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Rank 1 generated spectrahedral cones

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let

- $\mathcal{S}^n$ be the space of real symmetric $n \times n$ matrices
- $\mathcal{S}_+^n \subset \mathcal{S}_n$ the cone of positive semi-definite matrices

a QCQP is a problem of the form [Ramana, Goldman 1995]

$$\min_{x \in \mathbb{R}^n} x^T S x \ : \quad x^T A_i x = 0, \ i = 1, \dots, k; \quad x^T B x = 1$$

$A_1, \dots, A_k; B; S \in \mathcal{S}_n$ define the homogeneous quadratic constraints, the inhomogeneous quadratic constraint, and the quadratic cost function

set $X = xx^T \in \mathcal{S}_+^n$

we get

$$\min_{X \in K} \langle S, X \rangle \ : \quad \langle B, X \rangle = 1, \quad \operatorname{rk} X = 1$$

here $K = L \cap \mathcal{S}_+^n$, where

$$L = \{X \in \mathcal{S}^n \mid \langle A_i, X \rangle = 0 \ \forall \, i = 1, \dots, k\}$$

Definition

Linear sections of the cone of positive semi-definite matrices $\mathcal{S}_+^n$ are called spectrahedral cones.

original QCQP:

$$\min_{X \in K} \langle S, X \rangle \ : \quad \langle B, X \rangle = 1, \quad \operatorname{rk} X = 1$$

K spectrahedral cone

can be relaxed to a semi-definite program (SDP) by dropping the rank constraint:

$$\min_{X \in K} \langle S, X \rangle \ : \quad \langle B, X \rangle = 1$$

this SDP is convex and can be efficiently solved by freely (CLP, LiPS, SDPT3, SeDuMi, ...) and commercially (CPLEX, MOSEK, ...) available solvers

Lemma

Let K be such that its extreme rays are generated by rank 1 matrices. Then either the two problems are both infeasible, or the SDP is unbounded, or both problems have the same optimal value.

Definition

We call a spectrahedral cone rank 1 generated (ROG) if its extreme rays are generated by rank 1 matrices.

Applications of QCQPs and its relaxations

numerous problems in statistics can be written as QCQP and tackled by its semi-definite relaxation

- MLE for angular synchronization problem [Bandeira, Boumal, Singer 2014]
- information theoretical clustering [Wang, Sha 2011]
- MAP assignment over discrete Markov random fields [Huang, Chen, Guibas 2014]
- robust PCA [McCoy, Tropp 2011]
- inference on graphs [Wainwright, Jordan 2003]
- sparse PCA [d'Aspremont, El Ghaoui, Jordan, Lanckriet 2004; d'Aspremont, Bach, El Ghaoui 2014; Krauthgamer, Nadler, Vilenchik 2015]
- sparse covariance selection, sparse SVD, sparse nonnegative matrix factorization [d'Aspremont et al 2007]
- high-dimensional sparse PCA [Amini, Wainwright 2009]
- . . .

Examples of rank 1 generated spectrahedral cones

- full positive semi-definite matrix cone $\mathcal{S}_+^n$
- cone of positive semi-definite $n \times n$ Hankel matrices Han_+^n
- cone of positive semi-definite $n \times n$ tridiagonal matrices Tri_+^n
- $$K = \left\{ \begin{pmatrix} a_1 & a_6 & a_5 & a_7 & a_{11} & a_{10} \\ a_6 & a_2 & a_4 & a_{13} & a_8 & a_{12} \\ a_5 & a_4 & a_3 & a_{15} & a_{14} & a_9 \\ a_7 & a_{13} & a_{15} & a_4 & a_9 & a_8 \\ a_{11} & a_8 & a_{14} & a_9 & a_5 & a_7 \\ a_{10} & a_{12} & a_9 & a_8 & a_7 & a_6 \end{pmatrix} \in \mathcal{S}_+^6, \quad a_1, \dots, a_{15} \in \mathbb{R} \right\}$$

the positive semi-definite Hankel matrices are the moment cone of the univariate polynomials of degree $2n$

the last 15-dimensional cone is the moment cone of the *ternary quartics*, which are nonnegative if and only if they can be represented as a sum of squares [Hilbert 1888]

Definition (Helton, Vinnikov 2007)

A closed set $C \subset \mathbb{R}^m$ is an algebraic interior if there exists a polynomial p on $\mathbb{R}^m$ such that C equals the closure of a connected component of the set $\{x \in \mathbb{R}^m \mid p(x) > 0\}$. Such a polynomial is called defining polynomial.

Lemma (Helton, Vinnikov 2007)

Let C be an algebraic interior. Then the defining polynomial p of C with minimal degree (the minimal defining polynomial) is unique up to multiplication by a positive constant. Any other defining polynomial of C is divisible by p.

every spectrahedral cone is a convex algebraic interior with a homogeneous minimal defining polynomial

Theorem

Let K be a ROG spectrahedral cone whose interior consists of positive definite matrices. Then the determinantal defining polynomial d of K is a minimal defining polynomial.

applicable to any non-degenerate spectrahedral cone $K \subset \mathcal{S}_+^n$ such that there exist linearly independent vectors $x_1, \dots, x_n \in \mathbb{R}^n$ satisfying $x_i x_i^T \in K, i = 1, \dots, n$

the degree of the minimal defining polynomial of an algebraic interior C is called the degree of C

Lemma

Let K be a ROG spectrahedral cone. Then the degree of K is given by $\deg K = \max_{X \in K} \operatorname{rk} X$.

