Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Canonical barriers on regular convex cones

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Outline

1 Conic optimization

- Definition
- Logarithmically homogeneous barriers
- Different barrier constructions

2 Geometric view on the canonical barrier

- Splitting of Hessian metric
- Para-Kähler space form
- Barriers and Lagrangian submanifolds

3 3-dimensional cones

- ubiquitous in science and engineering
- main division: convex vs. non-convex optimization problems

convex programs minimize objective function with respect to constraints

$$\min_{x \in X} f(x)$$

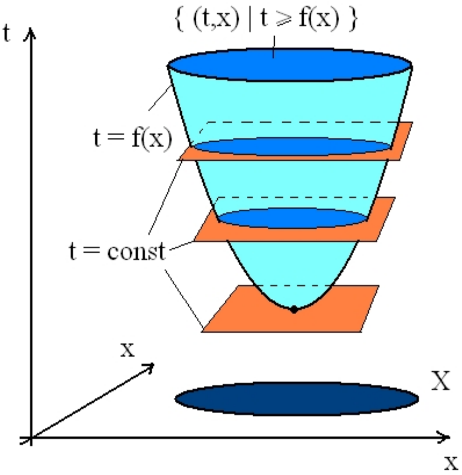
f and X are assumed convex
$X \subset \mathbb{R}^n$ is called the feasible set

examples

- linear programs (LP)
- second-order cone programs (SOCP)
- semi-definite programs (SDP)
- geometric programs (GP)

properties

- duality theory
- global solutions



$f(x)$ can be assumed linear

otherwise minimize t over the epigraph

Definition

A regular convex cone $K \subset \mathbb{R}^n$ is a closed convex cone having nonempty interior and containing no lines.

The dual cone

$$K^* = \{y \in \mathbb{R}_n \mid \langle x, y \rangle \geq 0 \quad \forall\, x \in K\}$$

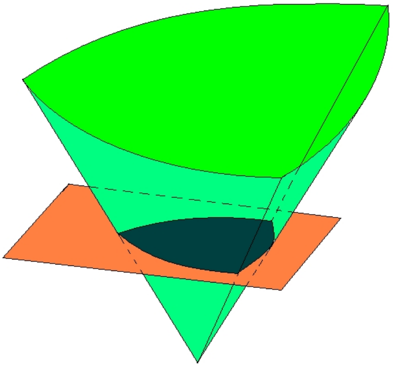
of a regular convex cone K is also regular.

Definition

A conic program over a regular convex cone $K \subset \mathbb{R}^n$ is an optimization problem of the form

$$\min_{x \in K} \langle c, x \rangle : \quad Ax = b.$$

every convex optimization problem can be written as a conic program



the feasible set is the intersection of K with an affine subspace

$$\min_x \langle c', x \rangle \; : \; A'x + b' \in K$$

explicit parametrization

Definition

A real symmetric $n \times n$ matrix A such that $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n_+$ is called copositive.

the set of all such matrices is a regular convex cone, the copositive cone $\mathcal{C}_n$

Theorem (Murty, Kabadi 1987)

Checking whether an $n \times n$ integer matrix is not copositive is NP-complete.

Theorem (Burer 2009)

Any *mixed binary-continuous optimization problem with linear constraints and (non-convex) quadratic objective function can be written as a copositive program*

$$\min_{x \in \mathcal{C}_n} \langle c, x \rangle : \qquad Ax = b$$

Definition (Nesterov, Nemirovski 1994)

Let $K \subset \mathbb{R}^n$ be a regular convex cone. A (self-concordant logarithmically homogeneous) barrier on K is a smooth function $F : K^o \to \mathbb{R}$ on the interior of K such that

- $F(\alpha x) = -\nu \log \alpha + F(x)$ (logarithmic homogeneity)
- $F''(x) \succ 0$ (convexity)
- $\lim_{x \to \partial K} F(x) = +\infty$ (boundary behaviour)
- $|F'''(x)[h,h,h]| \leq 2(F''(x)[h,h])^{3/2}$ (self-concordance)

for all tangent vectors h at x.

The homogeneity parameter ν is called the barrier parameter.

Theorem (Nesterov, Nemirovski 1994)

Let $K \subset \mathbb{R}^n$ be a regular convex cone and $F : K^o \to \mathbb{R}$ a barrier on K with parameter ν. Then the Legendre transform F^* is a barrier on $-K^*$ with parameter ν.

