Weierstrass Institute for
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Leibniz Association

Periodic discrete dynamical systems and copositive matrices with circulant zero patterns

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Outline

1. **Copositive matrices**
   - Approximations and extreme rays
   - Zeros and zero patterns

2. **Periodic dynamical systems and extreme matrices**
   - Periodic systems
   - Zero sets with circulant supports

Definition

A real symmetric $n \times n$ matrix A such that $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n_+$ is called copositive.

the set of all such matrices is a regular convex cone, the copositive cone $\mathcal{C}_n$

related cones

- *completely positive* cone $\mathcal{C}_n^* = conv\{xx^T \,|\, x \geq 0\}$
- sum $\mathcal{N}_n + \mathcal{S}_n^+$ of nonnegative and positive semi-definite cone
- *doubly nonnegative* cone $\mathcal{N}_n \cap \mathcal{S}_n^+$

$$\mathcal{C}_n^* \subset \mathcal{N}_n \cap \mathcal{S}_n^+ \subset \mathcal{N}_n + \mathcal{S}_n^+ \subset \mathcal{C}_n$$

$\mathcal{N}_n + \mathcal{S}_n^+$ is an inner approximation of $\mathcal{C}_n$

Theorem (Murty, Kabadi 1987)

Checking whether an $n \times n$ integer matrix is not copositive is NP-complete.

Theorem (Burer 2009)

Any mixed binary-continuous optimization problem with linear constraints and (non-convex) quadratic objective function can be written as a copositive program

$$\min_{x \in \mathcal{C}_n} \langle c, x \rangle : \qquad Ax = b$$

the approximation $\mathcal{N}_n + \mathcal{S}_n^+$ is semi-definite representable

Theorem (Diananda 1962)

Let $n \leq 4$. Then the copositive cone $\mathcal{C}_n$ equals the sum of the nonnegative cone $\mathcal{N}_n$ and the positive semi-definite cone $\mathcal{S}_n^+$.

the Horn form (discovered by Alfred Horn)

$$H = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix}$$

is an example of a matrix in $\mathcal{C}_5 \setminus (\mathcal{N}_5 + \mathcal{S}_5^+)$

matrices in $\mathcal{C}_n \setminus (\mathcal{N}_n + \mathcal{S}_n^+)$ are called exceptional

Definition

Let $K \subset \mathbb{R}^n$ be a regular convex cone. An nonzero element $u \in K$ is called extreme if it cannot be decomposed into a non-trivial sum of linearly independent elements of K.

in [Hall, Newman 63] the extreme rays of $\mathcal{C}_n$ belonging to $\mathcal{N}_n + \mathcal{S}_n^+$ have been described:

- the extreme rays of $\mathcal{N}_n$, generated by E_{ii} and $E_{ij} + E_{ji}$
- rank 1 matrices $A = xx^T$ with x having both positive and negative elements

in [Hoffman, Pereira 1973] the extreme elements of $\mathcal{C}_n$ with entries in $\{-1, 0, +1\}$ have been described

every feasible copositive program has an extremal solution

exceptional extreme rays of particular importance: their knowledge allows to check whether inner approximations of $\mathcal{C}_n$ are exact

Theorem (H. 2012)

The extreme elements $A \in \mathcal{C}_5 \setminus (\mathcal{N}_5 + \mathcal{S}_5^+)$ of $\mathcal{C}_5$ are exactly the matrices $DPMP^TD$, where D is a diagonal positive definite matrix, P is a permutation matrix, and M is either the Horn form H or is given by a matrix

$$T = \begin{pmatrix} 1 & -\cos\psi_4 & \cos(\psi_4+\psi_5) & \cos(\psi_2+\psi_3) & -\cos\psi_3 \\ -\cos\psi_4 & 1 & -\cos\psi_5 & \cos(\psi_5+\psi_1) & \cos(\psi_3+\psi_4) \\ \cos(\psi_4+\psi_5) & -\cos\psi_5 & 1 & -\cos\psi_1 & \cos(\psi_1+\psi_2) \\ \cos(\psi_2+\psi_3) & \cos(\psi_5+\psi_1) & -\cos\psi_1 & 1 & -\cos\psi_2 \\ -\cos\psi_3 & \cos(\psi_3+\psi_4) & \cos(\psi_1+\psi_2) & -\cos\psi_2 & 1 \end{pmatrix},$$

where $\psi_k > 0$ for $k = 1, \dots, 5$ and $\sum_{k=1}^5 \psi_k < \pi$.

