4 Linear programs

4.1 Standard form

Consider the problem of minimization of a linear objective function under linear equality and inequality constraints,

$$\min_{x\in\mathbb{R}^n} c^T x \ : \qquad Ax = b,\ Cx \le d.$$

Let m be the number of inequalities. We introduce an additional variable $y \in \mathbb{R}^m$ and reformulate the problem as

$$\min_{x,y} \langle (c, \mathbf{0}), (x, y)\rangle \ : \qquad \begin{pmatrix} A & 0 \\ C & I \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}, \quad y \in \mathbb{R}^m_+.$$

The *slack variable* y allowed to turn the inequalities into equalities, at the cost of introducing the *conic constraint* $y \in \mathbb{R}^m_+$.

The variable x can now be eliminated using the linear equality constraints. Indeed, if the kernel L of the matrix $\begin{pmatrix} A \\ C \end{pmatrix}$ is trivial, then there exist n equalities which determine x completely as linear functions of y.

If the kernel of this matrix is non-trivial, then x has degrees of freedom which are not determined by the equality constraints. For every $p \in L$ we have that $(x + p, y)$ is a feasible point whenever (x, y) is feasible. If the linear functional c does not vanish on the kernel identically, then the problem is either infeasible or unbounded. If c vanishes on the kernel, then these degrees of freedom are redundant for the problem.

We can hence assume the linear program (LP) in the following *standard form*:

$$\min_{x\ge 0} c^T x \ : \qquad Ax = b. \tag{1}$$

Without loss of generality A can be assumed of full row rank, otherwise the problem is infeasible or equality constraints are redundant. We shall also assume that $b \ge 0$, by possibly multiplying some rows of A by -1.

4.2 Duality

Consider the LP in standard form

$$\min_{x\ge 0} c^T x \ : \qquad Ax = b \tag{2}$$

with the matrix A being of full row rank of size $m \times n$. Let now $s \in \mathbb{R}^m$ such that $c \ge A^T s$. We claim that the optimal value of (2) is bounded from *below* by the quantity $s^T b$.

Indeed, we have for every feasible x that

$$s^T b = s^T Ax \le c^T x. \tag{3}$$

In this way s provides a *certificate* that the optimal value of LP (2) is not below a certain value. It is then natural to ask what the *best* such certificate is and which value it provides. We may formulate this question as another optimization problem:

$$\max_s b^T s \ : \qquad A^T s \le c.$$

Observe that this problem is also a linear program. It is called the *dual* LP, in contrast to (2) which is called the *primal*.

The dual program is hence a *maximization* problem, and every feasible point for the dual problem yields a *lower bound* on the objective value of the primal problem. Vice versa, from (3) it also follows that every feasible point for the primal problem yields an *upper bound* on the objective value of the dual problem.

Introduce the slack variable $p \ge 0$ and reformulate the dual problem as

$$\max_{p\ge 0, s} b^T s \ : \qquad p + A^T s = c. \tag{4}$$

If (x, p) is a primal-dual feasible pair, then the difference of the respective objective values is given by

$$c^T x - b^T s = c^T x - x^T A^T s = c^T x - x^T (c - p) = \langle p, x\rangle.$$

Hence the *complementarity* condition $p^T x = 0$ implies that both x, p are optimal solutions of the respective LP. This condition can equivalently be written as $x_i p_i = 0$ for all $i = 1, \dots, n$ or for every i, $x_i = 0$ or $p_i = 0$.

The dual points p and the objective functional c can be thought of as elements of the dual space to $\mathbb{R}^n$.

We now come to the main result in the duality theory of linear programs.

Theorem 4.1. *(Strong duality for LP) If both problems* (2) *and* (4) *are feasible, then their objective values coincide and are attained at a complementary primal-dual pair of feasible points.*

Proof. If the dual problem is feasible, then the optimal value of the primal problem is lower bounded. Since the primal is also feasible, its optimal value is finite. Likewise, the dual optimal value is finite. Let v^* be the primal optimal value.

Let now $v \in \mathbb{R}$ be arbitrary and suppose that the half-space $\{x \mid c^T x \leq v\}$ has an empty intersection with the non-empty polyhedron $C = \{x \mid -x \leq 0,\ Ax = b\}$. By the Theorem on the Alternative there exist $\lambda_0, \lambda \geq 0$, μ such that $\lambda_0 c - \lambda^T I - \mu^T A = 0$, $\lambda_0 v - \mu^T b = -1$. If $\lambda_0 = 0$, then $C = \emptyset$, leading to a contradiction. Hence $\lambda_0 > 0$. Set $s = \frac{\mu}{\lambda_0}$. Then the relations can be rewritten as $c - s^T A \geq 0$, $v + \frac{1}{\lambda_0} = s^T b$. But this means that (4) has a feasible solution with value $v + \lambda_0^{-1} > v$.

