Linear group representations in the service of conic optimization

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Outline

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1. Conic optimization
   - Conic programs
   - Programs over symmetric cones
   - Robust conic programs
2. Symmetries and semi-definite representations
   - Semi-definite representations
   - Semi-definite approximations
   - Automorphism groups
   - Applications of automorphisms
3. Lorentz-positive maps

Optimization problems

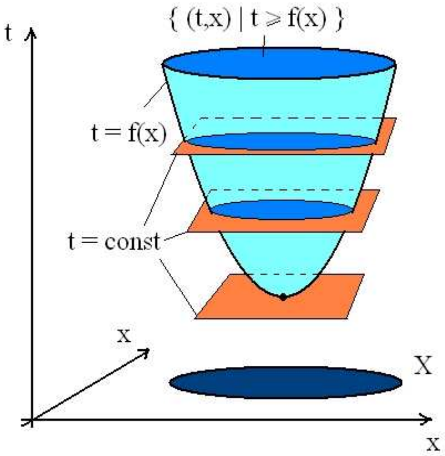
minimize objective function with respect to constraints

$$\min_{x \in X} f(x)$$

in convex optimization problems, f and X are assumed convex

$X \subset \mathbb{R}^n$ is called the feasible set

Linear objective function



$f(x)$ can be assumed linear

otherwise minimize t over the epigraph

Regular convex cones

Definition

A regular convex cone $K \subset \mathbb{R}^n$ is a closed convex cone having nonempty interior and containing no lines.

The dual cone

$$K^* = \{y \in \mathbb{R}_n \mid \langle x, y \rangle \geq 0 \quad \forall x \in K\}$$

of a regular convex cone K is also regular.

$$K_1 \subset K_2 \quad \Leftrightarrow \quad K_1^* \supset K_2^*$$

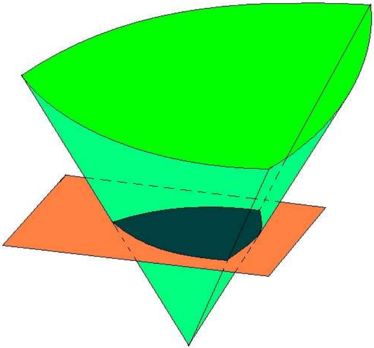
Conic programs

Definition

A conic program over a regular convex cone $K \subset \mathbb{R}^n$ is an optimization problem of the form

$$\min_{x \in K} \langle c, x \rangle : \quad Ax = b.$$

Geometric interpretation



the feasible set is the intersection of K with an affine subspace

$$\min_x \langle c', x \rangle : \; A'x + b' \in K$$

explicit parametrization

Projections

Lemma

Let $K \subset \mathbb{R}^n$, $K' \subset \mathbb{R}^{n'}$ be regular convex cones, $n' \geq n$, $\Pi : \mathbb{R}^{n'} \to \mathbb{R}^n$ a linear map such that $\Pi[K'] = K$. Then the conic program

$$\min_{x \in K} \langle c, x \rangle : \quad Ax = b$$

is equivalent to the conic program

$$\min_{y \in K'} \langle c, \Pi(y) \rangle : \quad A\Pi(y) = b.$$

Sections

Lemma

Let $K \subset \mathbb{R}^n$, $K' \subset \mathbb{R}^{n'}$ be regular convex cones, $n' \geq n$, $\mathcal{I} : \mathbb{R}^n \to \mathbb{R}^{n'}$ an injective linear map such that $\mathcal{I}^{-1}[K'] = K$. Then the conic program

$$\min_x \langle c', x \rangle : \quad A'x + b' \in K$$

is equivalent to the conic program

$$\min_x \langle c', x \rangle : \quad \mathcal{I}(A'x + b') \in K'.$$

Projections of sections

If we are able to solve conic programs over a cone K, then we are also able to solve conic programs over linear projections of linear sections of K.