- the set of matrices $DPHP^TD$ has codimension 10
- the set of matrices $DPTP^TD$ has codimension 5

let $A \in \mathcal{C}_n$ be a copositive matrix

- a non-zero vector $x \geq 0$ is called a zero of A if $x^T Ax = 0$
- the set $\operatorname{supp} x = \{i \,|\, x_i > 0\}$ is called the support of x
- the set $\mathcal{V}_A = \{\operatorname{supp} x \,|\, x \text{ is a zero of } A\}$ is called the zero pattern of A

Example: Horn form

$$A = H = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix}: \qquad x = \begin{pmatrix} a \\ a+b \\ b \\ 0 \\ 0 \end{pmatrix}, \quad \begin{matrix} a, b \geq 0, \\ a + b > 0 \end{matrix}$$

and cyclically permuted vectors

$\mathcal{V}_H$ consists of $\{1,2\}, \{1,2,3\}$ and cyclically permuted sets

$$T = \begin{pmatrix} 1 & -\cos\psi_4 & \cos(\psi_4+\psi_5) & \cos(\psi_2+\psi_3) & -\cos\psi_3 \\ -\cos\psi_4 & 1 & -\cos\psi_5 & \cos(\psi_5+\psi_1) & \cos(\psi_3+\psi_4) \\ \cos(\psi_4+\psi_5) & -\cos\psi_5 & 1 & -\cos\psi_1 & \cos(\psi_1+\psi_2) \\ \cos(\psi_2+\psi_3) & \cos(\psi_5+\psi_1) & -\cos\psi_1 & 1 & -\cos\psi_2 \\ -\cos\psi_3 & \cos(\psi_3+\psi_4) & \cos(\psi_1+\psi_2) & -\cos\psi_2 & 1 \end{pmatrix}$$

has zeros given by the columns of the matrix

$$\begin{pmatrix} \sin\psi_5 & 0 & 0 & \sin\psi_2 & \sin(\psi_3+\psi_4) \\ \sin(\psi_4+\psi_5) & \sin\psi_1 & 0 & 0 & \sin\psi_3 \\ \sin\psi_4 & \sin(\psi_1+\psi_5) & \sin\psi_2 & 0 & 0 \\ 0 & \sin\psi_5 & \sin(\psi_1+\psi_2) & \sin\psi_3 & 0 \\ 0 & 0 & \sin\psi_1 & \sin(\psi_2+\psi_3) & \sin\psi_4 \end{pmatrix}$$

and homothetic images

the zero pattern is $\mathcal{V}_T = \{\{1,2,3\},\{2,3,4\},\{3,4,5\},\{4,5,1\},\{5,1,2\}\}$

let $A \in \mathcal{C}_n$ be a copositive matrix

- a zero u of a A is called minimal if there exists no zero v of A such that the inclusion $\operatorname{supp} v \subset \operatorname{supp} u$ holds strictly
- the set $\mathcal{V}_{\min}(A) = \{\operatorname{supp} x \mid x \text{ is a minimal zero of } A\}$ is called the minimal zero pattern of A

every zero of A is a convex combination of minimal zeros

Lemma (H. 2014)

Let A be a copositive matrix, and let u, v be minimal zeros of A with $\operatorname{supp} u = \operatorname{supp} v$. Then u, v differ by a positive multiplicative factor.
In particular, the number of minimal zeros of A is finite up to homothety.

- Horn form: $\mathcal{V}_{\min}(H) = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,5\}\}$
- T-matrices: $\mathcal{V}_{\min}(T) = \{\{1,2,3\},\{2,3,4\},\{3,4,5\},\{4,5,1\},\{5,1,2\}\}$

Generalization to higher dimensions?

scalar discrete-time time-variant dynamical system

$$x_{t+d} + \sum_{i=0}^{d-1} c_{t,i} x_{t+i} = 0, \qquad t \geq 1$$

coefficients n-periodic, $c_{t+n,i} = c_{t,i}$

- solution space $\mathcal{L}$ is d-dimensional, $n > d$
- $\mathcal{L}$ can be parameterized by initial values $x_1, \dots, x_d$
- if $c_{t,0} \neq 0$ for all t, then the system is time-reversible
- system may or may not have n-periodic solutions

let $\mathcal{L}_{per}$ be the subspace of n-periodic solutions let $x = (x_t)_{t\geq 1}$ be a solution

then $y = (x_{t+n})_{t\geq 1}$ is also a solution

Definition

The linear map $\mathfrak{M} : \mathcal{L} \to \mathcal{L}$ taking x to y is called the monodromy of the periodic system.
Its eigenvalues are called Floquet multipliers.

