Abstract cones of positive polynomials and their sums of squares relaxations

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May 13, 2011

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Cones of positive polynomials

$\mathcal{L}_\mathcal{A}$ — linear space of polynomials

$$p(x_1,\dots,x_n) = \sum_{\alpha\in\mathcal{A}} c_\alpha(p) x^\alpha$$

$\mathcal{A} \subset \mathbb{N}^n$ — ordered set of multi-indices $\alpha^k = (\alpha_1^k,\dots,\alpha_n^k)$

$\dim \mathcal{L}_\mathcal{A} = \#\mathcal{A} = m$

define $m \times n$ matrix $M_\mathcal{A} = (\alpha_l^k)_{k=1,\dots,m;l=1,\dots,n}$

with $\alpha_0^k = 1$, $M'_\mathcal{A} = (\alpha_l^k)_{k=1,\dots,m;l=0,\dots,n}$ — $m \times (n+1)$ matrix

$\mathcal{I}_\mathcal{A} : p \mapsto (c_\alpha(p))_{\alpha\in\mathcal{A}} \subset \mathbb{R}^m$ linear isomorphism

$\mathcal{P}_\mathcal{A}$ cone of positive polynomials

$\mathcal{I}_\mathcal{A}[\mathcal{P}_\mathcal{A}] \subset \mathbb{R}^m$ its image

Example

Motzkin polynomial $p_M(x, y, z) = x^2y^4 + x^4y^2 + z^6 - 3x^2y^2z^2$

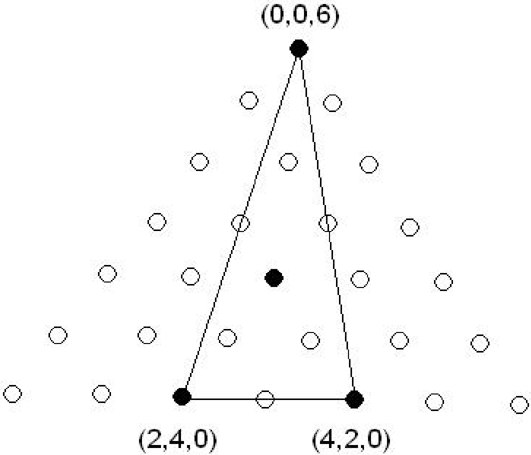
$\mathcal{A} = \{(2, 4, 0), (4, 2, 0), (0, 0, 6), (2, 2, 2)\}$, $n = 3$, $m = \#\mathcal{A} = 4$

$$M_{\mathcal{A}} = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 6 \\ 2 & 2 & 2 \end{pmatrix}$$

$$M'_{\mathcal{A}} = \begin{pmatrix} 1 & 2 & 4 & 0 \\ 1 & 4 & 2 & 0 \\ 1 & 0 & 0 & 6 \\ 1 & 2 & 2 & 2 \end{pmatrix}$$

$p_M \in \mathcal{P}_{\mathcal{A}}$, $\mathcal{I}_{\mathcal{A}}(p_M) = (1, 1, 1, -3)^T \in \mathcal{I}_{\mathcal{A}}[\mathcal{P}_{\mathcal{A}}]$

Example cont'd



Newton polytope and integer lattices

for $p \in \mathcal{L}_\mathcal{A}$

$$N(p) = \text{conv}\{\alpha \,|\, c_\alpha(p) \neq 0\}$$

Newton polytope associated with p

$$N_\mathcal{A} = \cup_{p \in \mathcal{L}_\mathcal{A}} N(p) = \text{conv}\mathcal{A}$$