Duality

primal program

$$\min_{x \in K} \langle c, x \rangle : \; Ax = b$$

dual program

$$\max_{s \in K^*} \langle c', s \rangle : \quad A's = b'$$

Complexity of conic programs

complexity depends on the complexity of the cone K

Example: copositive cone

$$\mathcal{C}_n = \{A \in \mathcal{S}(n) \,|\, x^T A x \geq 0 \quad \forall\, x \in \mathbb{R}^n_+\}$$

Theorem (Murty, Kabadi, 1987)

Deciding membership in the copositive cone is co-NP-complete.

Linear programs

example: conic programs over $K = \mathbb{R}^n_+$

feasible set is a convex polyhedron $\to$ **linear program** (LP)

- efficient solution algorithms since the 50s (simplex method)
- widely used in operations research, micro- and macroeconomics
- limited descriptive power

What is the "correct" generalization of LPs?

Symmetric cones

$\mathbb{R}^n_+$ is self-dual: $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$

and homogeneous: $\mathrm{Aut}(\mathbb{R}^n_+)$ acts transitively on $\mathbb{R}^n_{++}$

Definition

A self-dual, homogeneous convex cone is called symmetric.

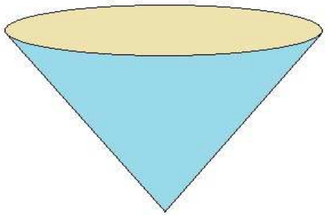
Classification of symmetric cones

Theorem (Vinberg, 1960; Koecher, 1962)

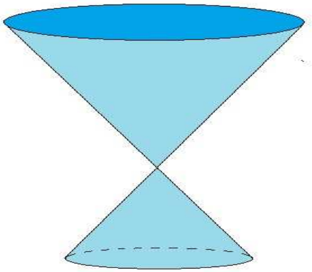
Every symmetric cone can be represented as a direct product of a finite number of the following irreducible symmetric cones:

- *Lorentz (or second order) cone* $L_n = \left\{ (x_0, \dots, x_{n-1}) \,|\, x_0 \geq \sqrt{x_1^2 + \dots + x_{n-1}^2} \right\}$
- *matrix cones* $S_+(n)$, $H_+(n)$, $Q_+(n)$ *of real, complex, or quaternionic hermitian positive semi-definite matrices*
- *Albert cone* $O_+(3)$ *of octonionic hermitian positive semi-definite* 3×3 *matrices*

Lorentz cones



spheric section $x_0 = const$



∂L_n contained in zero set of $J = \text{diag}(1, -1, \ldots, -1)$

Programs over symmetric cones

conic programs over symmetric cones are efficiently solvable by interior-point methods [Nesterov, Nemirovski, 1994]

- linear programs (LP) over $\mathbb{R}^n_+$ $\sim 10^6$ variables
- conic quadratic programs (CQP) over L_n $\sim 10^4$ variables
- semi-definite programs (SDP) over $S_+(n)$ $\sim 10^2$ variables

structure can greatly increase tractable sizes

free (CLP, LiPS, SDPT3, SeDuMi, ...) and commercial (CPLEX, MOSEK, ...) solvers available

increasingly used in engineering sciences and industry

Conic programs with uncertain data

conic program in explicit parametrization

$$\min_x \langle c, x \rangle : Ax + b \in K$$

suppose used data (A, b) are noisy and deviate from real data $A' = A + \delta A$, $b' = b + \delta b$

then actual constraint $A'x + b' \in K$ might be violated by the nominal optimal solution $x^*(A, b)$

Robust counterpart

assume data (A, b) is in a convex uncertainty region U

Definition (Nemirovski, 2007)

The *robust counterpart* (RC) of the conic program

$$\min_x \langle c, x\rangle : Ax + b \in K$$

is the optimization problem

$$\min_x \langle c, x\rangle : \quad Ax + b \in K \quad \forall\, (A, b) \in U.$$