- x is periodic if and only if it is an eigenvector of $\mathfrak{M}$ with eigenvalue 1
- $\det \mathfrak{M} = (-1)^{nd} \prod_{t=1}^{n} c_{t,0}$

Evaluation functionals

let $x = (x_t)_{t \geq 1}$ be a solution

for every t, define a linear map $\mathbf{e}_t$ by $\mathbf{e}_t(x) = x_t$

- $\mathbf{e}_t$ belongs to the dual space $\mathcal{L}^*$
- $\mathbf{e}_{t+n} = \mathfrak{M}^* \mathbf{e}_t$
- $\mathbf{e}_1, \ldots, \mathbf{e}_d$ span $\mathcal{L}^*$

$\mathbf{e}_t$ evolves according to

$$\mathbf{e}_{t+d} + \sum_{i=0}^{d-1} c_{t,i} \mathbf{e}_{t+i} = 0$$

- a linear form on $\mathcal{L}^*$ is a solution $x \in \mathcal{L}$
- a bilinear form on $\mathcal{L}^*$ is a linear combination of tensor products $x \otimes y$, $x, y \in \mathcal{L}$
- a symmetric bilinear form on $\mathcal{L}$ is a linear combination of $x \otimes x$, $x \in \mathcal{L}$

Definition

A symmetric bilinear form B on $\mathcal{L}^*$ is called shift-invariant if

$$B(\mathbf{e}_{t+n}, \mathbf{e}_{s+n}) = B(\mathbf{e}_t, \mathbf{e}_s) \qquad \forall\, t, s \geq 1$$

- B is shift-invariant if and only if $B(w, w') = B(\mathfrak{M}^* w, \mathfrak{M}^* w')$ for all $w, w' \in \mathcal{L}^*$
- $B = x \otimes x$ for x periodic are shift-invariant

let $n \geq 5$ and let $\mathbf{u} = \{u^1, \dots, u^n\} \subset \mathbb{R}^n_+$ with

$$\begin{aligned} \operatorname{supp} u^1 &= \{1, 2, \dots, n-2\} =: I_1 \\ \operatorname{supp} u^2 &= \{2, 3, \dots, n-1\} =: I_2 \\ &\vdots \\ \operatorname{supp} u^n &= \{n, 1, \dots, n-3\} =: I_n \end{aligned}$$

Problem: Characterize copositive matrices with zeros $u^1, \dots, u^n$.

Definition

Let $A \in \mathcal{C}_n$ be exceptional, and let $u^1, \dots, u^n$ be among its zeros.
We call A regular if every zero of A is proportional to one of the zeros u^j.
We call A degenerate if there are zeros of A which are not proportional to one of the zeros u^j.

Correspondence between systems and matrices : overview

- let $F_{\mathbf{u}}$ be the face of $\mathcal{C}_n$ of matrices having $u^1, \dots, u^n$ among their zeros
- let $P_{\mathbf{u}}$ be the sub-face of positive semi-definite matrices

Matrices	Systems
zero subset $\mathbf{u}$	periodic dynamical system $\mathbf{S}_{\mathbf{u}}$
copositive matrices $A \in F_{\mathbf{u}}$	bilinear symmetric forms $B \in \mathcal{F}_{\mathbf{u}}$ satisfying a certain LMI
entry A_{ij}	value $B(\mathbf{e}_i, \mathbf{e}_j)$ on evaluation functionals
subset $P_{\mathbf{u}}$ of positive semi-definite matrices	convex hull $\mathcal{P}_{\mathbf{u}}$ of $\{x \otimes x \mid x \in \mathcal{L}_{per}\}$
regular matrices	positive definite forms
degenerate matrices	corank 1 positive semi-definite forms

