Centro-affine differential geometry, Lagrangian submanifolds of the reduced paracomplex projective space, and conic optimization

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June 5, 2012 / Differential Geometry 2012, Będlewo

Outline

1. Conic optimization and barriers
   - Convex programs
   - Barriers on convex sets
   - Conic programs
   - Logarithmically homogeneous barriers
2. Barriers and centro-affine geometry
   - Splitting theorem
   - Centro-affine equivalents of barriers
   - Applications
3. Lagrangian submanifolds in para-Kähler space
   - Cross-ratio manifold
   - Objects defined by cones
   - Barriers and Lagrangian submanifolds
   - Applications

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Convex optimization problems

minimize linear objective function with respect to convex constraints

$$\min_{x \in X} f(x)$$

$f = \langle c, x \rangle$, X convex

$X \subset \mathbb{R}^n$ is called the feasible set

Regular convex sets

Definition

A regular convex set $X \subset \mathbb{R}^n$ is a closed convex set having nonempty interior and containing no lines.

can assume the feasible set to be regular

Definition of barriers

Definition

Let $X \subset \mathbb{R}^n$ be a regular convex set. A ν-self-concordant barrier for X is a smooth function $F : X^o \to \mathbb{R}$ such that

- $F''(x) \succ 0$ (convexity)
- $\lim_{x \to \partial X} F(x) = +\infty$ (boundary behaviour)
- $|F_{,i}h^i|^2 \leq \nu F_{,ij}h^i h^j$ for all $h \in T_x\mathbb{R}^n$ (gradient inequality)
- $|F_{,ijk}h^i h^j h^k| \leq 2(F_{,ij}h^i h^j)^{3/2}$ for all $h \in T_x\mathbb{R}^n$ (self-concordance)

- F'' defines a Hessian metric on X^o
- uses only the affine connection on $\mathbb{R}^n$ $\Rightarrow$ affine invariance

Interior-point methods using barriers

$$\min_{x \in X} \langle c, x \rangle$$

constrained convex program

let $F(x) = +\infty$ for all $x \notin X^o$

$$\min_{x} \tau \langle c, x \rangle + F(x)$$

unconstrained program, $\tau > 0$ a parameter
by convexity and boundary behaviour of F this program is convex

the minimizer x_τ^* of the unconstrained program tends to the minimizer x^* of the constrained program as $\tau \to +\infty$

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Purpose of self-concordance

$|F_{,i}h^i|^2 \leq \nu F_{,ij}h^i h^j$ for all $h \in T_x\mathbb{R}^n$ (gradient inequality)
$|F_{,ijk}h^i h^j h^k| \leq 2(F_{,ij}h^i h^j)^{3/2}$ for all $h \in T_x\mathbb{R}^n$ (self-concordance)

self-concordance ensures good behaviour of the Newton method for computing x_τ^* [Nesterov, Nemirovski 1994]

the smaller ν, the faster the algorithm

Regular convex cones

now specializing to cones ...

Definition

A regular convex cone $K \subset \mathbb{R}^n$ is a convex cone which is regular as a set.

dual cone

$$K^* = \{y \in \mathbb{R}_n \,|\, \langle x, y \rangle \geq 0 \quad \forall\, x \in K\}$$

is also regular

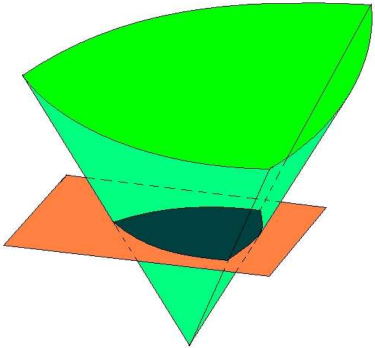
Conic programs

Definition

A conic program over a regular convex cone $K \subset \mathbb{R}^n$ is an optimization problem of the form

$$\min_{x \in K} \langle c, x \rangle : \quad Ax = b.$$

Geometric interpretation



the feasible set is the intersection of K with an affine subspace

Symmetric cones

A regular convex cone K is called self-dual if it is linearly isomorphic to its dual K^*.