"cost of robustness" is usually negligible

Robust counterpart as conic program

how to describe the feasible set

$$x: \qquad Ax + b \in K \quad \forall\, (A,b) \in U \quad ? \qquad\qquad (*)$$

$U \to$ homogenization $K_U = \cup_{\alpha \geq 0}\, \alpha U$

define linear map $\mathcal{A}_x : \ (A,b) \mapsto Ax + b$

$(*)$ becomes

$$x: \qquad \mathcal{A}_x(u) \in K \quad \forall\, u = (A,b) \in K_U$$

$\Leftrightarrow$ $\mathcal{A}_x$ maps K_U into K

$\Leftrightarrow$ feasible set is intersection of an affine subspace with the cone of all linear maps taking K_U to K

Robust counterpart as conic program

how to describe the feasible set

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Cones of positive linear maps

Definition

Let $K_1 \subset \mathbb{R}^{n_1}$, $K_2 \subset \mathbb{R}^{n_2}$ be regular convex cones. Call a linear map $A : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ *K_1-to-K_2 positive* if $A[K_1] \subset K_2$.

The cone $\mathcal{P}(K_1, K_2) \subset \mathbb{R}^{n_1 n_2}$ of K_1-to-K_2 positive maps is itself a regular convex cone.

[Barker, Loewy, 1975]
[Loewy, Schneider 1975]
[Horne, 1978]
[Tam, 1981, 1990, 1992, 1995]

Separable cones

$$\begin{aligned} A[K_1] \subset K_2 &\Leftrightarrow Ax \in K_2 \qquad \forall\, x \in K_1 \\ &\Leftrightarrow y^T A x = x^T A^T y \geq 0 \quad \forall\, x \in K_1,\ y \in K_2^* \\ &\Leftrightarrow \langle A, xy^T \rangle \geq 0 \qquad \forall\, x \in K_1,\ y \in K_2^* \end{aligned}$$

Theorem (Tam, 1977)

The cone $\mathcal{P}(K_1, K_2)$ is isomorphic to the cone $\mathcal{P}(K_2^*, K_1^*)$. The dual of $\mathcal{P}(K_1, K_2)$ is isomorphic to the cone $K_2^* \otimes K_1$ given by the convex hull of the set $\{x \otimes y \,|\, x \in K_2^*,\ y \in K_1\}$.

cones of the form $K \otimes K'$ will be called separable

Complexity of robust counterpart

solvability of robust counterpart depends on availability of a tractable description for the cone $\mathcal{P}(K_U, K)$ (or $K^* \otimes K_U$)

ellipsoidal uncertainty $\Rightarrow$ K_U is a Lorentz cone

- $K = \mathbb{R}^n_+$: RC is a CQP
- $K = L_n$: RC is a SDP
- $K = S_+(n)$: RC is a SDP for $n \leq 3$ [H., 2007], NP-hard for general n [Nesterov, 2003]

Problem formulation

Given a regular convex cone K, **how to convert a conic program over K into a semi-definite program?**

particularly interested in the situation when $K = K' \otimes L_n$, with K' another symmetric cone

Semi-definite representability

Definition

A cone K is called semi-definite representable if it is linearly isomorphic to a linear projection of a linear section of $S_+(n)$ for some n.

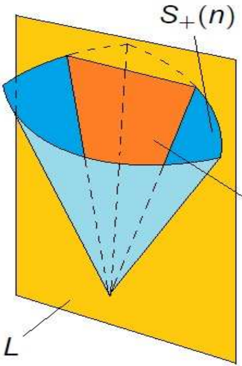
- linear intersection with subspace $L \subset \mathcal{S}(n)$
- linear projection along subspace $L' \subset L$

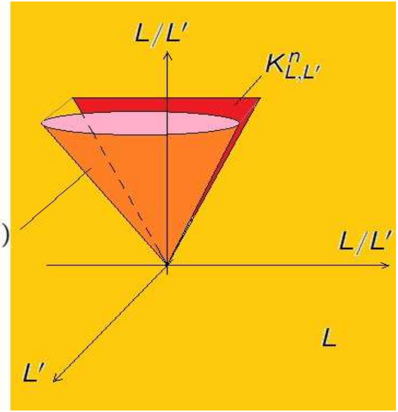
assume $L \cap S_{++}(n) \neq \emptyset$