- the map $x \mapsto F'(x)$ takes the level surfaces of F to the level surfaces of F^*
- the map $x \mapsto -F'(x)$ is an isometry between K^o and $(K^*)^o$ with respect to the Hessian metrics defined by F'', $(F^*)''$

let $K \subset \mathbb{R}^n$ be a regular convex cone
let $F : K^o \to \mathbb{R}$ be a barrier on K
consider the conic program

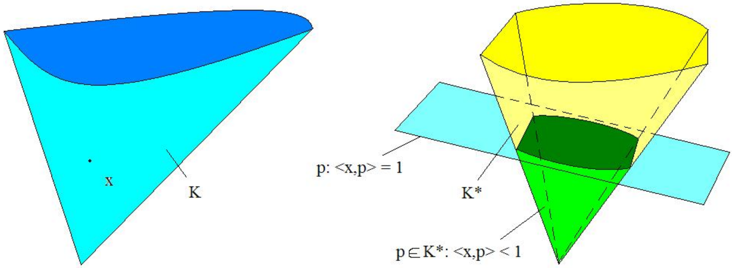
$$\min_{x \in K} \langle c, x \rangle : \quad Ax = b$$

for $\tau > 0$, solve instead the unconstrained problem

$$\min_{x \in \mathbb{R}^n} \tau \langle c, x \rangle + F(x) : \quad Ax = b$$

- unique minimizer $x^*(\tau) \in K^o$ for every $\tau > 0$
- solution depends continuously on τ (*central path*)
- $x^*(\tau) \to x^*$ as $\tau \to \infty$

path-following methods:
alternate Newton steps and increments of τ
the smaller the barrier parameter ν, the faster we can increase τ safely



volume function $V : K^o \ni x \mapsto Vol\{p \in K^* \mid \langle x, p \rangle < 1\}$

Theorem (Nesterov, Nemirovski 1994)

There exists an absolute constant $c > 0$ such that

$$F(x) = c \log V(x)$$

is a $(c \cdot n)$-self-concordant barrier on $K \subset \mathbb{R}^n$.

Lemma (Güler 1996)

The universal barrier equals $c \log \varphi(x)$ up to an additive constant, where

$$\varphi(x) = \int_{K^*} e^{-\langle x,p \rangle} dp$$

is the characteristic function of the cone.

- invariant under the action of $SL(\mathbb{R}, n)$
- fixed under unimodular automorphisms of K
- additive under the operation of taking products
- has barrier parameter $O(n)$

not invariant under duality

Theorem (Bubeck, Eldan 2014)

Let $K \subset \mathbb{R}^n$ be a convex body (compact with non-empty interior). Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(\theta) = \log\left(\int_{x \in K} e^{\langle \theta, x \rangle}\, dx\right).$$

Then the Fenchel dual $f^* : K^o \to \mathbb{R}$ defined by $f^*(x) = \sup_{\theta \in \mathbb{R}^n} \langle \theta, x \rangle - f(\theta)$ is a $(1 + \varepsilon_n) \cdot n$-self-concordant barrier on K, with $\varepsilon_n \leq 100\sqrt{\frac{\log n}{n}}$, for any $n \geq 80$.

originally defined for convex bodies, but can be extended to cones by homogenization

main ingredient of proof: Brunn-Minkowski inequality

- invariant under the action of $SL(\mathbb{R}, n)$
- fixed under unimodular automorphisms of K
- additive under the operation of taking products
- has barrier parameter $n + O(\log n\sqrt{n})$

not invariant under duality

homogeneous cones:

if the automorphism group of K acts transitively on K^o, then K is called homogeneous
homogeneous cones are related to T-algebras and a rank can be associated with them [Vinberg 1962]

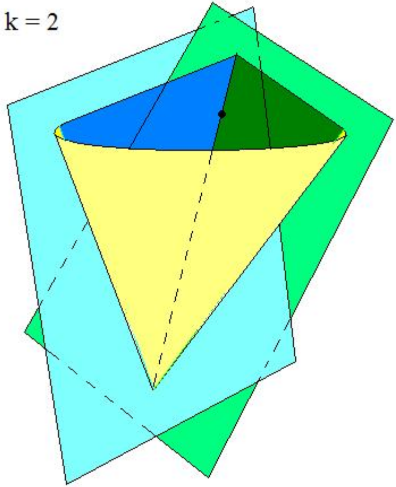
Lemma (Güler, Tunçel 1998)

Let K be a homogeneous convex cone with rank r.
Then the optimal barrier parameter on K equals r.

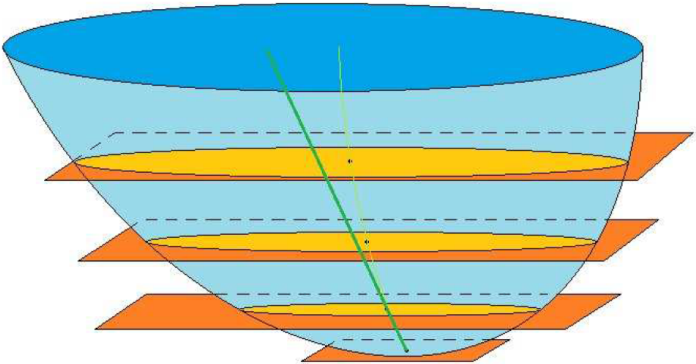
cones with corners:

Lemma (Nesterov, Nemirovski 1994)

Let K be a regular convex cone, $z \in \partial K$, $U \subset \mathbb{R}^n$ a neighbourhood of z, $A_1, \dots, A_k \subset \mathbb{R}^n$ closed affine half-spaces with $z \in \partial A_i$ for all i such that the normals to the half-spaces at z are linearly independent and the intersection $U \cap K$ equals the intersection $U \cap A_1 \cap \dots \cap A_k$.
Then a lower bound on the barrier parameter of any barrier on K is given by $\nu_* = k$.