A regular convex cone K is called homogeneous if its automorphism group $\text{Aut}(K)$ acts transitively on it.

Definition

A self-dual, homogeneous convex cone is called symmetric.

theory of conic programs over symmetric cones particularly well developed

Logarithmically homogeneous functions

let $K \subset \mathbb{R}^n$ be a regular convex cone
a logarithmically homogeneous function $F : K^o \to \mathbb{R}^n$ satisfies

$$F(\alpha x) = -\nu \log \alpha + F(x) \qquad \forall\, \alpha > 0,\ x \in K^o$$

$\nu > 0$ is called homogeneity parameter

- $F \mapsto cF \Rightarrow \nu \mapsto c\nu$
- $F_{,i}x^i = -\nu$
- $F_{,ij}x^j = -F_{,i}$
- $F_{,ij}x^i x^j = \nu$
- $F^{,ij}F_{,i}F_{,j} = \nu$

for F locally strongly convex the gradient inequality $|F_{,i}h^i|^2 \leq \nu F_{,ij}h^i h^j$ is satisfied

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Logarithmically homogeneous barriers

Definition (Nesterov, Nemirovski 1994)

Let $K \subset \mathbb{R}^n$ be a regular convex cone. A self-concordant logarithmically homogeneous barrier on K is a self-concordant barrier which at the same time is a logarithmically homogeneous function. The homogeneity parameter ν is called the barrier parameter.

- $F(\alpha x) = -\nu \log \alpha + F(x)$ (logarithmic homogeneity)
- $F''(x) \succ 0$ (convexity)
- $\lim_{x \to \partial K} F(x) = +\infty$ (boundary behaviour)
- $|F_{,ijk} h^i h^j h^k| \leq 2(F_{,ij} h^i h^j)^{3/2}$ (self-concordance)

invariant with respect to linear transformations

Duality

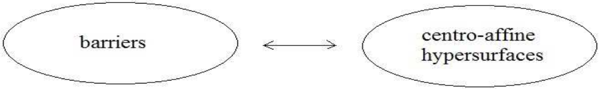
Theorem (Nesterov, Nemirovski 1994)

Let $K \subset \mathbb{R}^n$ be a regular convex cone and $F : K^o \to \mathbb{R}$ a self-concordant logarithmically homogeneous barrier on K with parameter ν. Then the Legendre transform F^* is a self-concordant logarithmically homogeneous barrier on $-K^*$ with parameter ν.

the map $x \mapsto F'(x)$ takes the level surfaces of F to the level surfaces of F^*

Examples

cone K	barrier $F(x)$	parameter ν
$\mathbb{R}_+^n$	$-\sum_{i=1}^n \log x_i$	n
L_n	$-\log(x_0^2 - \sum_{i=1}^{n-1} x_i^2)$	2
$S_+(n)$ $H_+(n)$ $Q_+(n)$	$-\log\det X$	n



- level surfaces of barrier F are centro-affine embeddings
- centro-affine embeddings define logarithmically homogeneous functions up to affine scaling $F \mapsto cF + b$

Normalization

F logarithmically homogeneous function with parameter ν

$$F \mapsto cF + b \qquad \Rightarrow \qquad \nu \mapsto c\nu$$

$$|F_{,ijk}h^i h^j h^k| \leq 2(F_{,ij}h^i h^j)^{3/2} \Leftrightarrow |cF_{,ijk}h^i h^j h^k| \leq 2c^{-1/2}(cF_{,ij}h^i h^j)^{3/2}$$

Convention: We divide the barriers by their parameter ν and consider functions with homogeneity parameter 1. The barrier parameter then appears in the self-concordance inequality

$$|F_{,ijk}h^i h^j h^k| \leq 2\sqrt{\nu}(F_{,ij}h^i h^j)^{3/2}$$

Splitting theorem

Theorem (Tsuji 1982; Loftin 2002)