Definition (Guler, Tunçel 1998)

Let K be a closed pointed convex cone. The Carathéodory number $\kappa(x)$ of a point $x \in K$ is the minimal number k such that there exist extreme elements $x_1, \dots, x_k$ of K satisfying $x = \sum_{i=1}^{k} x_i$.
The Carathéodory number $\kappa(K)$ of the cone K is the maximum of $\kappa(x)$ over $x \in K$.

Lemma

Let K be a ROG spectrahedral cone. For every $X \in K$, its Carathéodory number is given by $\kappa(X) = \operatorname{rk} X$.

Corollary

The Carathéodory number of a ROG cone equals its degree.

let $L \subset \mathcal{S}^n$, $L' \subset \mathcal{S}^{n'}$ be linear subspaces of matrix spaces $n \leq n'$
call L, L' isomorphic if there exists a full rank matrix A such that the map $X \mapsto AXA^T$ takes L onto L'
such isomorphisms preserve rank and signature

Definition

We call spectrahedral cones $K \subset \mathcal{S}_+^n$, $K' \subset \mathcal{S}_+^{n'}$ isomorphic if they can be represented as intersections $K = L \cap \mathcal{S}_+^n$, $K' = L' \cap \mathcal{S}_+^{n'}$ with isomorphic subspaces $L \subset \mathcal{S}^n$, $L' \subset \mathcal{S}^{n'}$.

spectrahedral cones which are linearly isomorphic as cones are not necessarily isomorphic in this sense
example $\mathbb{R}_+^2$:

$$K = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathcal{S}_+^2, \quad a, b \in \mathbb{R} \right\}, \qquad K' = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a+b & 0 \\ 0 & 0 & b \end{pmatrix} \in \mathcal{S}_+^3, \quad a, b \in \mathbb{R} \right\}$$

Theorem

Two ROG cones are isomorphic in the sense above if and only if they are linearly isomorphic as cones.

geometric structure determines **algebraic** structure (all ROG representations of a cone are isomorphic)

let $K_1 \subset \mathbb{R}^{n_1}, \dots, K_m \subset \mathbb{R}^{n_m}$ be convex cones
the convex cone $K = \{(x_1, \dots, x_m) \in \mathbb{R}^{n_1 + \dots n_m} \,|\, x_1 \in K_1, \dots, x_m \in K_m\}$ is called the direct sum of $K_1, \dots, K_m$

Theorem

Let K be a ROG cone which is representable as a direct sum of cones $K_1, \dots, K_m$. Then

- $K_1, \dots, K_m$ *are also ROG,*
- K *possesses a block-diagonal representation corresponding to the decomposition,*
- *the k-th block is a representation of the factor cone K_k.*

On the other hand, if $K_1, \dots, K_m$ are ROG cones, then the corresponding block-diagonal representation of their direct sum is a ROG representation.

Definition

We call a ROG cone which is not a non-trivial direct sum of other cones a simple ROG cone.

Lemma

Each ROG cone decomposes into a finite number of simple ROG cones which are unique up to permutation.

Lemma

Let K be a spectrahedral cone. Then the spectrahedral cone

$$\left\{ \begin{pmatrix} X & * \\ * & * \end{pmatrix} \succeq 0, \quad X \in K \right\}$$

is a ROG cone if and only if K is ROG.

we call K' a full extension of K if it is isomorphic to a cone of the above form

Lemma

A ROG cone $K \subset \mathcal{S}_+^n$ is a full extension of some smaller ROG cone if and only if there exist nontrivial linear subspaces $L \subset \mathcal{S}^n$ and $H \subset \mathbb{R}^n$ such that $K = L \cap \mathcal{S}_+^n$ and $xy^T + yx^T \in L$ for all $x \in H$, $y \in \mathbb{R}^n$.

the full extension of a ROG cone is simple

Lemma

Let F_1, F_2 be faces of the positive semi-definite matrix cone $\mathcal{S}_+^n$ and L_1, L_2 their linear hulls. Let $L \subset \mathcal{S}^n$ be a linear subspace such that $L_1 \cap L_2 \subset L = (L \cap L_1) + (L \cap L_2)$. Then the spectrahedral cone $K = L \cap \mathcal{S}_+^n$ equals the sum of its faces $K_1 = L_1 \cap K$, $K_2 = L_2 \cap K$. Moreover, K is a ROG cone if and only if K_1, K_2 are ROG cones.