k = 2

consider a non-degenerate convex hypersurface in $\mathbb{R}^n$



the affine normal is the tangent to the curve made of the gravity centers of the sections

a (proper) affine hypersphere is a hypersurface such that all affine normals meet at a point

for a hyperbolic affine hypersphere the affine normals meet outside of the convex hull hyperbolic affine hyperspheres:

- equi-affinely invariant
- solutions to a certain Monge-Ampère equation
- invariant under the *conormal map*
- Ricci curvature is non-positive (Calabi 1972)

Theorem (Calabi conjecture; Fefferman 76, Cheng-Yau 86, Li 90, and others)

Let $K \subset \mathbb{R}^n$ be a regular convex cone. Then there exists a unique foliation of K^o by a homothetic family of affine complete and Euclidean complete hyperbolic affine hyperspheres which are asymptotic to ∂K.

Every affine complete, Euclidean complete hyperbolic affine hypersphere is asymptotic to the boundary of a regular convex cone.

Theorem (H., 2014; independently D. Fox, 2015)

Let $K \subset \mathbb{R}^n$ be a regular convex cone. Then there exists a logarithmically homogeneous self-concordant barrier F on K^o with parameter $\nu = n$ such that the level surfaces of F are hyperbolic affine hyperspheres.

this barrier is called the canonical barrier

main idea of proof:

use non-positivity of the Ricci curvature

$$4F_{,ij} \succeq F^{,uv} F^{,rs} F_{,iur} F_{,jvs}$$

convex solution of the PDE

$$\log\det F'' = 2F, \qquad F|_{\partial K} = +\infty$$

already conjectured by O. Güler

- invariant under the action of $SL(\mathbb{R}, n)$
- fixed under unimodular automorphisms of K
- additive under the operation of taking products
- has barrier parameter n
- invariant under duality

Property	Universal barrier	Entropic barrier	Canonical barrier
$SL(\mathbb{R}, n)$-invariance	Yes	Yes	Yes
$\mathrm{Aut}(K)$-invariance	Yes	Yes	Yes
product additivity	Yes	Yes	Yes
parameter	$O(n)$	$n + O(\log n\sqrt{n})$	n
duality	No	No	Yes
computability	No	No	No

for $K \subset \mathbb{R}^3$ with non-trivial automorphism group, the canonical barrier is given generically by elliptic integrals (H., 2014)

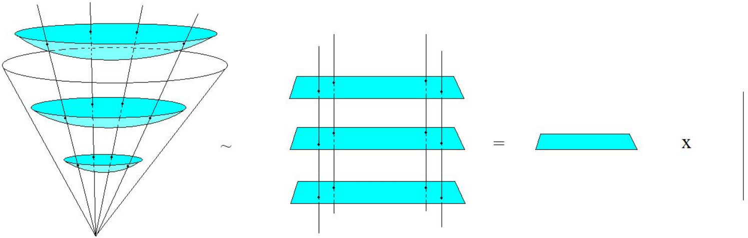
for homogeneous cones all three constructions coincide

Outline

Theorem (Tsuji 1982; Loftin 2002)

Let $K \subset \mathbb{R}^{n+1}$ be a regular convex cone, and $F : K^o \to \mathbb{R}$ a locally strongly convex logarithmically homogeneous function.
Then the Hessian metric on K^o splits into a direct product of a radial 1-dimensional part and a transversal n-dimensional part. The submanifolds corresponding to the radial part are rays, the submanifolds corresponding to the transversal part are level surfaces of F.

all nontrivial information contained in the transversal part



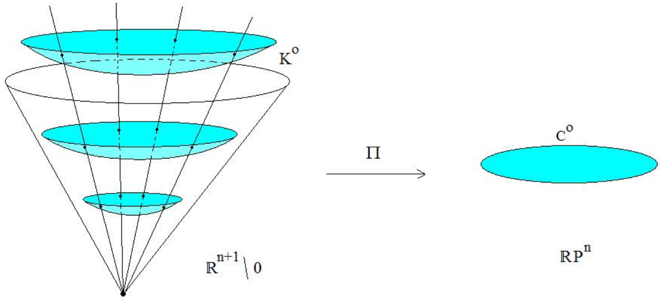
let $\mathbb{R}P^n, \mathbb{R}P_n$ be the primal and dual real projective space — lines and hyperplanes through the origin of $\mathbb{R}^{n+1}$

let $F : K^o \to \mathbb{R}$ be a barrier on a regular convex cone $K \subset \mathbb{R}^{n+1}$

the canonical projection $\Pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ maps $K \setminus \{0\}$ to a compact convex subset $C \subset \mathbb{R}P^n$

the canonical projection $\Pi^* : \mathbb{R}_{n+1} \setminus \{0\} \to \mathbb{R}P_n$ maps $K^* \setminus \{0\}$ to a compact convex subset $C^* \subset \mathbb{R}P_n$