Let $K \subset \mathbb{R}^{n+1}$ be a regular convex cone, and $F : K^o \to \mathbb{R}$ a locally strongly convex logarithmically homogeneous function with homogeneity parameter 1.
Then the Hessian manifold $(K^o, F_{,ij})$ splits into a direct product of a radial 1-dimensional part and a transversal n-dimensional part. The submanifolds corresponding to the radial part are rays, the submanifolds corresponding to the transversal part are level surfaces of F. The metric on the level surfaces is the centro-affine metric.

Centro-affine objects on K

let $M \subset K^o \subset \mathbb{R}^{n+1}$ be a level surface of F

consider M as centro-affine embedding

extend forms on T_xM to T_xK^o by putting them equal to zero on the radial part

g_{ij}	$F_{,ij} - F_{,i}F_{,j}$
C_{ijk}	$F_{,ijk} - 2F_{,ij}F_{,k} - 2F_{,ik}F_{,j} - 2F_{,jk}F_{,i} + 4F_{,i}F_{,j}F_{,k}$
$T_i = C_{ijk}g^{jk}$	$F_{,ijk}F^{,jk} - \frac{2}{n+1}F_{,i}$
$\nabla_l C_{ijk}$	$F_{,ijkl} - \frac{1}{2}F^{,rs}(F_{,ijr}F_{kls} + F_{,ikr}F_{,jls} + F_{,ilr}F_{,jks})$

Centro-affine pendants of barriers

let F be a logarithmically homogeneous function on $K \subset \mathbb{R}^{n+1}$ and M a level surface of F

Question: Which conditions has the centro-affine immersion to satisfy in order for F to be a self-concordant barrier?

convexity: $F_{,ij} \succ 0 \Leftrightarrow g_{ij} \succ 0$

boundary behaviour: $\lim_{x \to \partial K} F(x) = \infty \Leftrightarrow M$ asymptotic to ∂K

Pendant of self-concordance

Lemma

The self-concordance concordance condition $|F_{,ijk}h^ih^jh^k| \leq 2\sqrt{\nu}(F_{,ij}h^ih^j)^{3/2}$ for all tangent vectors $h \in T_xK^o$ is equivalent to the condition

$$|C_{ijk}u^iu^ju^k| \leq 2\gamma(g_{ij}u^iu^j)^{3/2}$$

for all tangent vectors $u \in T_xM$, where

$$\gamma = \frac{\nu - 2}{\sqrt{\nu - 1}}.$$

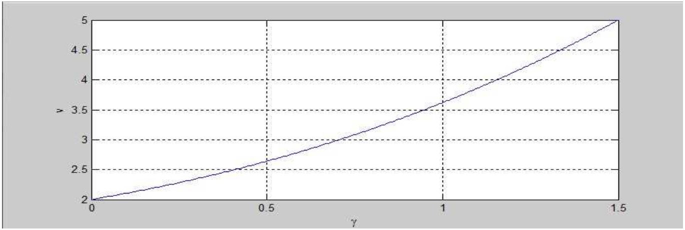
self-concordant functions correspond to centro-affine hypersurfaces with bounded cubic form

Summary

Theorem

Let $K \subset \mathbb{R}^{n+1}$ be a regular convex cone.
Every barrier F on K with parameter ν defines by its level surfaces a homothetic family of locally strongly convex centro-affine hypersurface embeddings of hyperbolic type, asymptotic to ∂K, with cubic form bounded by γ on the unit sphere.
Conversely, every locally strongly convex centro-affine hypersurface embedding $M \subset \mathbb{R}^{n+1}$ of hyperbolic type, asymptotic to ∂K, with cubic form bounded by γ on the unit sphere, defines up to an additive constant a unique barrier with parameter ν on K which is constant on M.
The bound γ and the parameter ν are related by $\gamma = \frac{\nu-2}{\sqrt{\nu-1}}$.

