An optimal control problem with control in a disc

MSIAM2 Modelling seminar project

Supervisor: Roland Hildebrand

We consider the optimal control problem

$$\min \frac{1}{2}\int_0^\infty \|x\|^2\,dt: \quad x(0) = x_0 \in \mathbb{R}^2,\ \dot{x}(0) = y(0) = y_0,\ \ddot{x} = \dot{y} = u \in U,$$

where U is the set of admissible controls, equal to the unit disc in the plane. The problem is a model for a certain kind of singular point in the theory of non-smooth Hamiltonian systems.

Let $B(x_0, y_0)$ be the optimal value of the problem in dependence on the initial point. The function $B(x, y)$ is called the *Bellman function*. It is the solution of the *Bellman equation*

$$\min_{u \in U}\left(\frac{\partial B}{\partial x}y + \frac{\partial B}{\partial y}u\right) = -\frac{1}{2}\|x\|^2,$$

a non-smooth PDE which cannot be integrated analytically.

Numeric computation is also difficult due to the inherent instability of the underlying Hamiltonian dynamical system. The optimal control in general runs an infinite number of times around the boundary of the control set U before the trajectory comes to rest at the origin.

In order to study the solution of the problem we need to know how it looks like approximately. To this end we wish to find estimates on the Bellman function from below and from above by different means. The goal of the project is to derive several such estimates:

Modification of the control set: If the control set U were a square, or more generally a rectangle R, then the control problem would decompose into independent problems on the components of the vectors x, y which are parallel to the edges of R. In this case the problem could be solved explicitly, and the corresponding Bellman function B_R could be described analytically.

Now if $U \subset R$, then the feasible set of the simpler decomposable problem with control set R is larger than the feasible set of the original problem, and its optimal value is correspondingly smaller. In this way we obtain an analytic lower bound on the Bellman function B. The best such bounds are obtained if R is chosen as a square Q centered on the origin and with edge length 2, such that U just fits inside Q. The corresponding lower bound $B_Q(x, y)$ on $B(x, y)$ for given (x, y) can be improved if the maximum of $B_Q(x, y)$ over all images Q obtained by rotation of the square are taken.

In a similar way, an upper bound on B can be obtained by the minimum over B_Q, where Q runs through all squares centered on the origin with edge length $\sqrt{2}$, or more generally all centrally symmetric rectangles barely contained in U.

The goal of this part of the project is to compute and compare the upper and lower bounds on B.

Modification of the objective: Another approach is to replace the objective function by another one leading to a simpler dynamical system. The standard choice is time-optimality, i.e., we consider the optimal control system

$$\min \int_0^\infty 1\,dt: \quad x(0) = x_0 \in \mathbb{R}^2,\ \dot{x}(0) = y(0) = y_0,\ \ddot{x} = \dot{y} = u \in U.$$

This system can be partially integrated, allowing a precise computation of the corresponding optimal control $\hat{u}_T(x, y) \in U$ as a function of the current state.

For the original system this control $\hat{u}_T$ leads to a feasible, but sub-optimal trajectory. The corresponding objective value will hence be an upper bound on the Bellman function B.

The goal of this part of the project will be to compute the bound analytically to the extent possible, and otherwise to obtain it numerically.

The project will allow the students to become acquainted with optimal control theory.