Parallelism conditions and algebras in affine differential geometry

Roland Hildebrand

LJK / CNRS

PADGE conference, Kath. Univ. Leuven
August 22, 2017

Affine differential geometry

studies hypersurface immersions

$$f : M^n \to \mathbb{A}^{n+1}$$

into real affine space, endowed with a transversal vector field ξ
one is interested in properties of the immersion that are invariant under affine transformations of $\mathbb{A}^{n+1}$

the following objects are induced on M

- a torsion-free connection ∇, the affine connection
- a symmetric bilinear form h, the affine fundamental form
- a 3-tensor $C = \nabla h$, the cubic form

if h is non-degenerate, then the immersion is called non-degenerate and h is called the affine metric (possibly indefinite)

Equiaffine immersions

if we fix a volume form ω on $\mathbb{A}^{n+1}$, we can in addition define on M

- the induced volume form θ

if $\nabla\theta = 0$, then the immersion is called equiaffine
for equiaffine immersions the cubic form C is totally symmetric in its three indices

examples of equiaffine immersions:

- graph immersions: ξ is a constant vector field
- centro-affine immersions: $\xi(x) = f(x) - a_0$, $a_0 \in \mathbb{A}^{n+1}$
- Blaschke immersions: induced volume form θ equals volume form $\sqrt{|\det h|}$ of affine metric

for centro-affine immersions, we may consider a_0 as the origin and the affine space $\mathbb{A}^{n+1}$ as a real vector space $\mathbb{R}^{n+1}$

Parallelism conditions

assume that the affine fundamental form is non-degenerate
then it has its own Levi-Civita connection $\hat{\nabla}$

the difference $K = \nabla - \hat{\nabla}$ is called the difference tensor,
$K_{ij}^{l} = -\frac{1}{2} h^{kl} C_{ijk}$

we are interested in classifying the immersions which satisfy one of the parallelism conditions

$$\begin{array}{ccc} \nabla C = 0 & & \nabla K = 0 \\ \hat{\nabla} C = 0 & \Leftrightarrow & \hat{\nabla} K = 0 \end{array}$$

research ongoing since late 80s [Vrancken 88; Nomizu, Pinkall 89; Bokan et al 90; Dillen, Vrancken 91,94,98; Dillen et al 94; Gigena 02,03,11; Hu et al 08,09,11; Hu, Li 11; Li 14; Fujioka et al 16; Cheng et al 17]

Outline of method

immersion $f : M^n \to \mathbb{A}^{n+1}$ with transversal field ξ

$$\Updownarrow$$

potential F

$$\Updownarrow$$

metrised algebra (A, σ)

Potentials for graph immersions

let $f : M \to \mathbb{A}^{n+1}$ be a graph immersion

for every $x \in M$ there exists a neighbourhood U of x and a potential $F : \mathbb{R}^n \supset D \simeq U \to \mathbb{R}$ such that

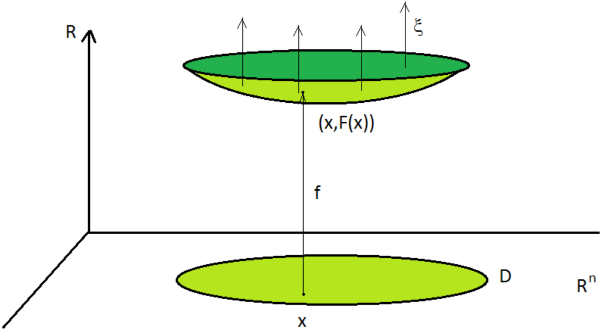
- f is locally isomorphic to the embedding $U \ni x \mapsto (x, F(x)) \in \mathbb{R}^n \times \mathbb{R}$
- the field ξ maps to $(\mathbf{0}, 1)$ under this isomorphism

then we get

$$h = F'', \qquad C = F''', \qquad \nabla = \partial$$

if the graph immersion is also Blaschke, then it is an improper affine hypersphere

this happens if and only if $\det F'' \equiv \pm 1$



R
ξ
(x,F(x))
f
D
R^n
x

Parallelism condition

the parallelism condition $\hat{\nabla} C = 0$ can be rewritten as a quasi-linear 4-th order partial differential equation on the potential F

$$F_{,\alpha\beta\gamma\delta} = \frac{1}{2} F^{,\rho\sigma} \left(F_{,\alpha\beta\rho} F_{,\gamma\delta\sigma} + F_{,\alpha\gamma\rho} F_{,\beta\delta\sigma} + F_{,\alpha\delta\rho} F_{,\beta\gamma\sigma} \right)$$

here $F^{,\rho\sigma}$ is the inverse of the Hessian F'' and $F_{,\alpha\beta\gamma}$ etc. are partial derivatives

