\},\{2,5,6\},\{3,5,6\},\{4,5,6\}$	26	$\{1,2,3\},\{1,2,4\},\{1,3,5\},\{1,4,5\},\{2,3,6\},\{2,4,6\}$
5	$\{1,2\},\{1,3\},\{2,4\},\{3,4,5\},\{1,5,6\},\{4,5,6\}$	27	$\{1,2,3\},\{1,2,4\},\{1,3,5\},\{1,4,5\},\{2,3,6\},\{3,4,6\}$
6	$\{1,2\},\{1,3\},\{1,4,5\},\{2,4,6\},\{3,4,6\},\{4,5,6\}$	28	$\{1,2,3\},\{1,2,4\},\{1,3,5\},\{2,4,5\},\{3,4,5\},\{2,3,6\}$
7	$\{1,2\},\{1,3\},\{2,4,5\},\{3,4,5\},\{2,4,6\},\{3,4,6\}$	29	$\{1,2,3\},\{1,2,4\},\{1,3,5\},\{2,4,5\},\{2,3,6\},\{2,5,6\}$
8	$\{1,2\},\{1,3\},\{2,4,5\},\{3,4,5\},\{2,4,6\},\{3,5,6\}$	30	$\{1,2,3\},\{1,2,4\},\{1,3,5\},\{2,4,5\},\{3,4,6\},\{3,5,6\}$
9	$\{1,2\},\{3,4\},\{1,3,5\},\{2,4,6\},\{1,5,6\},\{4,5,6\}$	31	$\{1,2,3\},\{1,2,4\},\{1,3,5\},\{2,4,5\},\{1,5,6\},\{2,5,6\}$
10	$\{1,2\},\{1,3,4\},\{1,3,5\},\{2,3,6\},\{3,4,6\},\{3,5,6\}$	32	$\{1,2,3\},\{1,2,4\},\{1,3,5\},\{2,4,5\},\{1,5,6\},\{4,5,6\}$
11	$\{1,2\},\{1,3,4\},\{1,3,5\},\{1,4,6\},\{2,5,6\},\{3,5,6\}$	33	$\{1,2,3\},\{1,2,4\},\{1,3,5\},\{2,4,5\},\{3,5,6\},\{4,5,6\}$
12	$\{1,2\},\{1,3,4\},\{1,3,5\},\{1,4,6\},\{3,5,6\},\{4,5,6\}$	34	$\{1,2,3\},\{1,2,4\},\{1,3,5\},\{2,4,6\},\{3,5,6\},\{4,5,6\}$
13	$\{1,2\},\{1,3,4\},\{1,3,5\},\{2,4,6\},\{3,4,6\},\{2,5,6\}$	35	$\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,6\},\{1,3,5,6\},\{2,4,5,6\},\{3,4,5,6\}$
14	$\{1,2\},\{1,3,4\},\{1,3,5\},\{2,4,6\},\{3,4,6\},\{3,5,6\}$	36	$\{1,2\},\{1,3\},\{1,4\},\{2,5\},\{4,5\},\{3,6\},\{5,6\}$
15	$\{1,2\},\{1,3,4\},\{1,3,5\},\{2,4,6\},\{3,4,6\},\{4,5,6\}$	37	$\{1,2\},\{1,3,4\},\{1,3,5\},\{1,4,6\},\{2,5,6\},\{3,5,6\},\{4,5,6\}$
16	$\{1,2\},\{1,3,4\},\{1,3,5\},\{2,4,6\},\{3,5,6\},\{4,5,6\}$	38	$\{1,2\},\{1,3,4\},\{1,3,5\},\{2,4,6\},\{3,4,6\},\{2,5,6\},\{3,5,6\}$
17	$\{1,2\},\{1,3,4\},\{2,3,5\},\{3,4,5\},\{2,4,6\},\{3,4,6\}$	39	$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,6\},\{1,4,6\},\{2,5,6\},\{3,5,6\}$
18	$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,6\},\{1,4,6\},\{1,5,6\}$	40	$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,6\},\{1,4,6\},\{3,5,6\},\{4,5,6\}$
19	$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,6\},\{1,4,6\},\{2,5,6\}$	41	$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,6\},\{2,4,6\},\{3,4,6\},\{3,5,6\}$
20	$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,6\},\{1,4,6\},\{3,5,6\}$	42	$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,6\},\{2,4,6\},\{3,5,6\},\{4,5,6\}$
21	$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,6\},\{2,4,6\},\{3,4,6\}$	43	$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,6\},\{1,4,6\},\{2,5,6\},\{3,5,6\},\{4,5,6\}$
22	$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,6\},\{2,4,6\},\{3,5,6\}$	44	$\{1,2,3\},\{1,2,4\},\{1,3,5\},\{1,4,5\},\{2,3,6\},\{2,4,6\},\{3,5,6\},\{4,5,6\}$