- Horn form H is the prototype of the degenerate matrices
- T-matrices are the prototype of the regular matrices

to a collection $\mathbf{u}$ of nonnegative vectors $u^1, \dots, u^n$ with $\operatorname{supp} u^k = I_k$ associate the n-periodic dynamical system $\mathbf{S_u}$ given by

$$\sum_{i=0}^{d} c_{t,i} x_{t+i} = 0$$

with $c_t = (u^t)_{I_t}, t = 1, \dots, n$

- order $d = n - 3$
- system is time-reversible
- all coefficients are positive
- $\det \mathfrak{M} = \prod_{j=1}^{n} u_j^j / u_{j+d}^j > 0$

let $\mathcal{A}_{\mathbf{u}} \subset \mathcal{S}_n$ be the linear subspace of symmetric $n \times n$ matrices A satisfying $(Au^k)_{I_k} = 0$

to every $A \in \mathcal{A}_{\mathbf{u}}$ associate a symmetric bilinear form B on the dual solution space $\mathcal{L}^*$ by

$$B(\mathbf{e}_t, \mathbf{e}_s) = A_{ts}, \qquad t, s = 1, \dots, d$$

let $\Lambda : A \mapsto B$ be the corresponding linear map

Λ maps quadratic forms on $\mathbb{R}^n$ to quadratic forms on $\mathbb{R}^d$

Lemma

The linear map Λ is injective and its image consists of those shift-invariant symmetric bilinear forms B which satisfy

$$B(\mathbf{e}_t, \mathbf{e}_s) = B(\mathbf{e}_{t+n}, \mathbf{e}_s) \qquad \forall\, t, s \geq 1 : \; 3 \leq s - t \leq n - 3$$

- the image of Λ may be $\{0\}$
- effectively finite number of linear conditions

Theorem

Let $\mathcal{F}_{\mathbf{u}}$ be the set of positive semi-definite shift-invariant symmetric bilinear forms B on $\mathcal{L}_{\mathbf{u}}^*$ satisfying the linear equality relations

$$B(\mathbf{e}_t, \mathbf{e}_s) = B(\mathbf{e}_{t+n}, \mathbf{e}_s), \qquad 1 \leq t < s \leq n : \ 3 \leq s - t \leq n - 3$$

and the linear inequalities

$$B(\mathbf{e}_t, \mathbf{e}_{t+2}) \geq B(\mathbf{e}_{t+n}, \mathbf{e}_{t+2}), \qquad t = 1, \dots, n.$$

Then the face of $\mathcal{C}_n$ defined by the zeros u^j, $j = 1, \dots, n$, is given by $F_{\mathbf{u}} = \Lambda^{-1}[\mathcal{F}_{\mathbf{u}}]$.

Let $\mathcal{P}_{\mathbf{u}}$ be the convex hull of the tensor products $x \otimes x$, $x \in \mathcal{L}_{per}$. Then $\mathcal{P}_{\mathbf{u}} \subset \mathcal{F}_{\mathbf{u}}$, and $P_{\mathbf{u}} = \Lambda^{-1}[\mathcal{P}_{\mathbf{u}}]$.

Corollary

Given a vector set $\mathbf{u} = \{u^1, \dots, u^n\} \subset \mathbb{R}_+^n$, the face $F_{\mathbf{u}}$ of the copositive cone $\mathcal{C}_n$ which consists of matrices having $u^1, \dots, u^n$ as zeros is semi-definite representable.

$n = 5$, $d = 2$, $\mathbf{u}$ given by columns of

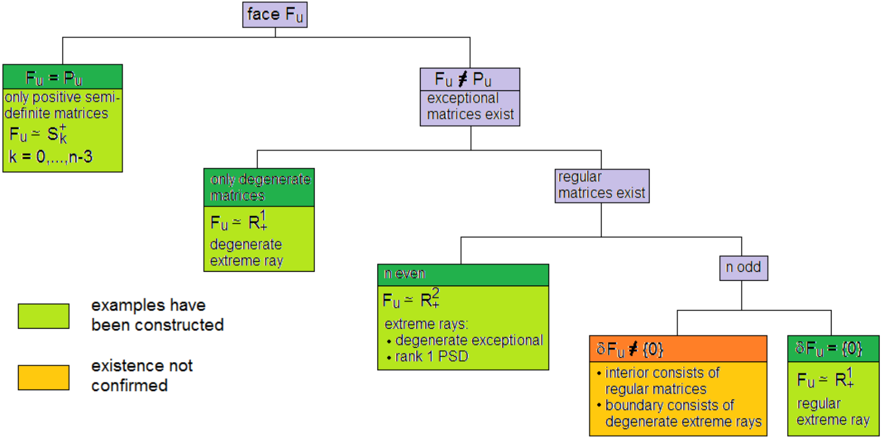
$$\begin{pmatrix} \sin\psi_5 & 0 & 0 & \sin\psi_2 & \sin(\psi_3+\psi_4) \\ \sin(\psi_4+\psi_5) & \sin\psi_1 & 0 & 0 & \sin\psi_3 \\ \sin\psi_4 & \sin(\psi_1+\psi_5) & \sin\psi_2 & 0 & 0 \\ 0 & \sin\psi_5 & \sin(\psi_1+\psi_2) & \sin\psi_3 & 0 \\ 0 & 0 & \sin\psi_1 & \sin(\psi_2+\psi_3) & \sin\psi_4 \end{pmatrix}$$