Dependence between γ and ν



Corollary

On cones $K \subset \mathbb{R}^{n+1}$, $n \geq 1$, there exist no logarithmically homogeneous self-concordant barriers with parameter $\nu < 2$.

Distinguished role of second-order cones

$$\gamma = 0 \qquad \Leftrightarrow \qquad \nu = 2$$

Theorem (Pick, Berwald)

Let M be an equiaffine hypersurface immersion with vanishing cubic form. Then M is a quadric.

Corollary

Let $K \subset \mathbb{R}^{n+1}$ be a regular convex cone and F a barrier on K with parameter ν. Then the following are equivalent.
1) $\nu = 2$.
2) K is a second-order cone and F is the canonical barrier on it.

Affine hyperspheres

Lemma (Schneider 1967)

Let M be a complete n-dimensional hyperbolic affine hypersphere with mean curvature $-H$. Then the length $C_{ijk}C^{ijk}$ of the cubic form on M is bounded from above by $4n(n-1)H$.

Lemma

The cubic form of M is bounded by

$$|C_{ijk}u^iu^ju^k| \leq \frac{2(n-1)\sqrt{-H}}{\sqrt{n}} \qquad \forall\, u \in T_xM : \; ||u|| = 1$$

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Einstein-Hessian barriers

affine hyperspheres with $H = -1$ are centro-affine embeddings

Corollary

Let $K \subset \mathbb{R}^n$ be a regular convex cone. The barrier F on K which has as its level surfaces the complete hyperbolic affine hyperspheres asymptotic to ∂K has barrier parameter $\nu \leq n$.

call this barrier the Einstein-Hessian barrier

Calabi product

Lemma (Sasaki 1980)

Let $K \subset \mathbb{R}^n$, $K' \subset \mathbb{R}^{n'}$ be regular convex cones and F, F' the Einstein-Hessian barriers on them. Then $\frac{nF+n'F'}{n+n'}$ is the Einstein-Hessian barrier on $K \times K'$. Its level surfaces are the Calabi product of the level surfaces of F and F'.

Calabi product of complete hyperbolic hyperspheres corresponds to direct product of convex cones

Parallel cubic form

Theorem (Hu, Li, Vrancken 2011)

A locally strongly convex affine hypersurface of $\mathbb{R}^{n+1}$, equipped with the Blaschke metric, and with parallel cubic form, is a quadric or a Calabi product, with factors being hyperboloids and standard immersions of $SL(m,R)/SO(m)$, $SL(m,C)/SU(m)$, $SU^*(2m)/Sp(m)$, or $E_{6(-26)}/F_4$.

Classification of symmetric cones

Theorem (Vinberg, 1960; Koecher, 1962)

Every symmetric cone can be represented as a direct product of a finite number of the following irreducible symmetric cones:

- *second-order cone*
- *matrix cones* $S_+(n)$, $H_+(n)$, $Q_+(n)$ *of real, complex, or quaternionic hermitian positive semi-definite matrices*
- *Albert cone* $O_+(3)$ *of octonionic hermitian positive semi-definite* 3×3 *matrices*

Simple characterization of $\hat{\nabla} C = 0$

Corollary

Let $M \subset \mathbb{R}^{n+1}$ be a locally strongly convex Blaschke hypersurface immersion with cubic form parallel with respect to the Levi-Civita connection.
Then either M is a quadric or M can be extended to a complete hyperbolic affine hypersphere which is asymptotic to a symmetric cone.
The determinant of the Jordan algebra generating the symmetric cone is constant on M.

Nonpositive sectional curvature

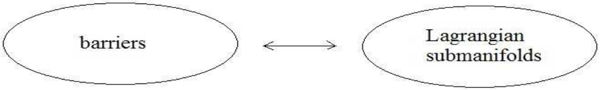
the affine hyperspheres asymptotic to second-order cones have nonpositive sectional curvature bounded away from zero

Question: How small must the cubic form of a centro-affine hypersurface immersion be to guarantee nonpositivity of the sectional curvature?