Website http://nadezhdaa.wix.com/copositiv

Exceptional 6x6 copositive forms

Contact

The purpose of this website is to classify the exceptional extreme 6x6 copositive forms. This task amounts to consider 80 cases of possible minimal zero support sets which such a form can have. A detailed description of the problem and an instruction how to solve a case is given here. Further information can be found here. Everybody who is familiar with matrix-vector multiplications, determinants, and trigonometry can participate, no knowledge of higher mathematics is needed.

Solutions should be sent in pdf format. A solution should mention the case number, describe all copositive forms with the given minimal zero support set if there are any, and state that there are none in the opposite case. If there are copositive forms, it should be determined whether they are extremal or not. All statements have to be mathematically proven. After a check for correctness the solutions will be published on the site.

1 {1,2},{1,3},{1,4},{2,5},{3,6},{5,6}

2 {1,2},{1,3},{1,4},{2,5},{3,6},{4,5,6}

3 {1,2},{1,3},{1,4},{2,5},{3,5,6},{4,5,6}

4 {1,2},{1,3},{1,4},{2,5,6},{3,5,6},{4,5,6}

5 {1,2},{1,3},{2,4},{3,5},{3,4,6},{2,5,6}

6 {1,2},{1,3},{2,4},{3,4,5},{1,5,6},{4,5,6}

colour code

white	the case is unsolved
green	exceptional extreme copositive forms with the corresponding minimal zero support set exist

Example of family of extreme rays

the minimal zero pattern
$\{\{1,2,3\},\{2,3,4\},\{3,4,5\},\{1,4,5\},\{1,2,5\},\{3,4,6\},\{1,4,6\},\{1,2,6\}\}$
corresponds to the extremal matrices

$$\begin{pmatrix} 1 & -\cos\phi_4 & \cos(\phi_4+\phi_5) & \cos(\phi_2+\phi_3) & -\cos\phi_3 & -\cos(\phi_3+\xi) \\ -\cos\phi_4 & 1 & -\cos\phi_5 & \cos(\phi_1+\phi_5) & \cos(\phi_3+\phi_4) & \cos(\phi_3+\phi_4+\xi) \\ \cos(\phi_4+\phi_5) & -\cos\phi_5 & 1 & -\cos\phi_1 & \cos(\phi_1+\phi_2) & \cos(\phi_1+\phi_2-\xi) \\ \cos(\phi_2+\phi_3) & \cos(\phi_1+\phi_5) & -\cos\phi_1 & 1 & -\cos\phi_2 & -\cos(\phi_2-\xi) \\ -\cos\phi_3 & \cos(\phi_3+\phi_4) & \cos(\phi_1+\phi_2) & -\cos\phi_2 & 1 & \cos\xi \\ -\cos(\phi_3+\xi) & \cos(\phi_3+\phi_4+\xi) & \cos(\phi_1+\phi_2-\xi) & -\cos(\phi_2-\xi) & \cos\xi & 1 \end{pmatrix}$$

with $\phi_1,\dots,\phi_5 > 0$, $\sum_{i=1}^5 \phi_i < \pi$, $\xi \in (-\phi_3, \phi_2)$

References

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Thank you!