K linearly isomorphic to

$$K^n_{L,L'} = \{x \in L/L' \,|\, \exists\, y \in x: \quad y \in L \cap S_+(n)\}$$

Semi-definite representable cones





Explicit representation

L/L' can be identified with subspace $L'' \subset L$ such that $L = L' \oplus L''$:

$$K^n_{L,L'} \simeq \{x \in L'' \,|\, \exists\, y \in L' : \quad x + y \in L \cap S_+(n)\}$$

same cone can have representations with different n, L, L'

Duality

Theorem

Let $L' \subset L \subset \mathcal{S}(n)$ be linear subspaces. Then

$$(K^n_{L,L'})^* = K^n_{L'^\perp,L^\perp}$$

Here $L'^\perp, L^\perp$ are the orthogonal complements of L', L.

we call $K^n_{L'^\perp,L^\perp}$ the dual representation of the representation $K^n_{L,L'}$

Equivalence of real and complex representations

$\mathcal{S}(n) \subset \mathcal{H}(n)$, $S_+(n) = H_+(n) \cap \mathcal{S}(n)$
$\Rightarrow$ real semi-definite representations can be considered as complex ones

$$S + iA \succeq 0 \quad \Leftrightarrow \quad \begin{pmatrix} S & A \\ -A & S \end{pmatrix} \succeq 0$$

$S \in \mathcal{S}(n)$, $A \in \mathcal{A}(n)$
$\Rightarrow$ complex semi-definite representations can be converted into real ones

Example

$$L_4 = \left\{ x = (x_0, x_1, x_2, x_3) \,|\, x_0 \geq \sqrt{x_1^2 + x_2^2 + x_3^2} \right\} \simeq H_+(2)$$

$$\begin{aligned} L_4 &= \left\{ x : \begin{pmatrix} x_0 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 - x_1 \end{pmatrix} \succeq 0 \right\} \\ &= \left\{ x : \begin{pmatrix} x_0 + x_1 & x_2 & 0 & x_3 \\ x_2 & x_0 - x_1 & -x_3 & 0 \\ 0 & -x_3 & x_0 + x_1 & x_2 \\ x_3 & 0 & x_2 & x_0 - x_1 \end{pmatrix} \succeq 0 \right\} \end{aligned}$$

denote these representations of L_4 by C_2 and R_4

Semi-definite approximations

many cones have no known semi-definite representation

- copositive cone $\mathcal{C}_n$ ($n \geq 5$)
- cones of multivariate nonnegative polynomials

Definition

An inner (outer) semi-definite approximation of a cone K is a semi-definite representable cone K' such that $K' \subset K$ ($K \subset K'$).
Approximation K'' is called tighter than approximation K' if $K' \subset K'' \subset K$ ($K \subset K'' \subset K'$).
The approximation K' is called exact if $K = K'$.

used for hard combinatorial and robust optimization problems

Approximations of separable cones

Theorem

Let K_1, K_2 be regular convex cones with explicit semi-definite representations R_1, R_2 given by $K^{n_1}_{L_1,L_1'}, K^{n_2}_{L_2,L_2'}$.
Then $K^{n_1 n_2}_{L_1 \otimes L_2, L_1 \otimes L_2' + L_1' \otimes L_2}$ is an outer semi-definite approximation of the separable cone $K_1 \otimes K_2$.

- underlying matrix cone is $S_+(n_1 n_2)$
- section with $L_1 \otimes L_2$
- projection on $(L_1/L_1') \otimes (L_2/L_2')$

Denote this approximation by $R_1 \otimes R_2$.