the interiors of C, C^* are isomorphic to the transversal factors of $K^o, (K^*)^o$ and acquire the metric of these factors



passing to the projective space removes the radial factor

Theorem

Let $K \subset \mathbb{R}^{n+1}$, $n \geq 2$, a regular convex cone and $F : K^o \to \mathbb{R}$ a logarithmically homogeneous locally strongly convex function with homogeneity parameter ν. Then F is self-concordant if and only if

$$|F'''(x)[h,h,h]| \leq 2\frac{\gamma}{\sqrt{\nu}} \left(F''(x)[h,h]\right)^{3/2}$$

for all tangent vectors h which are parallel to the level surfaces of F. Here $\gamma = \frac{\nu-2}{\sqrt{\nu-1}}$.

this is a condition on the transversal factor only

Corollary

Let $K \subset \mathbb{R}^n$ be a regular convex cone, and $n \geq 3$. Let $F : K^o \to \mathbb{R}$ be a self-concordant barrier on K. Then F has parameter $\nu \geq 2$, with equality if and only if K is the Lorentz cone and F the canonical barrier on K.

between elements of $\mathbb{R}P^n, \mathbb{R}P_n$ there is no scalar product, but an orthogonality relation

the set

$$\mathcal{M} = \{(x,p) \in \mathbb{R}P^n \times \mathbb{R}P_n \,|\, x \not\perp p\}$$

is dense in $\mathbb{R}P^n \times \mathbb{R}P_n$

$$\partial\mathcal{M} = \{(x,p) \in \mathbb{R}P^n \times \mathbb{R}P_n \,|\, x \perp p\}$$

is a submanifold of $\mathbb{R}P^n \times \mathbb{R}P_n$ of codimension 1

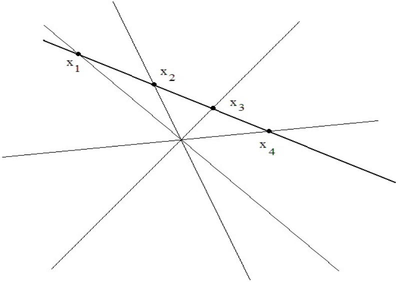
Theorem (Gadea, Montesinos Amilibia 1989)

The space $\mathcal{M}$ is a para-Kähler space form, it carries a natural para-Kähler structure with constant sectional curvature.

para-Kähler manifold:

- even dimension
- pseudo-metric of neutral signature
- symplectic structure satisfying $\nabla\omega = 0$
- para-complex structure J satisfying $g(X,Y) = \omega(JX,Y)$

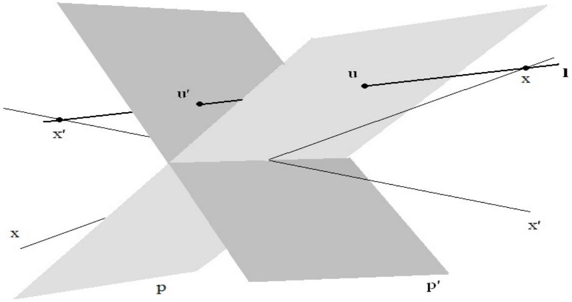
J is an involution of $T_x\mathcal{M}$ with the ± 1 eigenspaces forming n-dimensional integrable distributions



x_1, x_2, x_3, x_4 points on the projective line $\mathbb{R}P^1$

$$(x_1, x_2; x_3, x_4) = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_2 - x_3)(x_1 - x_4)}$$

[Ariyawansa, Davidon, McKennon 1999]: instead of 4 collinear points use 2 points and 2 dual points



$(u, x'; u', x)$ — quadra-bracket of x, p, x', p'

let $z = (x, p), z' = (x', p') \in \mathcal{M} \subset \mathbb{R}P^n \times \mathbb{R}P_n$

define a symmetric function $(\cdot;\cdot) : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ by

$$(z; z') = (z'; z) := (u, x'; u', x)$$

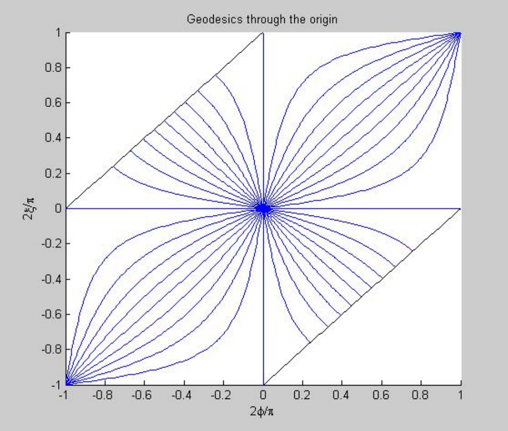
$(z; z')$ is the only projective invariant of a pair of points in $\mathcal{M}$
$\lim_{z \to \partial\mathcal{M}} (z; z') = \pm\infty$

the pseudo-metric g on $\mathcal{M}$ is such that

- If $(z; z') > 0$, then the velocity vector of the geodesic linking z, z' has positive square and $d(z, z') = \arcsin \sqrt{(z; z')}$.
- If $(z; z') = 0$ the geodesic linking z, z' is light-like.
- If $(z; z') < 0$, then the velocity vector of the geodesic linking z, z' has negative square and $d(z, z') = \text{arc}\sinh \sqrt{-(z; z')}$.