Lemma

Let M be a hyperbolic centro-affine hypersurface immersion with cubic form bounded by $|C_{ijk}u^iu^ju^k| \leq \sqrt{2}$ for all unit length vectors $u \in T_xM$. Then M has nonpositive sectional curvature.

saturated by affine hypersphere asymptotic to $\mathbb{R}^3_+$


barriers
Lagrangian submanifolds

Product of projective spaces

let $\mathbb{R}P^n, \mathbb{R}P_n$ be the primal and dual real projective space

no scalar product, but orthogonality

$$\mathcal{M} = \{(x,p) \in \mathbb{R}P^n \times \mathbb{R}P_n \,|\, x \not\perp p\}$$

is a dense subset of $\mathbb{R}P^n \times \mathbb{R}P_n$

para-Kähler space form isomorphic to reduced paracomplex projective space [Gadea, Amilibia 1992]

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Contact structure on $\partial\mathcal{M}$

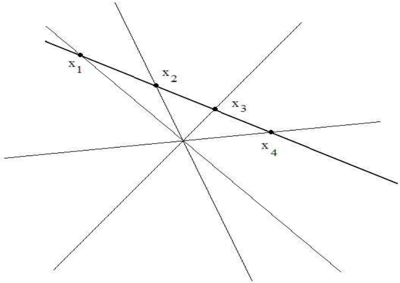
the projections π, π^* of $\mathbb{R}P^n \times \mathbb{R}P_n$ onto the factors define n-dimensional distributions $J_\pm$ on $\mathbb{R}P^n \times \mathbb{R}P_n$

traces $\tilde{J}_\pm$ on $\partial\mathcal{M}$ are of dimension $n-1$

Lemma

The manifold $\partial\mathcal{M}$ equipped with the distribution $\tilde{J}_+ + \tilde{J}_-$ is a contact manifold.

Cross-ratio

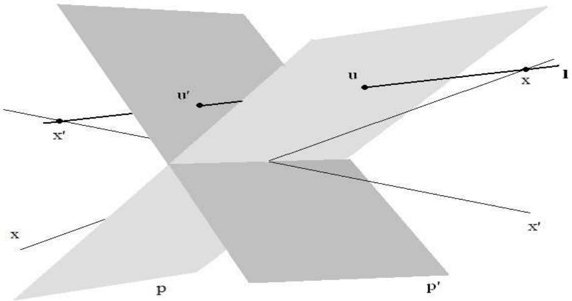


x_1, x_2, x_3, x_4 points on the projective line $\mathbb{R}P^1$

$$(x_1, x_2; x_3, x_4) = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_2 - x_3)(x_1 - x_4)}$$

Generalization to *n* dimensions

[Ariyawansa, Davidon, McKennon 1999]: instead of 4 collinear points use 2 points and 2 dual points



$(u, x'; u', x)$ — quadra-bracket of x, p, x', p'

Two-point function on $\mathcal{M}$

let $z = (x,p), z' = (x',p') \in \mathcal{M} \subset \mathbb{R}P^n \times \mathbb{R}P_n$

$$(z;z') = (z';z) := (u,x';u',x)$$

defines a symmetric function $(\cdot;\cdot) : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$

$$\lim_{z \to \partial\mathcal{M}} (z;z') = \pm\infty$$

Cross-ratio and geodesic distance

Theorem

Let $z, z' \in \mathcal{M}$ be two points and $d(z, z')$ their geodesic distance in $\mathcal{M}$.

