Robust conic quadratic programming with ellipsoidal uncertainties

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Uncertain conic programs

$$\min_x \langle c, x \rangle : Ax + b \in K$$

$K \subset R^N$ regular convex cone, $x \in \mathbb{R}^n$ vector of decision variables

data A, b, c may be uncertain and vary in an uncertainty set U

let x^* be the (nominal) optimal solution

for perturbed data $A' = A + \delta A$, $b' = b + \delta b$ the constraint might be violated:

$$A'x^* + b' \notin K$$

Robust counterpart

Example 1 (Nemirovski, SP XI Vienna, 2007):

in 19 (13) of the 90 NETLIB LP test programs (`http://www.netlib.org/lp/data/`), perturbation of the data by 0.01% leads to violation by 5% (50%) of some constraints

remedy : solve *robust counterpart*

$$\min_x \tau \;\; : \langle c, x\rangle \leq \tau, \;\; Ax + b \in K \quad \forall\; (A, b, c) \in U$$

in the sequel we consider the cost vector c to be certain

Example 1 (continued)

”cost of robustness” is usually negligible

in all of the 90 NETLIB LP problems, cost of robust optimal solution is $< 0.4\%$ ($< 1\%$) worse than that of the nominal optimal solution if robustified against perturbations of 0.01% (0.1%) magnitude

Example 2

Ben-Tal & Nemirovski, ”Robust convex optimization”, 1998:

truss topology design optimized with respect to a nominal load f^* highly unstable: application of a small force (10% of f^*) leads to an 3000-fold increase of the compliance

compliance of the robustified design is only 0.24% larger than that of the nominal one

Uncertainty description

complexity of robust conic program depends both on K and U

we suppose uncertainty set U given by

$$(A, b) = (A^0, b^0) + \sum_{k=1}^{m-1} u_k \cdot (A^k, b^k), \quad u \in B$$

$B \subset \mathbb{R}^{m-1}$ compact convex set

trivial case: finite number of scenarios

$\Leftrightarrow$ B convex polyhedral set with small number of vertices

Robust counterpart : reformulation

define cone

$$K_B = \{(\tau; \tau u) \in \mathbb{R}^m \,|\, \tau \geq 0,\ u \in B\}$$

then robust counterpart becomes

$$\min_x \langle c, x \rangle : \ \left(\sum_{k=0}^{m-1} u_k A^k\right) x + \sum_{k=0}^{m-1} u_k b^k \in K \quad \forall\, u \in K_B$$

or equivalently

$$\min_x \langle c, x \rangle : \ \mathcal{A}_x[K_B] \subset K,$$

where $\mathcal{A}_x : \mathbb{R}^m \to \mathbb{R}^N$ given by

$$\mathcal{A}_x(u) = \left(\sum_{k=0}^{m-1} u_k A^k\right) x + \sum_{k=0}^{m-1} u_k b^k$$

coefficients of linear map $\mathcal{A}_x$ affine in x

Positive maps

for regular convex cones $K_1 \subset \mathbb{R}^{n_1}$, $K_2 \subset \mathbb{R}^{n_2}$, call a linear map $A : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ *K_1-to-K_2 positive* if $A[K_1] \subset K_2$

cone of positive maps is itself a regular convex cone in $\mathbb{R}^{n_1 n_2}$

K-to-$\mathbb{R}_+$ positive cone is the dual cone K^*

$(K_1 \times \cdots \times K_m)$-to-$(K'_1 \times \cdots \times K'_{m'})$ positive cone is the product $\prod_{k=1}^{m} \prod_{k'=1}^{m'}$ of K_k-to-$K'_{k'}$ positive cones

nice description of robust counterpart depends on availability of nice description of the K_B-to-K positive cone

Choice of uncertainty B

L_1-ball (hyper-octahedron) ok for small number of uncertain variables, but in higher dimensions it becomes "spiky"

L_2-ball well-balanced uncertainty naturally occurring when data is obtained from parametric estimation

L_∞-ball (box) occurs if we have interval uncertainty, often intractable due to large number of vertices

robust LP with box-constrained uncertainty is an LP

(Ben-Tal & Nemirovski, "Robust convex optimization", 1998)

Ellipsoidal uncertainty

Lorentz cone

$$L_m = \{(u_0, \ldots, u_{m-1})^T \mid u_0 \geq ||(u_1, \ldots, u_{m-1})^T||_2\}$$

robust counterpart for ellipsoidal uncertainty can be written as

$$\min_x \langle c, x \rangle : \ \mathcal{A}_x \ L_m\text{-to-}K \text{ positive}$$