the tangent space to $\mathcal{M}$ at $z = (x, p) \in \mathcal{M}$ is a direct product of two subspaces which are parallel to the factors $\mathbb{R}P^n, \mathbb{R}P_n$: $h = (h_x, h_p)$

the para-complex structure $J : T\mathcal{M} \to T\mathcal{M}$ acts by $h = (h_x, h_p) \mapsto (h_x, -h_p)$



$\mathbb{R}P^1 \sim S^1$
$\mathbb{R}P^1 \times \mathbb{R}P_1 \sim T^2$

$\mathbb{R}P^1$ parameterized by ϕ
$\mathbb{R}P_1$ parameterized by ξ
$(\phi, \xi) \in [-\frac{\pi}{2}, \frac{\pi}{2})^2$

$\partial\mathcal{M} = \{(\phi, \xi) \,|\, \xi = \phi \pm \frac{\pi}{2}\}$

$\partial\mathcal{M} \sim S^1$
$\mathcal{M} \sim S^1 \times \mathbb{R}$

$g = \cos^{-2}(\phi - \xi) d\phi d\xi$

the canonical projection $\Pi \times \Pi^* : (\mathbb{R}^{n+1} \setminus \{0\}) \times (\mathbb{R}_{n+1} \setminus \{0\}) \to \mathbb{R}P^n \times \mathbb{R}P_n$ maps the set

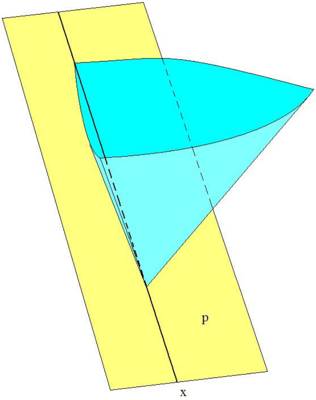
$$\Delta_K = \{(x,p) \in (\partial K \setminus \{0\}) \times (\partial K^* \setminus \{0\}) \,|\, x \perp p\}$$

to a set $\delta_K \subset \partial\mathcal{M}$

Lemma

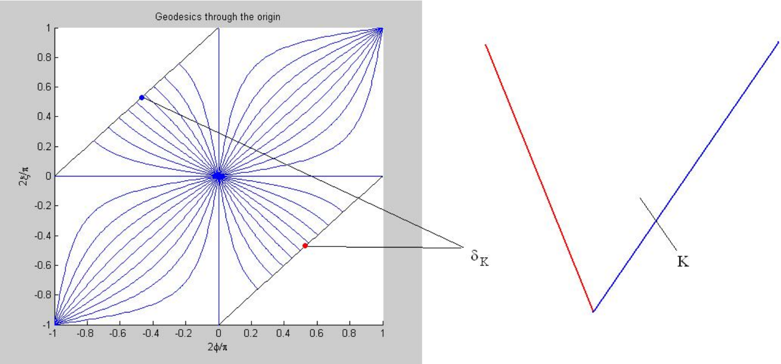
The projections π, π^* of $\mathbb{R}P^n \times \mathbb{R}P_n$ to the factors map δ_K onto ∂C and ∂C^*, respectively.
If K is smooth, then δ_K is homeomorphic to S^{n-1}.

call δ_K the boundary frame corresponding to the cone K



the boundary frame δ_K consists of pairs $z = (x, p) \in \partial\mathcal{M}$ where

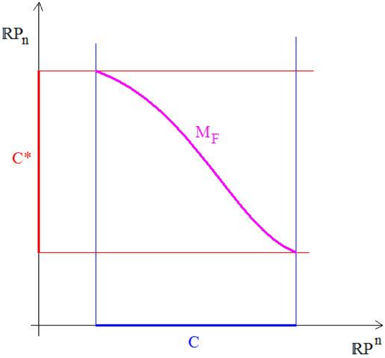
- the line x contains a ray in ∂K
- p is a supporting hyperplane at x



the boundary frame of a 2-dimensional cone consists of 2 points which can be linked by a (complete) geodesic with negative squared velocity

let $K \subset \mathbb{R}^{n+1}$ be a regular convex cone and $F : K^o \to \mathbb{R}$ a barrier on K
the bijection $x \mapsto -F'(x)$ factors through to an isometry between C^o and $(C^*)^o$