- *If the geodesic linking z, z' is of elliptic type, then $(z; z') > 0$ and $d(z, z') = \arcsin \sqrt{(z; z')}$.*
- *If the geodesic linking z, z' is light-like, then $(z; z') = 0$.*
- *If the geodesic linking z, z' is of hyperbolic type, then $(z; z') < 0$ and $d(z, z') = arc\sinh \sqrt{-(z; z')}$.*

$(z; z')$ is the only projective invariant of a pair of points in $\mathcal{M}$.

call $\mathcal{M}$ the cross-ratio manifold

Projective images of cones

let $K \subset \mathbb{R}^{n+1}$ be a regular convex cone

the canonical projection $\Pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ maps $K \setminus \{0\}$ to a compact convex subset $C \subset \mathbb{R}P^n$

the canonical projection $\Pi^* : \mathbb{R}_{n+1} \setminus \{0\} \to \mathbb{R}P_n$ maps $K^* \setminus \{0\}$ to a compact convex subset $C^* \subset \mathbb{R}P_n$

Images of conic boundaries

let $K \subset \mathbb{R}^{n+1}$ be a regular convex cone

the canonical projection
$\Pi \times \Pi^* : (\mathbb{R}^{n+1} \setminus \{0\}) \times (\mathbb{R}_{n+1} \setminus \{0\}) \to \mathbb{R}P^n \times \mathbb{R}P_n$ maps the set

$$\Delta_K = \{(x,p) \in (\partial K \setminus \{0\}) \times (\partial K^* \setminus \{0\}) \,|\, x \perp p\}$$

to a set $\delta_K \subset \partial \mathcal{M}$

Lemma

The set δ_K is Legendrian with respect to the contact structure on $\partial \mathcal{M}$.
The projections π, π^* of $\mathbb{R}P^n \times \mathbb{R}P_n$ to the factors map δ_K onto ∂C and ∂C^*, respectively.
If K is smooth, then δ_K is homeomorphic to S^{n-1}.

Images of barriers

let $K \subset \mathbb{R}^{n+1}$ be a regular convex cone and $F : K^o \to \mathbb{R}$ a barrier on K

$$M = \Pi \times \Pi^* \left[\left\{ (x, -F'(x)) \,|\, x \in K^o \right\} \right]$$

is a smooth nondegenerate Lagrangian submanifold of $\mathcal{M}$ with boundary δ_K

invariant with respect to duality

Bijection with level surfaces

there is a canonical bijection between a level surface of F (or F^*) and the submanifold M

Lemma

The canonical bijection between M and the level surfaces of F is an isometry.
The image of the cubic form under this bijection can be expressed through the second fundamental form II of M by

$$C(X, Y, Z) = -2\omega(II(X, Y), Z).$$

Equivalence to centro-affine geometry

applicable to any nondegenerate centro-affine immersion: project pair (position vector, image of conormal map)

Theorem

Nondegenerate Lagrangian submanifolds of $\mathcal{M}$ are in one-to-one correspondence with homothetic families of nondegenerate centro-affine hypersurface immersions in $\mathbb{R}^{n+1}$. The centro-affine metric on the immersion equals the metric on the submanifold inherited from $\mathcal{M}$. The cubic form on the immersion C and the second fundamental form on the submanifold obey the relation

$$C(X,Y,Z) = -2\omega(II(X,Y),Z).$$

Affine hyperspheres

Corollary (independently obtained by D. Fox)

A nondegenerate Lagrangian submanifold $M \subset \mathcal{M}$ is minimal if and only if the corresponding family of centro-affine immersions are affine hyperspheres.

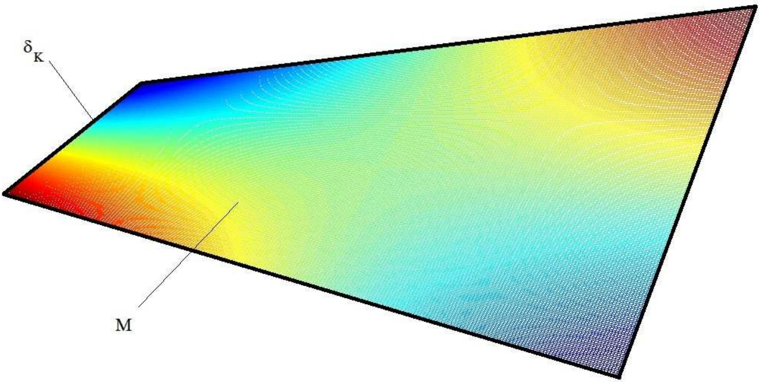
no convexity or completeness assumptions