Exactness of approximations

known exact approximations for cones $L_n \otimes K$

- $L_2 \otimes K$, K semi-def. representable (trivial)
- $L_3 \otimes S_+(n)$ [Terpstra, 1939]
- $L_3 \otimes H_+(n)$ [Yakubovich, 1970]
- $L_4 \otimes H_+(2)$ [Størmer, 1951]
- $L_4 \otimes H_+(3)$ [Woronowicz, 1976]
- $L_n \otimes S_+(3)$ [H., 2007]
- $L_n \otimes L_m$ — this talk

Partial order of representations

Definition

Let R, R' be semi-definite representations of a regular convex cone K.
We call R tighter than R' if for every semi-definite representable cone $\tilde{K}$ and every semi-definite representation $\tilde{R}$ of $\tilde{K}$ the approximation $R \otimes \tilde{R}$ of $K \otimes \tilde{K}$ is tighter than the approximation $R' \otimes \tilde{R}$.

partial order on the set of representations of K

Example (submatrix technique)

representation R_3 of L_4

$$L_4 = \left\{ x : \begin{pmatrix} x_0 + x_1 & x_2 & x_3 \\ x_2 & x_0 - x_1 & 0 \\ x_3 & 0 & x_0 - x_1 \end{pmatrix} \succeq 0 \right\}$$

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approximation $R_4 \otimes R$ of $L_4 \otimes K$

$$\left\{ (X_0, X_1, X_2, X_3) : \begin{pmatrix} X_0 + X_1 & X_2 & 0 & X_3 \\ X_2 & X_0 - X_1 & -X_3 & 0 \\ 0 & -X_3 & X_0 + X_1 & X_2 \\ X_3 & 0 & X_2 & X_0 - X_1 \end{pmatrix} \succeq 0 \right\}$$

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R_4 is tighter than R_3

Automorphism group of a cone

Definition

Let $K \subset \mathbb{R}^n$ be a regular convex cone. The *automorphism group* $\mathrm{Aut}(K)$ of K is the group of invertible linear maps $A \in GL(n, \mathbb{R})$ such that $A[K] = K$.

- $\mathrm{Aut}(K)$ preserves the facial structure of K
- $\mathbb{R}_{++} \subset \mathrm{Aut}(K)$ for every K
- $\mathrm{Aut}(K)$ has a canonical faithful linear representation

Automorphisms of symmetric cones

- $\mathrm{Aut}(\mathbb{R}^n_+) = \mathbb{R}^n_{++} \times S_n$
- $\mathrm{Aut}(L_n) = O^+(n-1,1) \times \mathbb{R}_{++}$
- $\mathrm{Aut}(S_+(n)) = GL(n,\mathbb{R})/\{-1,+1\}$
- $\mathrm{Aut}(H_+(n)) = GL(n,\mathbb{C})/\{e^{i\varphi}\}$ and complex conjugation

$A \in GL(n,\mathbb{R})$, $X \in \mathcal{S}(n)$: $X \mapsto AXA^T$
$A \in GL(n,\mathbb{C})$, $X \in \mathcal{H}(n)$: $X \mapsto AXA^*$

K	$\mathbb{R}^n_+$	L_n	$S_+(n)$	$H_+(n)$
$\dim K$	n	n	$\frac{n(n+1)}{2}$	n^2
$\dim \mathrm{Aut}(K)$	n	$\frac{n(n-1)}{2}+1$	n^2	$2n^2-1$

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$\dim \mathrm{Aut}(K)$	n	$\frac{n(n-1)}{2}+1$	n^2	$2n^2-1$

Automorphisms of semi-definite representable cones

- $L' \subset L \subset \mathcal{S}(n)$ — linear subspaces
- $\mathcal{A}_{L,L'} \subset \mathrm{Aut}(S_+(n))$ — automorphisms with L, L' invariant

Theorem

There exists a canonical homomorphism $\mathcal{A}_{L,L'} \to \mathrm{Aut}(K^n_{L,L'})$ into the automorphism group of the cone

$$K^n_{L,L'} = \{x \in L/L' \,|\, \exists\, y \in x : \; y \in L \cap S_+(n)\}.$$

need not be injective nor surjective

Automorphisms of separable cones

Theorem

Let K_1, K_2 be regular convex cones. Then there exists a canonical homomorphism $\mathrm{Aut}(K_1) \times \mathrm{Aut}(K_2) \to \mathrm{Aut}(K_1 \otimes K_2)$, given by $(g_1, g_2) \mapsto g_1 \otimes g_2$, for all $g_1 \in \mathrm{Aut}(K_1)$, $g_2 \in \mathrm{Aut}(K_2)$.