Integrability condition

differentiating with respect to x^η and substituting the fourth order derivatives by the right-hand side, we get

$$
\begin{aligned}
F_{,\alpha\beta\gamma\delta\eta} = \frac{1}{4} F^{,\rho\sigma} F^{,\mu\nu} & (F_{,\beta\eta\nu} F_{,\alpha\rho\mu} F_{,\gamma\delta\sigma} + F_{,\alpha\eta\mu} F_{,\rho\beta\nu} F_{,\gamma\delta\sigma} \\
+ & F_{,\gamma\eta\nu} F_{,\alpha\rho\mu} F_{,\beta\delta\sigma} + F_{,\alpha\eta\mu} F_{,\rho\gamma\nu} F_{,\beta\delta\sigma} + F_{,\beta\eta\nu} F_{,\gamma\rho\mu} F_{,\alpha\delta\sigma} \\
+ & F_{,\gamma\eta\mu} F_{,\rho\beta\nu} F_{,\alpha\delta\sigma} + F_{,\beta\eta\nu} F_{,\delta\rho\mu} F_{,\alpha\gamma\sigma} + F_{,\delta\eta\mu} F_{,\rho\beta\nu} F_{,\alpha\gamma\sigma} \\
+ & F_{,\delta\eta\nu} F_{,\alpha\rho\mu} F_{,\beta\gamma\sigma} + F_{,\alpha\eta\mu} F_{,\rho\delta\nu} F_{,\beta\gamma\sigma} + F_{,\delta\eta\nu} F_{,\gamma\rho\mu} F_{,\alpha\beta\sigma} \\
+ & F_{,\gamma\eta\mu} F_{,\rho\delta\nu} F_{,\alpha\beta\sigma})
\end{aligned}
$$

anti-commuting δ, η gives the integrability condition

$$
\begin{aligned}
F^{,\rho\sigma} F^{,\mu\nu} & (F_{,\beta\eta\nu} F_{,\delta\rho\mu} F_{,\alpha\gamma\sigma} + F_{,\alpha\eta\mu} F_{,\rho\delta\nu} F_{,\beta\gamma\sigma} + F_{,\gamma\eta\mu} F_{,\rho\delta\nu} F_{,\alpha\beta\sigma} \\
& - F_{,\beta\delta\nu} F_{,\eta\rho\mu} F_{,\alpha\gamma\sigma} - F_{,\alpha\delta\mu} F_{,\rho\eta\nu} F_{,\beta\gamma\sigma} - F_{,\gamma\delta\mu} F_{,\rho\eta\nu} F_{,\alpha\beta\sigma}) = 0
\end{aligned}
$$

Algebraic formulation

raising the index η we get

$$K^{\eta}_{\alpha\mu}K^{\mu}_{\delta\rho}K^{\rho}_{\beta\gamma} + K^{\eta}_{\beta\mu}K^{\mu}_{\delta\rho}K^{\rho}_{\alpha\gamma} + K^{\eta}_{\gamma\mu}K^{\mu}_{\delta\rho}K^{\rho}_{\alpha\beta}$$
$$- \quad K^{\mu}_{\alpha\delta}K^{\eta}_{\rho\mu}K^{\rho}_{\beta\gamma} - K^{\mu}_{\beta\delta}K^{\eta}_{\rho\mu}K^{\rho}_{\alpha\gamma} - K^{\mu}_{\gamma\delta}K^{\eta}_{\rho\mu}K^{\rho}_{\alpha\beta} = 0$$