linearly independent solutions of the associated dynamical system are given by

$$x^1 = (1, -\cos\psi_4, \cos(\psi_4+\psi_5), -\cos(\psi_4+\psi_5+\psi_1), \cos(\psi_4+\psi_5+\psi_1+\psi_2), \dots)$$
$$x^2 = (0, \sin\psi_4, -\sin(\psi_4+\psi_5), \sin(\psi_4+\psi_5+\psi_1), -\sin(\psi_4+\psi_5+\psi_1+\psi_2), \dots)$$

- the T-matrix corresponds to positive definite bilinear form $B = \Lambda(T) = x^1 \otimes x^1 + x^2 \otimes x^2$
- the monodromy $\mathfrak{M}$ is a rotation by $\pi - \sum_{j=1}^5 \psi_j$
- $\mathcal{F}_{\mathbf{u}}$ is the conic hull of B
- $\mathcal{P}_{\mathbf{u}} = \{0\}$

Classification of faces



Theorem

Let $n \geq 5$. Let $K \subset \mathbb{R}^3$ be a regular polyhedral cone with n extreme rays. Let $U \in \mathbb{R}^{n\times n}$ be a slack matrix of K, such that the j-th column u^j of U has support I_j. Set $\mathbf{u} = \{u^1, \dots, u^n\}$.

Then $F_{\mathbf{u}} = P_{\mathbf{u}} \simeq \mathcal{S}^+_{n-3}$.

For every collection $\mathbf{u} = \{u^1, \dots, u^n\}$ such that $\operatorname{supp} u^j = I_j$ and $F_{\mathbf{u}} = P_{\mathbf{u}} \simeq \mathcal{S}^+_{n-3}$, the zeros u^j are the columns of a slack matrix of a regular polyhedral cone $K \subset \mathbb{R}^3$ with n extreme rays.

faces $F_{\mathbf{u}} = P_{\mathbf{u}} \simeq \mathcal{S}^+_k$ with $k < n - 3$ can be constructed by perturbing some of the zeros u^j

Theorem

Let $A \in F_{\mathbf{u}}$ be an exceptional copositive matrix and set $B = \Lambda(A)$. Then the following are equivalent:

- *A is regular;*
- *the minimal zero pattern of A is $\{I_1, \dots, I_n\}$, with minimal zeros $u^1, \dots, u^n$;*
- *B is positive definite;*
- *the corank of the submatrices A_{I_j} equals 1, $j = 1, \dots, n$.*

For even n the matrix A is the sum of a degenerate exceptional copositive matrix and a rank 1 positive semi-definite matrix.
If n is odd and the monodromy operator $\mathfrak{M}$ has no eigenvalue equal to -1, then A is extremal.

The matrix A is embedded in a submanifold of codimension n, consisting of regular exceptional matrices. If A is extremal, then the matrices in the submanifold are also extremal.

no example of a non-extremal regular matrix for odd n found so far

Theorem

Let $A \in F_{\mathbf{u}}$ be an exceptional copositive matrix and set $B = \Lambda(A)$. Then the following are equivalent:

- *A is degenerate;*
- *the corank of B equals 1;*
- *the corank of the submatrices A_{I_j} equals 2, $j = 1, \dots, n$;*
- *the support of any minimal zero of A is a strict subset of one of the index sets $I_1, \dots, I_n$, and every index set I_j has exactly two subsets which are supports of minimal zeros of A;*
- *every non-minimal zero of A has support equal to I_j for some $j = 1, \dots, n$ and is a sum of two minimal zeros.*

In addition, A is extremal.