K	$L_n \otimes L_m$	$L_n \otimes S_+(m)$	$L_n \otimes H_+(m)$
$\dim K$	nm	$\frac{nm(m+1)}{2}$	nm^2
$\dim \mathrm{Aut}(K)$	$\frac{n^2+m^2-n-m}{2} + 1$	$\frac{n(n-1)}{2} + m^2$	$\frac{n(n-1)}{2} + 2m^2 - 2$

Methods using automorphisms

- block-diagonalization
- group averaging
- canonical forms

Block-diagonalization

Theorem (Schur)

Let $G \subset U(n)$ be unitary representation of a compact group, decomposing into irreducible representations $R_1, \dots, R_l$ of dimensions $d_1, \dots, d_l$ and multiplicities $m_1, \dots, m_l$.
Then there is a matrix $U_0 \in U(n)$ such that for every complex matrix A commuting with the action of G the matrix $A_0 = U_0 A U_0^*$ has a block-diagonal structure. Each irreducible representation R_k gives rise to d_k identical blocks of size m_k.

- for $K = K^n_{L,L'}$, applicable if $gA = Ag$ for all $A \in L$, $g \in G$
- used to block-diagonalize semi-definite representations [Gatermann, Parrilo 2004]

Example

C_n — $n \times n$ complex representation of a cone K
R_{2n} — its real form of size $2n \times 2n$

$$L = \left\{ \begin{pmatrix} S & A \\ -A & S \end{pmatrix} \ : \ S \in \mathcal{S}(n),\ A \in \mathcal{A}(n) \right\}$$

$$G = \left\{ \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \otimes I_n \ : \ \varphi \in (-\pi, \pi] \right\}$$

$$U_0 = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \otimes I_n$$

$$U_0 \begin{pmatrix} S & A \\ -A & S \end{pmatrix} U_0^* = \begin{pmatrix} S - iA & 0 \\ 0 & S + iA \end{pmatrix}$$

$\bar{C}_n \cap C_n \simeq R_{2n} \quad \Rightarrow R_{2n}$ tighter than C_n and $\bar{C}_n$

Group averaging

Theorem

Let $L' \subset L$ be subspaces of $\mathcal{S}(n)$ and let $G \subset \mathrm{Aut}(S_+(n))$ be a compact subgroup of $\mathcal{A}_{L,L'}$ giving rise only to the trivial automorphism of $K^n_{L,L'}$. Let $F_G \subset \mathcal{S}(n)$ be the subspace of fixed elements under the action of G.
Then $K^n_{L,L'} = K^n_{L\cap F_G, L'\cap F_G}$.

proof based on group averaging

results in reduction of the dimension of projection

group averaging technique used in [Gatermann, Parrilo 2004] to reduce size of semi-definite programs

Example

approximation $R_4 \otimes I$ of $L_4 \otimes S_+(n)$

$$(X_0, X_1, X_2, X_3) : \begin{pmatrix} X_0 + X_1 & X_2 & 0 & X_3 \\ X_2 & X_0 - X_1 & -X_3 & 0 \\ 0 & -X_3 & X_0 + X_1 & X_2 \\ X_3 & 0 & X_2 & X_0 - X_1 \end{pmatrix} \succeq 0$$

taking dual ...

Example

dual approximation $(R_4 \otimes I)^*$ of $\mathcal{P}(L_4, S_+(n))$

(X_0, X_1, X_2, X_3): there exist symmetric S_k, skew-symmetric A_k such that

$$\begin{pmatrix} X_0 + X_1 + S_1 & X_2 + S_2 + A_1 & S_4 + A_3 & X_3 + S_5 + A_4 \\ X_2 + S_2 - A_1 & X_0 - X_1 + S_3 & -X_3 + S_5 + A_5 & S_6 + A_6 \\ S_4 - A_3 & -X_3 + S_5 - A_5 & X_0 + X_1 - S_1 & X_2 - S_2 + A_2 \\ X_3 + S_5 - A_4 & S_6 - A_6 & X_2 - S_2 - A_2 & X_0 - X_1 - S_3 \end{pmatrix}$$