(recall that $K^{\alpha}_{\beta\gamma} = -\frac{1}{2}F^{,\alpha\delta}F_{,\beta\gamma\delta}$)

this is satisfied if and only if

$$K^{\eta}_{\alpha\mu}K^{\mu}_{\delta\rho}K^{\rho}_{\beta\gamma}u^{\alpha}u^{\beta}u^{\gamma}v^{\delta} = K^{\mu}_{\alpha\delta}K^{\eta}_{\rho\mu}K^{\rho}_{\beta\gamma}u^{\alpha}u^{\beta}u^{\gamma}v^{\delta}$$

for all tangent vectors u, v

consider K as structure tensor of a commutative algebra A with multiplication $\bullet$ on the tangent space T_xU, then the above is equivalent to

$$(u^2 \bullet v) \bullet u = (u \bullet v) \bullet u^2$$

Jordan algebras

Definition

An algebra J is a Jordan algebra if

- $x \bullet y = y \bullet x$ for all $x, y \in J$ (commutativity)
- $x^2 \bullet (x \bullet y) = x \bullet (x^2 \bullet y)$ for all $x, y \in J$ (Jordan identity)

where $x^2 = x \bullet x$.

Definition

A pair (A, σ) of an algebra A with multiplication $\bullet$ and a non-degenerate quadratic form σ is called metrised algebra if $\sigma(u, v \bullet w) = \sigma(u \bullet v, w)$ for all $u, v, w \in A$.

if we take $\sigma = h$ on the tangent spaces $T_x U$, then the above relation is satisfied by the symmetry of the cubic form $C_{\alpha\beta\gamma} = -2K^{\delta}_{\alpha\beta} h_{\gamma\delta}$

Main result (graph immersions)

Theorem

A graph immersion $f : M \to \mathbb{R}^{n+1}$ satisfying $\hat{\nabla} C = 0$ defines a metrised Jordan algebra (A, h) with structure tensor K on the tangent spaces $T_x M$.
Isomorphism classes of algebras are in one-to-one correspondence with isomorphism classes of graph immersions.
M is an improper affine sphere if and only if A is nilpotent and $\det h = \pm 1$.

- metrised algebras at different points isomorphic by parallelism condition $\hat{\nabla} K = 0$
- recovery of graph immersion from the algebra via the potential $F(x) = \sum_{k=2}^{\infty} \frac{(-1)^k}{k} h(x, x^{k-1})$

classification of improper affine hyperspheres with $\hat{\nabla} C = 0$ equivalent to classification of nilponent metrised Jordan algebras (A, σ) with $\det \sigma = \pm 1$

Potentials for centro-affine immersions

let $f : M \to \mathbb{R}^{n+1}$ be a centro-affine immersion, let $x \in M$ and U a neighbourhood of x

define the potential F on $\bigcup_{\alpha>0} \alpha U$ by

$$F(\alpha f(x)) = \log \alpha$$

then

$$h = f^* F'', \qquad C = f^* F''', \qquad -\frac{1}{2} F^{,\gamma\delta} F_{,\alpha\beta\gamma} f^\beta = \delta_\alpha^\gamma$$

but now F is defined on $D \subset \mathbb{R}^{n+1}$

Parallelism condition

the parallelism condition $\hat{\nabla} C = 0$ is again equivalent to the 4-th order PDE

$$F_{,\alpha\beta\gamma\delta} = \frac{1}{2} F^{,\rho\sigma} \left(F_{,\alpha\beta\rho} F_{,\gamma\delta\sigma} + F_{,\alpha\gamma\rho} F_{,\beta\delta\sigma} + F_{,\alpha\delta\rho} F_{,\beta\gamma\sigma} \right)$$

integrability condition is again the Jordan identity

but

- the Jordan algebra A is defined on $T_x M \times \mathbb{R}$
- f is a unit element

if the centro-affine immersion is also Blaschke, then it is a proper affine sphere

Main result (centro-affine immersions)

Theorem

A centro-affine immersion $f : M \to \mathbb{R}^{n+1}$ satisfying $\hat{\nabla} C = 0$ defines a metrised unital Jordan algebra (A, σ) with structure tensor $-\frac{1}{2} F^{,\gamma\delta} F_{,\alpha\beta\gamma}$ and quadratic form F'' on the spaces $T_x M \times \mathbb{R}$. Isomorphism classes of algebras are in one-to-one correspondence with isomorphism classes of centro-affine immersions. M is a proper affine sphere if and only if σ is the trace form $\tau(u, v) = \operatorname{tr} L_{u \bullet v}$ of the algebra.