$$\begin{pmatrix} X_{11} & X_{12} & 0 \\ X_{12}^T & X_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in L_1 \cap L, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & X_{22} & X_{23} \\ 0 & X_{23}^T & X_{33} \end{pmatrix} \in L_2 \cap L,$$

$$\begin{pmatrix} X_{11} & X_{12} & 0 \\ X_{12}^T & X_{22} & X_{23} \\ 0 & X_{23}^T & X_{33} \end{pmatrix} \in L, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \end{pmatrix} \in L.$$

we call K an intertwining of K_1, K_2

- an intertwining of K_1, K_2 is a projection of the direct sum $K_1 \oplus K_2$
- any two ROG cones can be intertwined along a 1-dimensional face
- example: the tridiagonal matrices are intertwinings of copies of $\mathcal{S}_+^2$

for mutually distinct angles $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in [0, \pi)$ define the cone $K_{\varphi_1,\varphi_2,\varphi_3,\varphi_4}$ by

$$\left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \cos\varphi_1 & \alpha_4 \cos\varphi_2 & \alpha_5 \cos\varphi_3 & \alpha_6 \cos\varphi_4 \\ \alpha_2 & \alpha_7 & \alpha_3 \sin\varphi_1 & \alpha_4 \sin\varphi_2 & \alpha_5 \sin\varphi_3 & \alpha_6 \sin\varphi_4 \\ \alpha_3 \cos\varphi_1 & \alpha_3 \sin\varphi_1 & \alpha_8 & 0 & 0 & 0 \\ \alpha_4 \cos\varphi_2 & \alpha_4 \sin\varphi_2 & 0 & \alpha_9 & 0 & 0 \\ \alpha_5 \cos\varphi_3 & \alpha_5 \sin\varphi_3 & 0 & 0 & \alpha_{10} & 0 \\ \alpha_6 \cos\varphi_4 & \alpha_6 \sin\varphi_4 & 0 & 0 & 0 & \alpha_{11} \end{pmatrix} \succeq 0, \quad \alpha_i \in \mathbb{R} \right\}$$

Lemma

The cone $K_{\varphi_1,\varphi_2,\varphi_3,\varphi_4}$ is a ROG cone. Two cones $K_{\varphi_1,\varphi_2,\varphi_3,\varphi_4}$, $K_{\varphi'_1,\varphi'_2,\varphi'_3,\varphi'_4}$ are isomorphic if and only if the corresponding quadruples of lines $l(\varphi_1), \dots, l(\varphi_4) \subset \mathbb{R}^2$ and $l(\varphi'_1), \dots, l(\varphi'_4) \subset \mathbb{R}^2$ define projectively equivalent quadruples of points in $\mathbb{R}P^1$.

$K_{\varphi_1,\varphi_2,\varphi_3,\varphi_4}$ is the intertwining of 5 copies of $\mathcal{S}^2_+$

codimension 1:

Lemma (Dines' theorem)

Let $L \subset \mathcal{S}^n$ be a linear subspace of codimension 1. Then the cone $K = L \cap \mathcal{S}_+^n$ is ROG.

codimension 2:

Theorem

Let $K = \{X \in \mathcal{S}_+^n \mid \langle X, Q_1 \rangle = \langle X, Q_2 \rangle = 0\}$ be a ROG cone of degree $n \geq 3$, where Q_1, Q_2 are linearly independent quadratic forms. Then K is isomorphic to the direct sum $\mathcal{S}_+^1 \oplus \mathcal{S}_+^2$ if $n = 3$ and to a full extension of this sum if $n > 3$.

low dimensions:

Theorem

Let K be a simple ROG cone of degree n. Then $\dim K \geq 2n - 1$.

examples:

- positive semi-definite Hankel matrices
- positive semi-definite tridiagonal matrices

Isolated extreme rays

Theorem

Let K be a ROG cone of degree n. Then the number of its isolated extreme rays does not exceed n. Let $R_1, \dots, R_k$ be the isolated extreme rays of K. Then K is isomorphic to a direct sum $K' \oplus \mathbb{R}_+^k$, where K' is a ROG cone of degree $n-k$ without isolated extreme rays, and the extreme rays $R_1, \dots, R_k$ correspond to the extreme rays of the summand $\mathbb{R}_+^k$.

isolated extreme rays split off as direct summands

consequence: simple cones of degree $\deg K \geq 2$ have no isolated extreme rays degree 1:

- dim 1: $\mathcal{S}_+^1$

degree 2:

- dim 3: $\mathcal{S}_+^2$

degree 3:

- dim 5: Tri_+^3, Han_+^3
- dim 6: $\mathcal{S}_+^3$

degree 4:

- dim 7: Han_+^4, full extension of $\mathcal{S}_+^1 \oplus \mathcal{S}_+^1 \oplus \mathcal{S}_+^1$, Tri_+^4, intertwining of Han_+^3 and $\mathcal{S}_+^2$
- dim 8: full extension of $\mathcal{S}_+^1 \oplus \mathcal{S}_+^2$
- dim 9: full extensions of $\mathcal{S}_+^1 \oplus \mathcal{S}_+^1$ and Han_+^3; $\mathcal{S}_+^2 \otimes \mathcal{S}_+^2$; $\{X \succeq 0 \mid \langle X, Q \rangle = 0\}$ with Q of signature $(+++-)$
- dim 10: $\mathcal{S}_+^4$

Thank you!