Therefore for every $v < v^*$ program (4) has a feasible solution with value $> v$. On the other hand, the optimal value of (4) is upper bounded by any number strictly greater than v^*. Hence the optimal value of (4) equals v^*. If LP (2) does not attain its optimal value, then we may set $v = v^*$ and there exists a dual feasible point with value $> v^*$, a contradiction. Hence (2) attains its optimum. In a similar manner, (4) attains its optimum.

Let x^*, s^* be the optimal solutions, and set $p^* = c - A^T s^*$. Then

$$0 = v^* - v^* = c^T x^* - b^T s^* = (x^*)^T p^*,$$

and x^*, p^* are complementary. □

If (4) is unbounded, then (2) must be infeasible, and when (2) is unbounded, then (4) must be infeasible. The converse is not true, however, as the following example shows.

Example: Consider the LP

$$\min_{x=(x_1,x_2)^T \geq 0} -x_2 \ : \qquad x_1 = -1.$$

Clearly this program is infeasible. The dual can be written as

$$\max_s (-s) \ : \qquad (s, 0) \leq (0, -1),$$

which is also infeasible.

4.3 Simplex algorithm

The simplex algorithm is the oldest algorithm to solve linear programs. Its appearance marked the advent of operations research as a discipline.

The feasible set of the LP is a polyhedron. The optimal value of the objective, if it exists, is attained at a vertex of the polyhedron. The simplex algorithm goes along the edges of the polyhedron from vertex to vertex, while decreasing monotonically the value of the objective function.

Consider an LP in standard form (1). Let m be the number of equality constraints and $n \geq m$ the number of variables. Then the feasible set C is the intersection of the polyhedron $\mathbb{R}^n_+$ with an affine subspace of dimension $n - m$. Since the coefficient matrix $A \in \mathbb{R}^{m \times n}$ has full row rank, there exists an invertible $m \times m$ submatrix of A. Note that the columns of A are associated with the variables x_i of the problem, while the rows are associated with the equality constraints.

Definition 4.2. A set $B \subset \{1, \dots, n\}$ of column indices of cardinality m is called *basic* if the corresponding submatrix A_B of A is invertible. The complementary index set $\bar{B}$ is called *non-basic*.

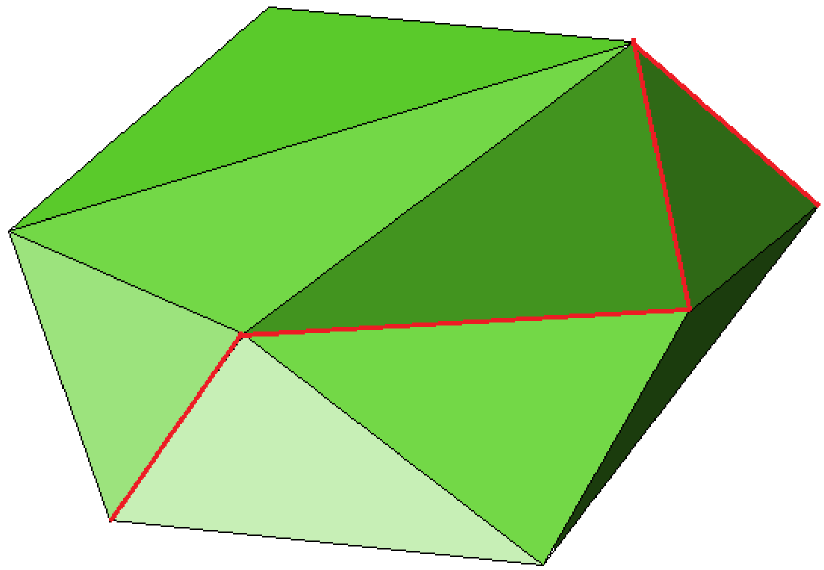

Figure 1: The simplex algorithm jumps from vertex to vertex along the edges of the polyhedron.

Division of the column indices $\{1, \dots, n\}$ into a basic set B and a non-basic set $\bar{B}$ induces a division of the vector $x \in \mathbb{R}^n$ of variables into a subvector $x_B \in \mathbb{R}^m$ of basic variables and a subvector $x_{\bar{B}} \in \mathbb{R}^{n-m}$ of non-basic variables. The equality constraints can then be written in the form

$$A_B x_B + A_{\bar{B}} x_{\bar{B}} = b \quad \Leftrightarrow \quad x_B = A_B^{-1} b - A_B^{-1} A_{\bar{B}} x_{\bar{B}}. \tag{5}$$

This means the basic variables can be expressed as an affine function of the non-basic variables. The whole feasible set C of the problem then projects bijectively to a polyhedron $C_{\bar{B}} \subset \mathbb{R}_+^{n-m}$ in the space of non-basic variables (recall that $x_{\bar{B}} \geq 0$ for every feasible $x \in C$).