Equivalents of barriers

Corollary

Let $K \subset \mathbb{R}^{n+1}$ be a regular convex cone.
The barriers on K with parameter ν correspond to positive definite Lagrangian submanifolds M of $\mathcal{M}$ of hyperbolic type inscribed in δ_K, with second fundamental form bounded by $\gamma = \frac{\nu-2}{\sqrt{\nu-1}}$.

Geometric interpretation



Local approximation

Lemma

Lagrangian geodesic submanifolds of $\mathcal{M}$ are totally geodesic.

let $F : K^o \to \mathbb{R}$ be a barrier with parameter ν and $M \subset \mathcal{M}$ the corresponding Lagrangian submanifold

at a given point $z \in M$ the tangent totally geodesic Lagrangian submanifold to M approximates M up to 1st order

Global approximation

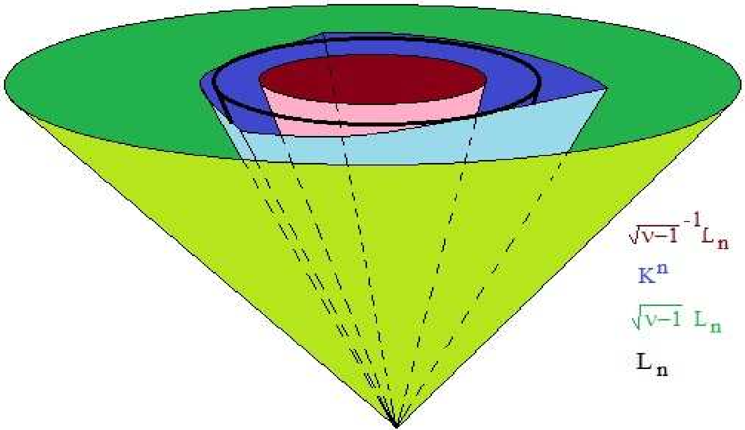
the tangent totally geodesic Lagrangian submanifold defines the barrier of a second-order cone L_{n+1}

L_{n+1} can be viewed as approximating K at the ray corresponding to z

Lemma

Let $K \subset \mathbb{R}^n$ be a regular convex cone, F a barrier on K with parameter ν and $z \in M$ a point on the corresponding Lagrangian submanifold. Let L_n be the second-order cone defined by the tangent totally geodesic submanifold at z.
If we pass to a coordinate system where L_n is centered and blow up (shrink) the horizontal affine section of L_n by a factor of $\sqrt{\nu - 1}$, we obtain an outer (inner) approximation of K.

Geometric interpretation



Upper bound on geodesic distance

in a Riemannian manifold, geodesic distances on a submanifold are not shorter than in the ambient manifold

not so in a pseudo-Riemannian manifold

Theorem

Let $M \subset \mathcal{M}$ be a definite Lagrangian submanifold of hyperbolic type. Suppose that for every two points $z, z' \in M$ there exists a path linking z, z' which projects bijectively to a line in the factor $\mathbb{R}P^n$ (or $\mathbb{R}P_n$).
Then the geodesic distance on M is bounded from above by the geodesic distance on $\mathcal{M}$, i.e., by arc $\sinh \sqrt{-(z; z')}$.

Bound on centro-affine immersions

Corollary

Let $x, x' \in M$ be two points on a locally strongly convex centro-affine hypersurface $M \subset \mathbb{R}^{n+1}$ of hyperbolic type that can be linked by a path on M that projects bijectively to a line segment in $\mathbb{R}P^n$. Let p, p' be the tangent spaces to M at x, x'. Then the geodesic distance $d(x, x')$ in the centro-affine metric is bounded from above by arc $\sinh \sqrt{-Q}$, where Q is the quadra-bracket of x, p, x', p'.