$\mathcal{A}_x : \mathbb{R}^m \to \mathbb{R}^N$ affine in x

due to possibility of taking products we can have

• independent ellipsoids on different data

• uncertainties which are convex hulls of different, possibly degenerated ellipsoids (e.g. L_1-L_2 hybrid ball)

Existing results

robust LP with ellipsoidal uncertainty (even for intersections of ellipsoids) is a CQP (Ben-Tal & Nemirovski, 1998)

L_m-to-$L_{m'}$ positive cone efficiently computable (Nemirovski)

hence robust CQP with ellipsoidal uncertainty computable with cutting-plane methods — practically unfeasible for $m \approx m' \geq 10$

if uncertainty on each constraint independent, and uncertainty on zero components independent of uncertainty on the other components, then the robust counterpart of a CQP is an SDP (Ben-Tal & Nemirovski, 1998)

Existing results

SDP with rank 2 ellipsoidal uncertainty is an SDP (Ben-Tal & Nemirovski, "Robust convex optimization", 1998)

$$\min_x \langle c, x \rangle :$$

$$A_0 + \sum_{k=1}^{n} x_k A_k + \sum_{j=1}^{m-1} u_j \left((b_j + x^T B_j) d^T + d(b_j^T + B_j^T x) \right) \succeq 0$$

$$\forall \; ||u||_2 \leq 1$$

with d fixed

L_m-to-$S_+(n)$ positive cone

$\mathcal{S}(n)$ — space of $n \times n$ real symmetric matrices

$\mathcal{A}(n)$ — space of $n \times n$ real skew-symmetric matrices

$S_+(n) \subset \mathcal{S}(n)$ — cone of PSD matrices

consider a map $A : \mathbb{R}^m \to \mathcal{S}(n)$ given by

$$x \mapsto \sum_{k=0}^{m-1} x_k A_k, \quad A_k \in \mathcal{S}(n)$$

Standard relaxation

define an associated matrix

$$\mathcal{M}_A = \begin{pmatrix} A_0 + A_1 & A_2 & \cdots & \cdots & A_{m-1} \\ A_2 & A_0 - A_1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & 0 \\ \vdots & 0 & 0 & A_0 - A_1 & 0 \\ A_{m-1} & 0 & \cdots & 0 & A_0 - A_1 \end{pmatrix}$$

suppose

$$\exists\ X \in \mathcal{A}(m-1) \otimes \mathcal{A}(n) : \quad \mathcal{M}_A + X \succeq 0 \qquad \text{(suf)}$$

then A is L_m-to-$S_+(n)$ positive

Proof let $z \in \mathbb{R}^n$ be arbitrary

let $x \in \partial L_m$ be normalized to $x_0 + x_1 = 1$

convex conic closure of such x is L_m

then with $\tilde{x} = (x_2, \dots, x_{m-1})^T$ we have $x_0^2 - x_1^2 = x_0 - x_1 = ||\tilde{x}||_2^2$

compute

$$[(1\ \tilde{x}^T) \otimes z^T]\ X\ \left[\begin{pmatrix} 1 \\ \tilde{x} \end{pmatrix} \otimes z \right] = 0$$

$$[(1\ \tilde{x}^T) \otimes z^T]\ \mathcal{M}_A \left[\begin{pmatrix} 1 \\ \tilde{x} \end{pmatrix} \otimes z \right] =$$

$$z^T[A_0 + A_1 + 2\sum_{k=2}^{m-1} x_k A_k + ||\tilde{x}||_2^2(A_0 - A_1)]z = 2z^T A(x) z \geq 0$$

hence $A(x) \succeq 0$ and A is L_m-to-$S_+(n)$ positive

LMI description

for $n = 1$ condition (suf) is trivially necessary

Theorem (Størmer, 1951) If $n = 2$, then condition (suf) is also necessary for positivity of the map A.

Theorem (Woronowicz, 1976) If $n = 3$ and $m \leq 4$, then condition (suf) is also necessary for positivity of the map A.