Definition 4.3. The basic index set B is called *feasible* if $x_{\bar{B}} = 0$ is the image of a feasible point $x \in C$.

Hence the inequality constraints involving the non-basic variables are active at the feasible point represented by a feasible basic set. The constraints at the basic variables may be active, but do not need to be. Clearly this feasible point must be a vertex of the polyhedron C. Hence every feasible basic index set defines a vertex of C. On the other hand, every vertex of C can be obtained from a feasible basic set. Indeed, if v is extremal in C, the columns $\{A_i \,|\, v_i > 0\}$ are linearly independent, otherwise we find $\delta \in \mathbb{R}^n$ with support contained in the support of v such that $v \pm \delta \in C$. Complete the set of indices $\{i \,|\, v_i > 0\}$ to a set B of cardinality m such that $\{A_i \,|\, i \in B\}$ is a basis of the column space of A. Then B is basic feasible.

Different basic feasible sets can yield, however, the same vertex of C.

Given a basic index set B, the feasible set C of the problem can be recovered from knowledge of the matrix $M = A_B^{-1} A_{\bar{B}}$ and the vector $\mu = A_B^{-1} b$, because the equality constraints (5) can be written as

$$x_B + M x_{\bar{B}} = \mu.$$

For a basic feasible set, the vector μ contains the elements of x_B in the corresponding vertex of C. We also get that a basic set B is feasible if and only if $\mu \geq 0$.

Recall that the whole $(n - m)$-dimensional affine hull of the feasible set C can be parameterized by the non-basic variables $x_{\bar{B}}$. Hence the objective function can also be expressed as an affine function of $x_{\bar{B}}$ only, namely

$$\langle c, x \rangle = c_B^T x_B + c_{\bar{B}}^T x_{\bar{B}} = c_B^T (\mu - M x_{\bar{B}}) + c_{\bar{B}}^T x_{\bar{B}} = (c_{\bar{B}} - M^T c_B)^T x_{\bar{B}} + c_B^T \mu = \xi^T x_{\bar{B}} + \gamma.$$

Here $c_B, c_{\bar{B}}$ are the basic and non-basic subvectors of c, and $\xi = c_{\bar{B}} - M^T c_B$, $\gamma = c_B^T \mu$. Note that γ equals the value of the cost function at the vertex of C which is represented by the basic feasible set B.

If $\xi \geq 0$, then the objective is not smaller than γ at every feasible point $x \in C$. Therefore B represents the optimal solution of the program if and only if $\mu \geq 0$, $\xi \geq 0$.

Given a basic set B and its complement $\bar{B}$, the whole information on the linear program can be coded by the *simplex table*

$-\gamma$	ξ^T
μ	M

Namely, the table corresponds to the program

$$\min_{(x_B, x_{\bar{B}}) \geq 0} \left(\langle \xi, x_{\bar{B}} \rangle + \gamma \right) : \qquad x_B + M x_{\bar{B}} = \mu. \tag{6}$$

The simplex table is called *feasible* if $\mu \geq 0$, and it is called *optimal* if in addition $\xi \geq 0$.

Changing one index in the basic set by dropping a basic index and including a non-basic one leads to either staying at the same vertex or moving along an edge of C to a new vertex. Let us consider how the simplex table changes when performing such an operation. Let $i \in B$, $j \in \bar{B}$, and define $\tilde{B} = B \setminus \{i\}$, $\tilde{N} = \bar{B} \setminus \{j\}$. Recall that the entries of μ and the rows of M are indexed by B, while the entries of ξ and the columns of M are indexed by $\bar{B}$. We have

$$x_{\tilde{B}} + M_{\tilde{B}\bar{B}} x_{\bar{B}} = x_{\tilde{B}} + M_{\tilde{B}\tilde{N}} x_{\tilde{N}} + M_{\tilde{B}j} x_j = \mu_{\tilde{B}}, \qquad x_i + M_{i\bar{B}} x_{\bar{B}} = x_i + M_{i\tilde{N}} x_{\tilde{N}} + M_{ij} x_j = \mu_i.$$

In order to obtain the new table, we have to express the variables $x_{\tilde{B}}, x_j$ as a function of $x_{\tilde{N}}, x_i$. Clearly this is possible only if $M_{ij} \neq 0$ and yields

$$x_j + M_{ij}^{-1} M_{i\tilde{N}} x_{\tilde{N}} + M_{ij}^{-1} x_i = M_{ij}^{-1} \mu_i.$$