Newton polytope associated with $\mathcal{L}_\mathcal{A}$

$$\Gamma_\mathcal{A} = \left\{ \sum_{\alpha \in \mathcal{A}} a_\alpha \alpha \,|\, a_\alpha \in \mathbb{Z} \ \forall \ \alpha \in \mathcal{A}, \ \sum a_\alpha = 1 \right\} \subset \mathbb{Z}^n$$

integer lattice generated by $\mathcal{A}$ in $\operatorname{aff} \mathcal{A}$

$$\Gamma^e_\mathcal{A} \subset \Gamma_\mathcal{A}$$

sublattice of even points

Regularity of $\mathcal{P}_{\mathcal{A}}$

Theorem [Reznick, 1978] Let $p \in \mathcal{P}_{\mathcal{A}}$ and let $\alpha^* \in \mathcal{A}$ be extremal in $N(p)$. Then $c_{\alpha^*}(p) > 0$ and α^* is even.

hence assume w.r.o.g. that $\mathrm{extr} N_{\mathcal{A}} \subset \Gamma^e_{\mathcal{A}}$
otherwise $\mathcal{P}_{\mathcal{A}}$ contained in proper subspace of $\mathcal{L}_{\mathcal{A}}$

Lemma $\mathcal{P}_{\mathcal{A}} \subset \mathcal{L}_{\mathcal{A}}$ is a regular convex cone.
$\mathcal{P}_{\mathcal{A}}$ closed and does not contain lines

$p(x) = \sum_{\alpha \in \mathrm{extr} N_{\mathcal{A}}} x^\alpha$ is in int$\mathcal{P}_{\mathcal{A}}$, since $p + q \in \mathcal{P}_{\mathcal{A}}$ for all
$q : \sum_{\alpha \in \mathcal{A}} |c_\alpha(q)| \leq 1$

Dual cone

vector of monomials $X_{\mathcal{A}}(x) = (x^{\alpha})_{\alpha \in \mathcal{A}}$

moment surface $\mathcal{X}_{\mathcal{A}} = \{X_{\mathcal{A}}(x) \,|\, x \in \mathbb{R}^n\}$

Lemma $(\mathcal{I}_{\mathcal{A}}[\mathcal{P}_{\mathcal{A}}])^* = \mathrm{conv}(\mathrm{con}\,\mathrm{cl}\mathcal{X}_{\mathcal{A}})$.

$p(x) = \langle \mathcal{I}_{\mathcal{A}}(p), X_{\mathcal{A}}(x) \rangle$

$$\mathrm{con}\,\mathrm{cl}\mathcal{X}_{\mathcal{A}} = \mathrm{cl}\{(-1)^{\delta} \circ \exp(y) \,|\, \delta \in Im\,\pi_2[M_{\mathcal{A}}],\ y \in Im\,M'_{\mathcal{A}}\}$$

$\pi_2 : \mathbb{Z} \to \mathbb{F}_2$ — ring homomorphism to $\mathbb{F}_2 = (\{0,1\}, +, \cdot)$

Equivalence relation

Theorem Let $\mathcal{A} = \{\alpha^1, \ldots, \alpha^m\} \subset \mathbb{N}^n$,
$\mathcal{A}' = \{{\alpha'}^1, \ldots, {\alpha'}^m\} \subset \mathbb{N}^{n'}$. Then the following are equivalent.

1) $\operatorname{con}\operatorname{cl}\mathcal{X}_{\mathcal{A}} = \operatorname{con}\operatorname{cl}\mathcal{X}_{\mathcal{A}'}$,

2) the order isomorphism $I_A : \mathcal{A} \to \mathcal{A}'$ can be extended to a lattice isomorphism $I_\Gamma : \Gamma_{\mathcal{A}} \to \Gamma_{\mathcal{A}'}$, and $I_\Gamma[\Gamma^e_{\mathcal{A}}] = \Gamma^e_{\mathcal{A}'}$.

1) — 2) imply

$\mathcal{I}_{\mathcal{A}}[\mathcal{P}_{\mathcal{A}}] = \mathcal{I}_{\mathcal{A}'}[\mathcal{P}_{\mathcal{A}'}]$.