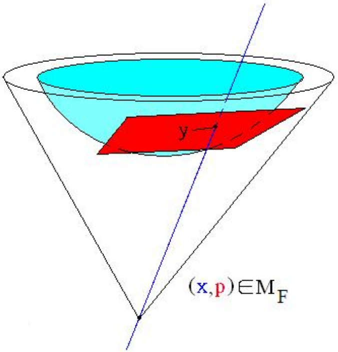
$$\begin{array}{ccc} K^o & \xrightarrow{-F'} & (K^*)^o \\ \Pi \downarrow & & \Pi^* \downarrow \\ C^o \sim K^o/\mathbb{R}_+ & \xrightarrow{\mathcal{I}_F} & (C^*)^o \sim (K^*)^o/\mathbb{R}_+ \end{array}$$



define the smooth submanifold M_F as the graph of the isometry $\mathcal{I}_F$

$$M_F = \Pi\times\Pi^* \left[\{(x, -F'(x)) \,|\, x \in K^o\}\right] \subset \mathcal{M}$$

$$\dim M_F = n = \frac{1}{2} \dim \mathcal{M}$$



the manifold M_F consists of pairs (x,p) where

- x is a line through a point $y \in K^o$
- p is parallel to the hyperplane which is tangent to the level surface of F at y

if $y \to \hat{y} \in \partial K$, then p tends to a supporting hyperplane at $\hat{y}$

let $M \subset \mathcal{M}$ be a submanifold of a (pseudo-)Riemannian space

choose a point $x \in M$ and a tangent vector $h \in T_x M$

consider the geodesics $\gamma_M, \gamma_{\mathcal{M}}$ in M and in $\mathcal{M}$ through x with velocity h

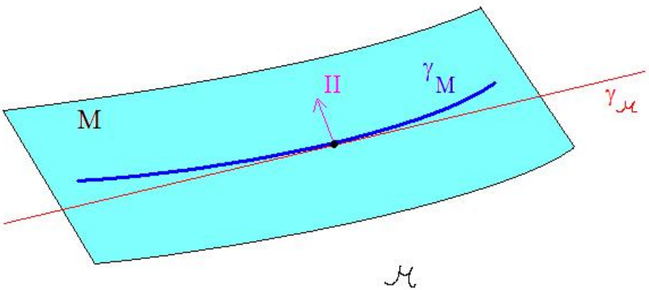
there is a second-order deviation

$$\gamma_M(t) - \gamma_{\mathcal{M}}(t) = \left(\left. \frac{d^2}{dt^2} \right|_{t=0} (\gamma_M - \gamma_{\mathcal{M}}) \right) \cdot \frac{t^2}{2} + O(t^3)$$

whose main term depends quadratically on h

the acceleration is called the second fundamental form II of M

$II_x : T_x M \times T_x M \to (T_x M)^{\perp}$



the second fundamental form measures the deviation of M from a geodesic submanifold

it is also called the extrinsic curvature

Theorem (H., 2011)

Let $F : K^o \to \mathbb{R}$ be a barrier on a regular convex cone $K \subset \mathbb{R}^{n+1}$ with parameter ν. The manifold $M_F \subset \mathbb{R}P^n \times \mathbb{R}P_n$ is

- *a nondegenerate Lagrangian submanifold of $\mathcal{M}$*
- *bounded by δ_K in $\mathbb{R}P^n \times \mathbb{R}P_n$*
- *its induced metric is $-\nu$ times the metric generated by $C^o, (C^*)^o$*
- *its second fundamental form II satisfies*

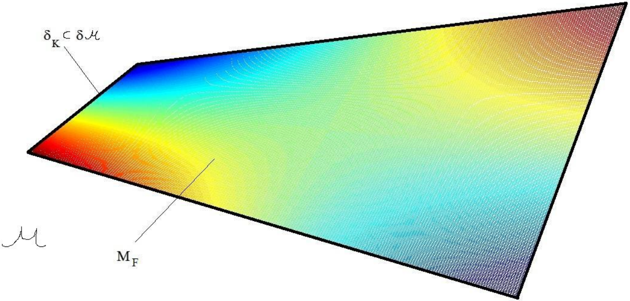
$$F'''[h,h,h'] = 2\omega(II(\tilde{h},\tilde{h}),\tilde{h}')$$

for all vectors h, h' tangent to the level surfaces of F and their images $\tilde{h}, \tilde{h}'$ on the tangent bundle TM_F.

Corollary

Let $K \subset \mathbb{R}^{n+1}$ be a regular convex cone and $F : K^o \to \mathbb{R}$ a locally strongly convex logarithmically homogeneous function with parameter ν.
Then F is self-concordant if and only if the Lagrangian submanifold $M_F \subset \mathcal{M}$ has its second fundamental form bounded by $\gamma = \frac{\nu-2}{\sqrt{\nu-1}}$.

the barrier parameter determines how close M_F is to a geodesic submanifold



- complete negative definite Lagrangian submanifold
- bounded by δ_K
- second fundamental form bounded by $\gamma = \frac{\nu-2}{\sqrt{\nu-1}}$

Definition

Let $\mathcal{M}$ be a pseudo-Riemannian manifold. Then $M \subset \mathcal{M}$ is a minimal submanifold if M is a stationary point of the volume functional with respect to variations with compact support.

a submanifold is minimal if and only if its *mean curvature* vanishes identically