is positive semi-definite

Example

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(X_0, X_1, X_2, X_3): there exist symmetric S_k, skew-symmetric A_k such that

$$\begin{pmatrix} X_0 + X_1 + S_1 & X_2 + S_2 + A_1 & S_4 + A_3 & X_3 + S_5 + A_4 \\ X_2 + S_2 - A_1 & X_0 - X_1 + S_3 & -X_3 + S_5 + A_5 & S_6 + A_6 \\ S_4 - A_3 & -X_3 + S_5 - A_5 & X_0 + X_1 - S_1 & X_2 - S_2 + A_2 \\ X_3 + S_5 - A_4 & S_6 - A_6 & X_2 - S_2 - A_2 & X_0 - X_1 - S_3 \end{pmatrix}$$

is positive semi-definite

symmetry group $G = \left\{ \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \otimes I_n : \ \varphi \in (-\pi, \pi] \right\}$

$$F_G = \left\{ \begin{pmatrix} S & A \\ -A & S \end{pmatrix} : \quad S \in \mathcal{S}(n), \ A \in \mathcal{A}(n) \right\}$$

Example

dual approximation $(R_4 \otimes I)^*$ of $\mathcal{P}(L_4, S_+(n))$

(X_0, X_1, X_2, X_3): there exist symmetric S_k, skew-symmetric A_k such that

$$\begin{pmatrix} X_0 + X_1 + S_1 & X_2 + S_2 + A_1 & S_4 + A_3 & X_3 + S_5 + A_4 \\ X_2 + S_2 - A_1 & X_0 - X_1 + S_3 & -X_3 + S_5 + A_5 & S_6 + A_6 \\ S_4 - A_3 & -X_3 + S_5 - A_5 & X_0 + X_1 - S_1 & X_2 - S_2 + A_2 \\ X_3 + S_5 - A_4 & S_6 - A_6 & X_2 - S_2 - A_2 & X_0 - X_1 - S_3 \end{pmatrix}$$

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Example

dual approximation $(R_4 \otimes I)^*$ of $\mathcal{P}(L_4, S_+(n))$

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$$\begin{pmatrix} X_0 + X_1 & X_2 + A_1 & A_3 & X_3 + A_4 \\ X_2 - A_1 & X_0 - X_1 & -X_3 + A_4 & A_6 \\ -A_3 & -X_3 - A_4 & X_0 + X_1 & X_2 + A_1 \\ X_3 - A_4 & -A_6 & X_2 - A_1 & X_0 - X_1 \end{pmatrix}$$

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Canonical forms

Lemma

Let $K_1, K_2 \subset \mathbb{R}^n$ be regular convex cones, and let $G \subset \mathrm{Aut}(K_1) \cap \mathrm{Aut}(K_2)$ be a subgroup of automorphisms of both cones. Let $H \subset \mathbb{R}^n$ be a subspace such that for every $x \in K_1^o$ there exists $g \in G$: $g(x) \in H$. If $H \cap K_1 \subset H \cap K_2$, then $K_1 \subset K_2$.

apply this lemma in the situation when

- one of K_1, K_2 is the semi-definite representable cone $K^n_{L,L'}$
- the other is a regular convex cone K

$\Rightarrow$ $K^n_{L,L'}$ is an outer (inner) semi-definite approximation of K

Example

- $L_n = \left\{ x = (x_0, \dots, x_{n-1}) \,|\, x_0 \geq \sqrt{x_1^2 + \dots + x_{n-1}^2} \right\}$
- G: orthogonal $(x_2, \dots, x_{n-1})$-transformations and hyperbolic (x_0, x_1)-rotations
- $H = \{x \,|\, x_1 = x_3 = \dots = x_{n-1} = 0\}$
- $L_n \cap H = \{x \in H \,|\, x_0 \geq |x_2|\}$

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$$K = \left\{ x : \begin{pmatrix} x_0 + x_1 & x_2 & \cdots & x_{n-1} \\ x_2 & & & \\ \vdots & & (x_0 - x_1) I_{n-2} & \\ x_{n-1} & & & \end{pmatrix} \succeq 0 \right\}$$