1) — 2) define an equivalence relation $\sim_P$ on the class of ordered multi-index sets of cardinality m

Abstract positive cones

$[\mathcal{A}]$ — equivalence class of $\mathcal{A}$ w.r. to $\sim_P$

Definition We call $P_{[\mathcal{A}]} = \mathcal{I}_{\mathcal{A}}[\mathcal{P}_{\mathcal{A}}] \subset \mathbb{R}^m$ an *abstract cone* of positive polynomials.

infinitely many cones of positive polynomials $P_{\mathcal{A}}$ generate the same abstract cone $P_{[\mathcal{A}]}$

Sums of squares

classical approach

$$\Sigma_{\mathcal{A}} = \left\{ p \in \mathcal{L}_{\mathcal{A}} \,|\, \exists\ N, q_k :\ p = \sum_{k=1}^{N} q_k^2 \right\}$$

$$\Sigma_{h,\mathcal{A}} = \left\{ p \in \mathcal{L}_{\mathcal{A}} \,|\, \exists\ N, q_k :\ ph = \sum_{k=1}^{N} q_k^2 \right\}$$

h nonzero positive polynomial
$\Sigma_{\mathcal{A}}, \Sigma_{h,\mathcal{A}}$ inner semidefinite relaxations of $\mathcal{P}_{\mathcal{A}}$

in general $\Sigma_{\mathcal{A}} \neq \mathcal{P}_{\mathcal{A}}$, e.g., $p_M \notin \Sigma_{\mathcal{A}}$

Theorem [Reznick, 1978] Let $p = \sum_{k=1}^{N} q_k^2$. Then $N(q_k) \subset N(p)/2\ \forall\ k = 1, \ldots, N$.

$\Rightarrow$ if $p = \sum_{k=1}^{N} q_k^2 \in \mathcal{P}_{\mathcal{A}}$, then $q_k \in \mathcal{L}_{N_{\mathcal{A}}/2 \cap \mathbb{N}^n}$

Structure of $\Sigma_{\mathcal{A}}$

$\mathcal{F} \subset \mathbb{N}^n$ — ordered multi-index set

$$\begin{aligned}\Sigma_{\mathcal{F},\mathcal{A}} &= \{p \in \mathcal{L}_{\mathcal{A}} \,|\, \exists\; N, q_k \in \mathcal{L}_{\mathcal{F}} : p(x) = \sum_{k=1}^{N} q_k^2\} \\ &= \{p \in \mathcal{L}_{\mathcal{A}} \,|\, \exists\; C \succeq 0 : p(x) = X_{\mathcal{F}}(x)^T C X_{\mathcal{F}}(x)\}\end{aligned}$$

is an inner semidefinite relaxation for $\mathcal{P}_{\mathcal{A}}$
$L_{\mathcal{F},\mathcal{A}} : \mathcal{S}(m') \to \mathcal{L}_{(\mathcal{F}+\mathcal{F})\cup\mathcal{A}}$, $L_{\mathcal{F},\mathcal{A}} : C \to p$ linear projection

$$\Sigma_{\mathcal{F},\mathcal{A}} = \mathcal{L}_{\mathcal{A}} \cap L_{\mathcal{F},\mathcal{A}}[\mathcal{S}_+(m')]$$

explicit semidefinite description

Order structure

$\mathcal{F}$ smaller $\Rightarrow$ relaxation $\Sigma_{\mathcal{F},\mathcal{A}}$ weaker

$\Sigma_{\mathcal{A}} = \Sigma_{N_{\mathcal{A}}/2 \cap \mathbb{N}^n, \mathcal{A}}$ — strongest among $\Sigma_{\mathcal{F},\mathcal{A}}$

w.r.o.g. $\mathcal{F} \subset \mathcal{F}_{\max}(\mathcal{A}) = N_{\mathcal{A}}/2 \cap \mathbb{N}^n$

inclusion (partial) ordering on the set of such $\mathcal{F}$ induces partial ordering on the set of SOS relaxations $\Sigma_{\mathcal{F},\mathcal{A}}$