The matrix A is embedded in a submanifold of codimension $2n$, consisting of degenerate extremal exceptional matrices.

in all examples, there are exactly n minimal zeros (up to multiplication by a positive scalar) with supports $I_j \cap I_{j+1}, j = 1, \dots, n-1$, and $I_1 \cap I_n$

let $n \geq 5$, then $A \in \mathcal{S}_n$ given by

$$A_{ij} = \begin{cases} 2(1 + 2\cos\frac{\pi}{n}\cos\frac{3\pi}{n}), & i = j, \\ -2(\cos\frac{\pi}{n} + \cos\frac{3\pi}{n}), & |i-j| \in \{1, n-1\}, \\ 1, & |i-j| \in \{2, n-2\}, \\ 0, & |i-j| \in \{3, \dots, n-3\}, \end{cases}$$

is degenerate extremal

let $n \geq 5$ be odd, then $A \in \mathcal{S}_n$ given by

$$A_{ij} = \begin{cases} 2(1 + 2\cos\frac{\pi}{n+1}\cos\frac{3\pi}{n+1}), & i = j, \\ -2(\cos\frac{\pi}{n+1} + \cos\frac{3\pi}{n+1}), & |i-j| \in \{1, n-1\}, \\ 1, & |i-j| \in \{2, n-2\}, \\ 0, & |i-j| \in \{3, \dots, n-3\}, \end{cases}$$

is regular extremal

every degenerate exceptional matrix can be scaled to a matrix of the form

$$\begin{pmatrix} 1 & -\cos\varphi_1 & \cos(\varphi_1+\varphi_2) & -\cos(\varphi_1+\varphi_2+\varphi_3) & \cos(\varphi_2+\varphi_3) & -\cos\varphi_3 \\ -\cos\varphi_1 & 1 & -\cos\varphi_2 & \cos(\varphi_2+\varphi_3) & -\cos(\varphi_1+\varphi_2+\varphi_3) & \cos(\varphi_1+\varphi_3) \\ \cos(\varphi_1+\varphi_2) & -\cos\varphi_2 & 1 & -\cos\varphi_3 & \cos(\varphi_1+\varphi_3) & -\cos(\varphi_1+\varphi_2+\varphi_3) \\ -\cos(\varphi_1+\varphi_2+\varphi_3) & \cos(\varphi_2+\varphi_3) & -\cos\varphi_3 & 1 & -\cos\varphi_1 & \cos(\varphi_1+\varphi_2) \\ \cos(\varphi_2+\varphi_3) & -\cos(\varphi_1+\varphi_2+\varphi_3) & \cos(\varphi_1+\varphi_3) & -\cos\varphi_1 & 1 & -\cos\varphi_2 \\ -\cos\varphi_3 & \cos(\varphi_1+\varphi_3) & -\cos(\varphi_1+\varphi_2+\varphi_3) & \cos(\varphi_1+\varphi_2) & -\cos\varphi_2 & 1 \end{pmatrix}$$

with $\varphi_1, \varphi_2, \varphi_3 > 0$, $\varphi_1 + \varphi_2 + \varphi_3 < \pi$

the minimal zero pattern is $\{\{1,2,3\},\{2,3,4\},\dots,\{6,1,2\}\}$ with zeros being the columns of

$$\begin{pmatrix} \sin\varphi_2 & 0 & 0 & 0 & \sin\varphi_2 & \sin(\varphi_1+\varphi_3) \\ \sin(\varphi_1+\varphi_2) & \sin\varphi_3 & 0 & 0 & 0 & \sin\varphi_3 \\ \sin\varphi_1 & \sin(\varphi_2+\varphi_3) & \sin\varphi_1 & 0 & 0 & 0 \\ 0 & \sin\varphi_2 & \sin(\varphi_1+\varphi_3) & \sin\varphi_2 & 0 & 0 \\ 0 & 0 & \sin\varphi_3 & \sin(\varphi_1+\varphi_2) & \sin\varphi_3 & 0 \\ 0 & 0 & 0 & \sin\varphi_1 & \sin(\varphi_2+\varphi_3) & \sin\varphi_1 \end{pmatrix}$$

preprint: "Copositive matrices with circulant zero pattern", arXiv 1603.05111

Thank you!