Theorem (H., 2011)

Let $K \subset \mathbb{R}^n$ be a regular convex cone and $F : K^o \to \mathbb{R}$ be a barrier on K.
Then the submanifold $M_F \subset \mathcal{M}$ is minimal if and only if the level surfaces of F are affine hyperspheres.

the canonical barrier is given by the unique minimal complete negative definite Lagrangian submanifold which can be inscribed in the boundary frame $\delta_K \subset \partial\mathcal{M}$

for three-dimensional cones K the submanifolds M_F are two-dimensional
hence M_F is a complete non-compact simply connected Riemann surface

Uniformization theorem: Every simply connected Riemann surface is conformally equivalent to either the unit disc $\mathbb{D}$, or the complex plane $\mathbb{C}$, or the Riemann sphere S, equipped with either the hyperbolic metric, or the flat (parabolic) metric, or the spherical (elliptic) metric, respectively.

due to Klein, Riemann, Schwarz, Koebe, Poincaré, Hilbert, Weyl, Radó ... 1880–1920

- there exists an oriented atlas of charts on M_F such that $h = e^{2\phi}(dx_1^2 + dx_2^2)$
- each chart parameterized by one complex parameter $z = x_1 + ix_2$, $h = e^{2\phi}|dz|^2$
- transition maps holomorphic (conformal $+$ oriented $=$ holomorphic)
- global chart with values in $\mathbb{D}, \mathbb{C}$, or S exists and is unique up to automorphisms

- $\mathbb{D}$: $h = e^{2\tilde{\phi}} \frac{4|dz|^2}{(1-|z|^2)^2}$ with $\tilde{\phi}$ uniquely defined scalar field on M_F
- $\mathbb{C}$: $h = e^{2\tilde{\phi}}|dz|^2$ with $\tilde{\phi}$ scalar field defined up to additive constant
- non-compactness of M rules out elliptic case S

consider a conformal chart on M_F such that $h = e^{2\phi}(dx_1^2 + dx_2^2)$

the cubic form $C = \nu^{-1}F'''$ can be decomposed as

$$C = \left[\begin{pmatrix} \frac{3}{4}e^{2\phi}T_1 + U_1 & \frac{1}{4}e^{2\phi}T_2 - U_2 \\ \frac{1}{4}e^{2\phi}T_2 - U_2 & \frac{1}{4}e^{2\phi}T_1 - U_1 \end{pmatrix}, \begin{pmatrix} \frac{1}{4}e^{2\phi}T_2 - U_2 & \frac{1}{4}e^{2\phi}T_1 - U_1 \\ \frac{1}{4}e^{2\phi}T_1 - U_1 & \frac{3}{4}e^{2\phi}T_2 + U_2 \end{pmatrix}\right]$$

T is the Tchebycheff form and represents the trace part of C; define $E = \frac{1}{4}(T_1 - iT_2)$
$U = U_1 + iU_2$ is a cubic differential representing the trace-free part of C, $U(w) = U(z)(\frac{dz}{dw})^3$

compatibility requirements on ϕ, C [Liu, Wang 1997]:
the form T is closed with (real) potential t, then $E = \frac{1}{2}\frac{\partial t}{\partial z}$ and

$$\frac{\partial U}{\partial \bar{z}} = e^{4\phi}\frac{\partial}{\partial z}(e^{-2\phi}E),$$

$$|U|^2 = 2e^{6\phi} + e^{4\phi}|E|^2 - 8e^{4\phi}\frac{\partial^2\phi}{\partial z\partial\bar{z}}$$

here $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$

the barrier F is canonical if and only if the Tchebycheff form T vanishes: $E = 0$ and U holomorphic

Theorem (follows from (Simon, Wang 1993))

- *Let $K \subset \mathbb{R}^3$ be a regular convex cone. Then the canonical barrier on K defines a unique complete canonical Riemannian metric $h = e^{2\phi}|dz|^2$ on the Riemann surface M_F and an associated holomorphic cubic differential U satisfying the relation*

$$|U|^2 = 2e^{6\phi} - 2e^{4\phi}\Delta\phi = 2e^{6\phi}(1 + \mathbf{K}),$$

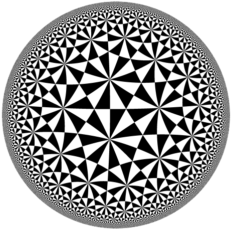
where Δ is the ordinary Laplacian and $\mathbf{K}$ the Gaussian curvature.