Example

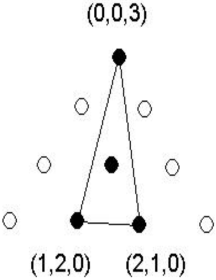
Motzkin polynomial

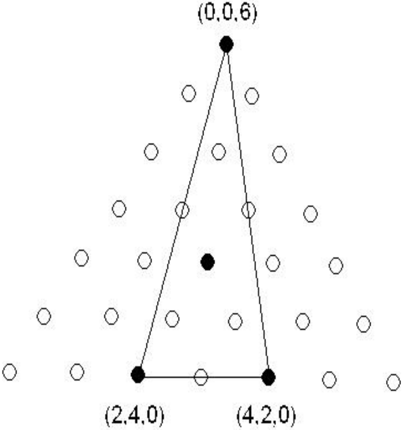
$\mathcal{A} = \{(2,4,0),(4,2,0),(0,0,6),(2,2,2)\}$

$\mathcal{F}_{\mathsf{max}}(\mathcal{A}) = \mathcal{F} = \{(1,2,0),(2,1,0),(0,0,3),(1,1,1)\}$

$$M_{\mathcal{F}} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

Example cont'd





Dimensions of $\Sigma_{\mathcal{A}}$ and $\mathcal{P}_{\mathcal{A}}$ equal?

$\Sigma_{\mathcal{F},\mathcal{A}} \subset \mathcal{L}_{(\mathcal{F}+\mathcal{F})\cap\mathcal{A}}$

$\Sigma_{\mathcal{F},\mathcal{A}}$ relaxation of $\mathcal{P}_{(\mathcal{F}+\mathcal{F})\cap\mathcal{A}}$ rather than of $\mathcal{P}_{\mathcal{A}}$

$\dim \Sigma_{\mathcal{F},\mathcal{A}} = \dim \mathcal{P}_{\mathcal{A}} \Rightarrow \mathcal{A} \subset \mathcal{F} + \mathcal{F}$

Is this necessary condition verified by $\mathcal{F} = \mathcal{F}_{\mathsf{max}}(\mathcal{A})$ for all $\mathcal{A}$?

NO!: [Reznick, 1978]
$\mathcal{A} = \{(2,0,0),(0,2,0),(2,2,0),(0,0,4),(1,1,1)\}$

not even dim $\Sigma_{\mathcal{A}} =$ dim $\mathcal{P}_{\mathcal{A}}$ for all $\mathcal{A}$

Equivalence relation

Theorem $\mathcal{F} = \{\beta^1, \dots, \beta^{m'}\}, \mathcal{A} = \{\alpha^1, \dots, \alpha^m\} \subset \mathbb{N}^n$, $\mathcal{F}' = \{\beta'^1, \dots, \beta'^{m'}\}, \mathcal{A}' = \{\alpha'^1, \dots, \alpha'^m\} \subset \mathbb{N}^{n'}$ s.t. $\mathcal{F} \subset \mathcal{F}_{\max}(\mathcal{A})$, $\mathcal{F}' \subset \mathcal{F}_{\max}(\mathcal{A}')$; $I_F : \mathcal{F} \to \mathcal{F}'$, $I_A : \mathcal{A} \to \mathcal{A}'$ order isomorphisms. If there exists a bijective map I that makes

$$\begin{array}{ccccc} \mathcal{F} \times \mathcal{F} & \stackrel{s_{\mathcal{F},\mathcal{A}}}{\longrightarrow} & (\mathcal{F} + \mathcal{F}) \cup \mathcal{A} & \stackrel{\mathrm{incl}_{\mathcal{A}}}{\longleftarrow} & \mathcal{A} \\ I_F \times I_F \downarrow & & I \downarrow & & I_A \downarrow \\ \mathcal{F}' \times \mathcal{F}' & \stackrel{s_{\mathcal{F}',\mathcal{A}'}}{\longrightarrow} & (\mathcal{F}' + \mathcal{F}') \cup \mathcal{A}' & \stackrel{\mathrm{incl}_{\mathcal{A}'}}{\longleftarrow} & \mathcal{A}' \end{array}$$

commutative, then $\mathcal{I}_{\mathcal{A}}[\Sigma_{\mathcal{F},\mathcal{A}}] = \mathcal{I}_{\mathcal{A}'}[\Sigma_{\mathcal{F}',\mathcal{A}'}]$.