Theorem (H., 2007) If $n = 3$, then condition (suf) is also necessary for positivity of the map A.

this yields a (lifted) LMI representation of the L_m-to-$S_+(n)$ positive cone for $n \leq 3$

L_m-to-L_n positive cone

consider a map $A : \mathbb{R}^m \to \mathbb{R}^n$ given by a real $n \times m$ matrix

interpret A as an element of $\mathbb{R}^n \otimes \mathbb{R}^m$

define a linear map $\mathcal{W}_r : \mathbb{R}^r \to \mathcal{S}(r-1)$ by

$$\mathcal{W}_r(x) = \begin{pmatrix} x_0 + x_1 & x_2 & \cdots & \cdots & x_{r-1} \\ x_2 & x_0 - x_1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & 0 \\ \vdots & 0 & 0 & x_0 - x_1 & 0 \\ x_{r-1} & 0 & \cdots & 0 & x_0 - x_1 \end{pmatrix}$$

Standard relaxation

suppose

$$\exists\ X \in \mathcal{A}(n-1) \otimes \mathcal{A}(m-1): \quad (\mathcal{W}_n \otimes \mathcal{W}_m)(A) + X \succeq 0 \qquad \text{(suf2)}$$

then A is L_m-to-L_n positive

Proof

let $x \in \partial L_n$ be normalized to $x_0 + x_1 = 1$

let $y \in \partial L_m$ be normalized to $y_0 + y_1 = 1$

define $\tilde{x} = (x_2, \ldots, x_{n-1})^T$, $\tilde{y} = (y_2, \ldots, y_{m-1})^T$

compute

$$[(1\ \tilde{x}^T) \otimes (1\ \tilde{y}^T)]\ X\ \left[\begin{pmatrix} 1 \\ \tilde{x} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \tilde{y} \end{pmatrix} \right] = 0$$

$$[(1\ \tilde{x}^T) \otimes (1\ \tilde{y}^T)]\ (\mathcal{W}_n \otimes \mathcal{W}_m)(A) \left[\begin{pmatrix} 1 \\ \tilde{x} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \tilde{y} \end{pmatrix} \right] = 4x^T Ay \geq 0$$

hence $A[L_m] \subset L_n$ by self-duality of L_n and A is L_m-to-L_n positive

LMI description

Theorem (Yakubovich, 1962) If $n = 3$ or $m = 3$, then condition (suf2) is also necessary for positivity of the map A.

Theorem (Størmer, 1951) If $n = 4$ or $m = 4$, then condition (suf2) is also necessary for positivity of the map A.

Theorem (H., 2008) Condition (suf2) is also necessary for positivity of the map A for arbitrary n, m.

this yields a (lifted) LMI representation of the L_m-to-L_n positive cone

Example $(\mathcal{W}_4 \otimes \mathcal{W}_4)(A) =$

$$\begin{pmatrix} A_{++} & A_{+2} & A_{+3} & A_{2+} & A_{22} & A_{23} & A_{3+} & A_{32} & A_{33} \\ A_{+2} & A_{+-} & & A_{22} & A_{2-} & & A_{32} & A_{3-} & \\ A_{+3} & & A_{+-} & A_{23} & & A_{2-} & A_{33} & & A_{3-} \\ A_{2+} & A_{22} & A_{23} & A_{-+} & A_{-2} & A_{-3} & & & \\ A_{22} & A_{2-} & & A_{-2} & A_{--} & & & & \\ A_{23} & & A_{2-} & A_{-3} & & A_{--} & & & \\ A_{3+} & A_{32} & A_{33} & & & & A_{-+} & A_{-2} & A_{-3} \\ A_{32} & A_{3-} & & & & & A_{-2} & A_{--} & \\ A_{33} & & A_{3-} & & & & A_{-3} & & A_{--} \end{pmatrix}$$

$A_{+\pm} = A_{00} \pm A_{01} + A_{10} \pm A_{11}$, $A_{-\pm} = A_{00} \pm A_{01} - A_{10} \mp A_{11}$,
$A_{\pm k} = A_{0k} \pm A_{1k}$, $A_{k\pm} = A_{k0} \pm A_{k1}$

LMI description of robust programs

robust counterpart of mixed LP/CQP/SDP with SDP individual block size not exceeding 3 for real symmetric blocks and 2 for complex hermitian blocks

$$K = \mathbb{R}_+^{N_{LP}} \times \prod_{i=1}^{N_{CQP}} L_{n_i} \times \prod_{i=1}^{N_{SDP}} S_+(3)$$

with uncertainty given by convex hulls of a finite number of ellipsoids is a mixed CQP/SDP

block structure is inherited from original program as well as from structure of uncertainty