- metrised algebras at different points isomorphic by parallelism condition
- recovery of centro-affine immersion as integral manifold of the 1-form $\zeta = \sigma(x^{-1}, \cdot)$

classification of proper affine hyperspheres with $\hat{\nabla} C = 0$ equivalent to classification of semi-simple metrised Jordan algebras

Proper affine spheres with $\hat{\nabla} C = 0$

Vector space	Real dimension	Range	Φ	ω	Affine sphere
$\mathbb{C}$	2		$Re(c \log x)$	$\|x\|^2$	$\|x\| = const$
$\mathbb{C}^m$	$2m$	$m \geq 3$	$Re(c \log x^T x)$	$\|x^T x\|^m$	$\|x^T x\| = const$
$S_m(\mathbb{C})$	$m(m+1)$	$m \geq 3$	$Re(c \log \det A)$	$\|\det A\|^{m+1}$	$\|\det A\| = const$
$M_m(\mathbb{C})$	$2m^2$	$m \geq 3$	$Re(c \log \det A)$	$\|\det A\|^{2m}$	$\|\det A\| = const$
$A_{2m}(\mathbb{C})$	$2m(2m-1)$	$m \geq 3$	$Re(c \log \mathrm{pf}\, A)$	$\|\mathrm{pf}\, A\|^{2(2m-1)}$	$\|\mathrm{pf}\, A\| = const$
$H_3(O, \mathbb{C})$	54		$Re(c \log \det A)$	$\|\det A\|^{18}$	$\|\det A\| = const$
$\mathbb{R}$	1		$\alpha \log \|x\|$	$\|x\|$	Point
$\mathbb{R}^m$	m	$m \geq 3$	$\alpha \log \|x^T Q x\|$	$\|x^T Q x\|^{m/2}$	Quadric
$M_m(\mathbb{R})$	m^2	$m \geq 3$	$\alpha \log \|\det A\|$	$\|\det A\|^m$	$\det A = const$
$M_m(\mathbb{H})$	$4m^2$	$m \geq 2$	$\alpha \log \det S$	$(\det S)^{2m}$	$\det S = const$
$S_m(\mathbb{R})$	$\frac{m(m+1)}{2}$	$m \geq 3$	$\alpha \log \|\det A\|$	$\|\det A\|^{(m+1)/2}$	$\det A = const$
$H_m(\mathbb{C})$	m^2	$m \geq 3$	$\alpha \log \|\det A\|$	$\|\det A\|^m$	$\det A = const$
$H_m(\mathbb{H})$	$m(2m-1)$	$m \geq 3$	$\alpha \log \det S$	$(\det S)^{m-1/2}$	$\det S = const$
$A_{2m}(\mathbb{R})$	$m(2m-1)$	$m \geq 3$	$\alpha \log \|\mathrm{pf}\, A\|$	$\|\mathrm{pf}\, A\|^{2m-1}$	$\mathrm{pf}\, A = const$
$SH_m(\mathbb{H})$	$m(2m+1)$	$m \geq 2$	$\alpha \log \det S$	$(\det S)^{m+1/2}$	$\det S = const$
$H_3(\mathbb{O})$	27		$\alpha \log \|\det A\|$	$\|\det A\|^9$	$\det A = const$
$H_3(O, \mathbb{R})$	27		$\alpha \log \|\det A\|$	$\|\det A\|^9$	$\det A = const$

Other parallelism conditions

commutative metrised algebras associated to different classes of equiaffine hypersurface immersions with parallelism conditions

	Blaschke	graph	centro-affine
$\nabla C = 0$	nilpotent L_u	general	quadratic factor
$\nabla K = 0$	nilpotent associative	associative	quadratic factor
$\hat{\nabla} C = 0$	semi-simple Jordan / nilpotent Jordan	Jordan	unital Jordan

Thank you