here $s_{\mathcal{F},\mathcal{A}}(\beta^k, \beta^{k'}) = \beta^k + \beta^{k'}$

defines an equivalence relation $\sim_\Sigma$ on the class of pairs $(\mathcal{F}, \mathcal{A})$ satisfying $\mathcal{F} \subset \mathcal{F}_{\max}(\mathcal{A})$

Abstract SOS cones

$[(\mathcal{F}, \mathcal{A})]$ — equivalence class of $(\mathcal{F}, \mathcal{A})$ w.r. to $\sim_\Sigma$

Definition We call $\Sigma_{[(\mathcal{F},\mathcal{A})]} = \mathcal{I}_\mathcal{A}[\Sigma_{\mathcal{F},\mathcal{A}}] \subset \mathbb{R}^m$ an *abstract SOS cone*.

infinitely many SOS cones $\Sigma_{\mathcal{F},\mathcal{A}}$ generate the same abstract cone $\Sigma_{[(\mathcal{F},\mathcal{A})]}$

SOS relaxations of abstract positive cones

Definition C — equivalence class w.r.t. to $\sim_P$, $\mathcal{P}_C$ corresponding abstract cone of positive polynomials. For every pair $(\mathcal{F}, \mathcal{A})$ s.t. $\mathcal{A} \in C$, $\mathcal{F} \subset \mathcal{F}_{\max}(\mathcal{A})$, we call the abstract cone $\Sigma_{[(\mathcal{F},\mathcal{A})]}$ an *SOS relaxation* of $\mathcal{P}_C$.

$\Sigma_{[(\mathcal{F},\mathcal{A})]} \subset \mathcal{P}_C$

Hierarchy

$\mathcal{A} \sim_P \mathcal{A}' \not\Rightarrow \Sigma_{\mathcal{A}} \sim_\Sigma \Sigma_{\mathcal{A}'}$ in general

no "standard" SOS relaxation for $\mathcal{P}_{[\mathcal{A}]}$

Definition $\mathcal{P}_C$ abstract positive cone, $\Sigma_{C_1}, \Sigma_{C_2}$ SOS relaxations of $\mathcal{P}_C$. If there exist $\mathcal{F}_1, \mathcal{F}_2, \mathcal{A}$ s.t. $(\mathcal{F}_1, \mathcal{A}) \in C_1$, $(\mathcal{F}_2, \mathcal{A}) \in C_2$, and $\mathcal{F}_1 \subset \mathcal{F}_2$, then Σ_{C_1} is *coarser* than Σ_{C_2}, or Σ_{C_2} is *finer* than Σ_{C_1}.

we do not require $\mathcal{A} \in C$
$\Sigma_{C_1}, \Sigma_{C_2}$ can be SOS relaxations for different cones $\mathcal{P}_C, \mathcal{P}_{C'}$, but order relation is independent of choice of positive cone

Construction of finer relaxations

Theorem $\mathcal{F}, \mathcal{A} \subset \mathbb{N}^n$ s.t. $\mathcal{F} \subset \mathcal{F}_{\max}(\mathcal{A})$, R — $n \times n$ integer matrix with odd determinant, $v \in \mathbb{Z}^n$ integer row vector. $(\mathcal{F}', \mathcal{A}')$ s.t. $M_{\mathcal{F}'} = M_{\mathcal{F}} R + \mathbf{1} v$, $M_{\mathcal{A}'} = M_{\mathcal{A}} R + 2\mathbf{1} v$. If $\mathcal{A}' \subset \mathbb{N}^n$, then $\mathcal{A} \sim_P \mathcal{A}'$. If $\mathcal{F}', \mathcal{A}' \subset \mathbb{N}^n$, then $(\mathcal{F}, \mathcal{A}) \sim_\Sigma (\mathcal{F}', \mathcal{A}')$.