- *Every simply connected non-compact Riemann surface with complete metric $h = e^{2\phi}|dz|^2$ and holomorphic cubic differential U satisfying above relation defines a regular convex cone $K \subset \mathbb{R}^3$ with its canonical barrier.*

- level surfaces of F can be recovered from (h, U) by solving a Cauchy initial value problem of a PDE
- [Simon, Wang 1993] gives a necessary and sufficient integrability condition on ϕ
- for given ϕ, U is determined up to a constant factor $e^{i\varphi}$
- for given U, there exists at most one solution ϕ (maximum principle)
- symmetry group of K = symmetry group of (h, U) times homothety subgroup

[Dumas, Wolf 2015] polynomials U of degree k correspond to polyhedral cones K with $k+3$ extreme rays
$U = z^k$ corresponds to the cone over the regular $(k+3)$-gon
Riemann surface conformally equivalent to $\mathbb{C}$

[Wang 1997; Loftin 2001; Labourie 2007] holomorphic functions on compact Riemann surface of genus $g \geq 2$ form finite-dimensional space
each such function U determines a unique metric h on the surface and its universal cover
the corresponding cone K has an automorphism group with cocompact action on the level surfaces on F
∂K is C^1, but in general nowhere C^2
Riemann surface conformally equivalent to $\mathbb{D}$



[Benoist, Hulin 2014] the following are equivalent:

- $k = \sup_{M_F} \mathbf{K} < 0$
- M_F is Gromov hyperbolic (geodesic triangles have bounded width)
- $\mathbb{R}^3_+$ is not in the closure of the orbit of K under $SL(3,\mathbb{R})$
- M_F is conformally equivalent to $\mathbb{D}$ and U is bounded in the hyperbolic metric
- ∂K is C^1 and quasi-symmetric

recall: the ∞-norm of the cubic form

$$\gamma = \sup |C(x)[h,h,h]| \; : \qquad x \in M, \; h \in T_x M, \; ||h|| = 1$$

relates to the barrier parameter ν of F by

$$\gamma = \frac{2(\nu - 2)}{\sqrt{\nu - 1}}, \qquad \nu = \frac{\gamma^2 + 16 + \gamma\sqrt{\gamma^2 + 16}}{8}$$

Lemma (Simon, Wang 1993)

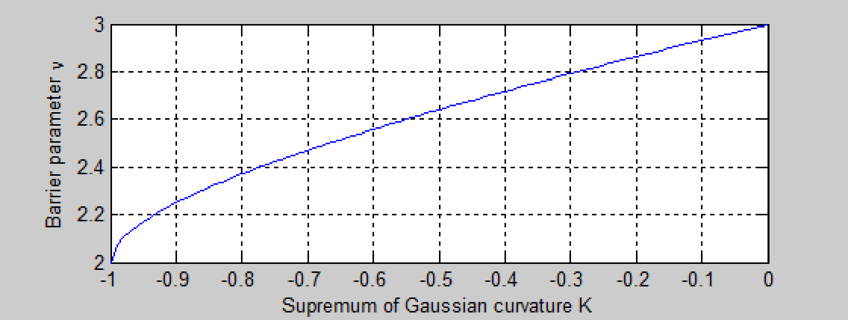
Let (h, U) be a compatible pair of a metric and a holomorphic cubic differential. Then $|U|^2 = 2(\mathbf{K} + 1)e^{6\phi}$, where $\mathbf{K}$ is the Gaussian curvature, $-1 \leq \mathbf{K} \leq 0$.

Corollary

Let $K \subset \mathbb{R}^3$ be a regular convex cone, F the canonical barrier on it, and (h, U) the metric and holomorphic cubic differential defined by F. Then

$$\nu = \frac{k + 9 + \sqrt{(k+1)(k+9)}}{4},$$

where $k = \sup_{M_F} \mathbf{K}$ is the supremum of the Gaussian curvature.



extreme cases:

- $\mathbf{K} \equiv 0$: flat metric, $K = \mathbb{R}^3_+$
- $\mathbf{K} \equiv -1$: hyperbolic metric, $K = L_3$

generalize to arbitrary dimension

Lemma (follows from (H., 2013; H., 2014))

Let $K \subset \mathbb{R}^n$ be a regular convex cone such that $\mathbb{R}^n_+$ is in the closure of the orbit of K under $SL(n,\mathbb{R})$. Then $\nu_{opt}(K) = n$.

Corollary

Let $K \subset \mathbb{R}^3$ be a regular convex cone. Then the following are equivalent:

- $\nu_{opt}(K) < 3$
- $\nu_{can}(K) < 3$
- $k = \sup_{M_F} \mathbf{K} < 0$
- M_F *is Gromov hyperbolic*
- $\mathbb{R}^3_+$ *is not in the closure of the orbit of* K *under* $SL(3,\mathbb{R})$
- M_F *is conformally equivalent to* $\mathbb{D}$ *and* U *is bounded in the hyperbolic metric*
- ∂K *is* C^1 *and quasi-symmetric*

There is a 1-to-1 correspondence between such cones and bounded holomorphic cubic differentials U on $\mathbb{D}$.

Open questions

Which cones allow barriers such that the corresponding Riemann surface is conformally equivalent to $\mathbb{C}$?

Which entire functions are cubic forms of an affine hypersphere?

Are there cones such that $\nu_{opt} < \nu_{can}$? (for $n > 3$ there are)

How to compute ν_{opt} or ν_{can} from K?

Hildebrand R. Canonical barriers on convex cones. *Math. Oper. Res.* **39**(3):841–850, 2014.

Thank you!