$G \subset \mathrm{Aut}(K)$

Example

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$$K \cap H = \left\{ x : \begin{pmatrix} x_0 & x_2 & \cdots & 0 \\ x_2 & & & \\ \vdots & & x_0 \cdot I_{n-2} & \\ 0 & & & \end{pmatrix} \succeq 0 \right\} = L_n \cap H$$

$\Rightarrow \qquad L_n \subset K$

Example

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$L_n \subset K$

repeating with the dual representation of K^* yields $L_n \subset K^*$
$\Leftrightarrow K \subset L_n$

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$\Leftrightarrow K \subset L_n$

$\Rightarrow L_n = K$

Example

$$L_n = \left\{ x : \begin{pmatrix} x_0 + x_1 & x_2 & \cdots & x_{n-1} \\ x_2 & & & \\ \vdots & & (x_0 - x_1) I_{n-2} & \\ x_{n-1} & & & \end{pmatrix} \succeq 0 \right\}$$

denote this representation of L_n by R_{n-1}

Main result

Theorem

The semi-definite approximation $R^*_{n-1} \otimes R^*_{m-1}$ of the separable cone $L_n \otimes L_m = (\mathcal{P}(L_m, L_n))^*$ is exact.

Lorentz-positive maps studied in [Loewy, Schneider 1975]

Clifford algebras

Definition

The Clifford algebra $Cl_{n-1}(\mathbb{R})$ is a real associative algebra generated by $e_1, \dots, e_{n-1}$ subject to $e_k e_l = -e_l e_k$, $k \neq l$, $e_k^2 = 1$.

Properties of $Cl_{n-1}(\mathbb{R})$

- transposition antiautomorphism, $(ab)^t = b^t a^t$
- n-dimensional linear subspace $Y \subset Cl_{n-1}(\mathbb{R})$
- quadratic form J of signature $(+ - \cdots -)$ on Y
- $\Rightarrow$ Lorentz cone $L_n \subset Y$
- spin group $\mathrm{Spin}_{1,n-1}(\mathbb{R})$ acting on Y, $y \mapsto gyg^t \in Y$
- action of $\mathrm{Spin}_{1,n-1}(\mathbb{R})$ preserves J and L_n
- induces $SO^+_{1,n-1}(\mathbb{R}) \subset \mathrm{Aut}(L_n)$
- complex matrix representation s.t. $x \mapsto X \Leftrightarrow x^t \mapsto X^*$
- real matrix representation s.t. $x \mapsto X \Leftrightarrow x^t \mapsto X^T$
- real rep. decomposes into copies of complex rep.

Semi-definite representation of $L_n \otimes L_m$

- $SO^+_{1,n-1}(\mathbb{R}) \times SO^+_{1,m-1}(\mathbb{R})$ brings interior of $L_n \otimes L_m$ to diagonal form
- canonical forms: complex representation of $Cl_{n-1}(\mathbb{R}) \otimes Cl_{m-1}(\mathbb{R})$ induces semi-definite representation of $L_n \otimes L_m$
- block-diagonalization: real representation of $Cl_{n-1}(\mathbb{R}) \otimes Cl_{m-1}(\mathbb{R})$ induces semi-definite representation of $L_n \otimes L_m$
- group averaging: reduction of projection dimension for the dual representation
- submatrix technique: size reduction of the dual representation

LAMA papers

Hildebrand R. An LMI description for the cone of Lorentz-positive maps. *Linear and Multilinear Algebra*, 55(6):551-573, 2007.

Hildebrand R. An LMI description for the cone of Lorentz-positive maps II. *Linear and Multilinear Algebra*, 59(7):719-731, 2011.

Open problem

Is every convex semi-algebraic regular cone semi-definite representable?

- $L_3 \otimes L_3 \otimes L_3$?
- $L_4 \otimes S_+(4)$?
- $S_+(3) \otimes S_+(3)$?
- $\mathcal{C}_5$?

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Thank you