nonnegativity of $\mathcal{F}', \mathcal{A}'$ can be enforced by choice of v

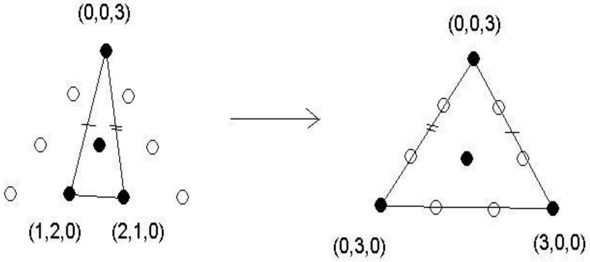
if $\det R = \pm 1$, then $\Gamma_{\mathcal{A}} \simeq \Gamma_{\mathcal{A}'}$, $\Gamma^e_{\mathcal{A}} \simeq \Gamma^e_{\mathcal{A}'}$, $\Sigma_{\mathcal{A}} \sim_\Sigma \Sigma_{\mathcal{A}'}$
if $\#\mathcal{A} > 1$, then strictly finer relaxations can always be obtained this way
hierarchy if infinite

Example: Motzkin polynomial

$M_{\mathcal{F}'} = M_{\mathcal{F}} R + \mathbf{1} v$, $\det R = -3$

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 0$$

$\mathcal{A}' = \{(6,0,0),(0,6,0),(0,0,6),(2,2,2)\}$



Example cont'd

$$\mathcal{I}_{\mathcal{A}'}^{-1} \circ \mathcal{I}_{\mathcal{A}} : p_M(x,y,z) = x^2y^4 + x^4y^2 + z^6 - 3x^2y^2z^2 \mapsto$$

$$\begin{aligned} p'_M &= x^6 + y^6 + z^6 - 3x^2y^2z^2 \\ &= (x^2+y^2+z^2)(x^4+y^4+z^4 - x^2y^2 - y^2z^2 - z^2x^2) \end{aligned}$$

$$\begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix} \succeq 0$$

p'_M is SOS

moreover: $\mathcal{P}_{\mathcal{A}'} = \Sigma_{\mathcal{A}'}$

$$\begin{aligned} p(c) &= c_1x^2y^4 + c_2x^4y^2 + c_3z^6 - c_4x^2y^2z^2 \\ p'(c) &= c_1x^6 + c_2y^6 + c_3z^6 - c_4x^2y^2z^2 \end{aligned}$$

$$p(c) \geq 0 \;\Leftrightarrow\; p'(c) \geq 0 \;\Leftrightarrow\; p'(c) \text{ is SOS}$$

not possible with $\Sigma_{h,\mathcal{A}}$ for any h [Reznick, 2005]

Example: $\mathcal{C}_5$

$$C \in \mathcal{C}_5 \Leftrightarrow \begin{pmatrix} x_1^2 \\ \vdots \\ x_5^2 \end{pmatrix}^T C \begin{pmatrix} x_1^2 \\ \vdots \\ x_5^2 \end{pmatrix} \geq 0$$

$\mathcal{C}_5 \sim \mathcal{P}_{\mathcal{A}}$ with

$$\mathcal{F} = \{2e_i \,|\, i = 1, \dots, 5\}$$

$$\mathcal{A} = \{2(e_i + e_j) \,|\, 1 \leq i \leq j \leq 5\}$$

$\mathcal{A}_k = k\mathcal{A} \sim_P \mathcal{A}$, $\mathcal{F}_k = k\mathcal{F}$