Einstein-Hessian metrics

Theorem (Cheng, Yau 1980)

Let $X \subset \mathbb{R}^n$ be a regular convex set. Then the boundary value problem

$$\det G'' = e^{2G}, \qquad G|_{\partial X} = +\infty$$

has a unique locally strongly convex solution.

Question: Is this solution or a multiple of it a barrier, i.e., is it self-concordant and does it satisfy the gradient inequality?

Rational solutions

let $K \subset \mathbb{R}^n$ be a regular convex cone and F the Einstein-Hessian barrier on K

Question: For which cones K the function $e^{2F} = \det F''$ is the inverse of a polynomial? (symmetric cones?)

For which cones K is it a rational function? (homogeneous cones?)

Optimal barrier parameter

let $K \subset \mathbb{R}^n$ be a regular convex cone

Question: What is the smallest possible parameter of a barrier on K?

What is the smallest possible bound on the second fundamental form of a Lagrangian submanifold of $\mathcal{M}$ inscribed in δ_K?

partial answer: lower bounds available

How small can the distance be?

let $z, z' \in \mathcal{M}$ be such that the geodesic $[z, z']$ is of hyperbolic type

with no further restrictions $\inf d(z, z') = 0$ (choose a path close to the distributions $J_\pm$)

Question: Does an upper bound on the second fundamental form of M imply a lower bound on the geodesic distance $d(z, z')$ on M?

let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and consider the cone

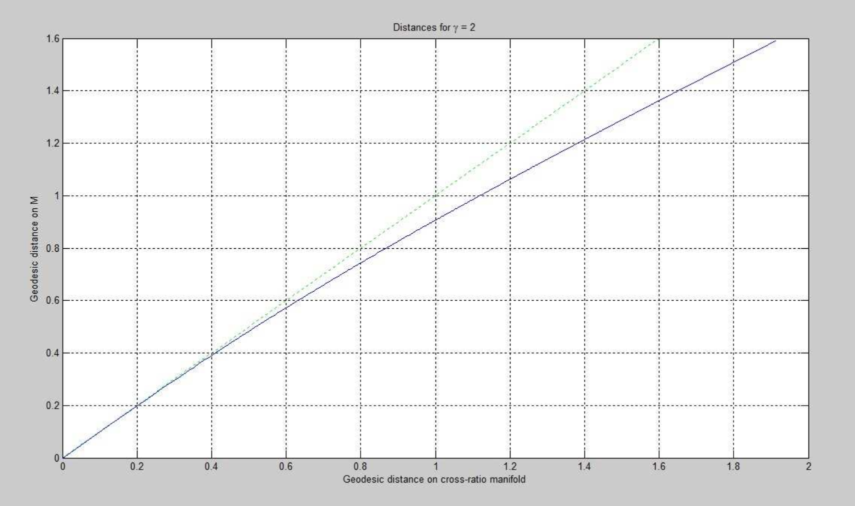
$$K = \left\{ (x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1} \,|\, x \geq 0,\ y \geq 0,\ ||z||^2 \leq x^{1/p} y^{1/q} \right\}$$

with $s = \frac{q}{p}$ the affine hypersphere with mean curvature -1 asymptotic to this cone has its cubic form bounded by

$$2\gamma = \frac{2(n-1)|s-1|}{\sqrt{(ns+1)(n+s)}},$$

for points z, z' on the geodesic corresponding to the 2-plane $z = 0$ we have

$$d(z, z') = \left(\frac{\gamma^2}{4} + 1 \right)^{-1/2} \operatorname{arc\,sinh} \left(\sqrt{\frac{\gamma^2}{4} + 1} \sqrt{-(z; z')} \right)$$


Distances for γ = 2
Geodesic distance on M
Geodesic distance on cross-ratio manifold
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

References

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Thank you