Inserting this expression for x_j into the equation for $x_{\tilde{B}}$, we get

$$\begin{aligned} & x_{\tilde{B}} + M_{\tilde{B}\tilde{N}} x_{\tilde{N}} + M_{\tilde{B}j} \left(M_{ij}^{-1} \mu_i - M_{ij}^{-1} x_i - M_{ij}^{-1} M_{i\tilde{N}} x_{\tilde{N}} \right) \\ =& x_{\tilde{B}} + \left(M_{\tilde{B}\tilde{N}} - M_{ij}^{-1} M_{\tilde{B}j} M_{i\tilde{N}} \right) x_{\tilde{N}} - M_{ij}^{-1} M_{\tilde{B}j} x_i + M_{ij}^{-1} M_{\tilde{B}j} \mu_i = \mu_{\tilde{B}}, \\ \Rightarrow \qquad & x_{\tilde{B}} + \left(M_{\tilde{B}\tilde{N}} - M_{ij}^{-1} M_{\tilde{B}j} M_{i\tilde{N}} \right) x_{\tilde{N}} - M_{ij}^{-1} M_{\tilde{B}j} x_i = \mu_{\tilde{B}} - M_{ij}^{-1} M_{\tilde{B}j} \mu_i. \end{aligned}$$

The objective function is given by

$$\begin{aligned} \langle \xi, x_{\bar{B}} \rangle + \gamma &= \xi_{\tilde{N}}^T x_{\tilde{N}} + \xi_j x_j + \gamma = \xi_{\tilde{N}}^T x_{\tilde{N}} + \xi_j \left(M_{ij}^{-1} \mu_i - M_{ij}^{-1} x_i - M_{ij}^{-1} M_{i\tilde{N}} x_{\tilde{N}} \right) + \gamma \\ &= \left(\xi_{\tilde{N}}^T - M_{ij}^{-1} \xi_j M_{i\tilde{N}} \right) x_{\tilde{N}} - M_{ij}^{-1} \xi_j x_i + \gamma + M_{ij}^{-1} \xi_j \mu_i. \end{aligned}$$

Thus the transformation of the table is given by

$$\begin{bmatrix} i & \leftarrow & j \\ j & \leftarrow & i \\ \mu_{\tilde{B}} & \leftarrow & \mu_{\tilde{B}} - M_{ij}^{-1} M_{\tilde{B}j} \mu_i \\ \mu_i & \leftarrow & M_{ij}^{-1} \mu_i \\ \xi_{\tilde{N}} & \leftarrow & \xi_{\tilde{N}} - M_{ij}^{-1} \xi_j M_{i\tilde{N}}^T \\ \xi_j & \leftarrow & -M_{ij}^{-1} \xi_j \\ -\gamma & \leftarrow & -\gamma - M_{ij}^{-1} \xi_j \mu_i \\ M_{\tilde{B}\tilde{N}} & \leftarrow & M_{\tilde{B}\tilde{N}} - M_{ij}^{-1} M_{\tilde{B}j} M_{i\tilde{N}} \\ M_{\tilde{B}j} & \leftarrow & -M_{ij}^{-1} M_{\tilde{B}j} \\ M_{i\tilde{N}} & \leftarrow & M_{ij}^{-1} M_{i\tilde{N}} \\ M_{ij} & \leftarrow & M_{ij}^{-1} \end{bmatrix}. \tag{7}$$

The simplex algorithm then evolves the basic set of indices by changing it one by one, while ensuring that the set always corresponds to a vertex of C and the cost function decreases monotonically. If $\mu_i > 0$, then in order for the simplex table to stay feasible we must have $M_{ij} > 0$ and $\mu_i^{-1}\mu_{\bar{B}} \geq M_{ij}^{-1}M_{\bar{B}j}$. For fixed j this essentially determines the index i. Namely, among all $i \in B$ such that $M_{ij} > 0$ it has to be the one which minimizes the ratio $\frac{\mu_i}{M_{ij}}$. Further, the cost function at the represented vertex decreases if $M_{ij}^{-1}\xi_j\mu_i \leq 0$, i.e., if $\xi_j \leq 0$. Therefore the index j is chosen from those indices satisfying $\xi_j < 0$. If no such index is available, then the simplex table is already optimal. If for given j we have $M_{ij} \leq 0$ for all i, then the problem is unbounded.

The algorithm terminates in a finite number of steps, by either

- finding that a solution cannot be improved and is optimal, or
- finding an edge with decreasing cost function that recedes to infinity, in which case the LP is unbounded.

The algorithm needs a starting vertex to commence. Such a vertex is found by solving the auxiliary LP

$$\min_{x\geq 0, z\geq 0} \mathbf{1}^T z \ : \qquad Ax + z = b.$$

Let C' be the feasible set of the auxiliary problem. Then the feasible set of the original problem is given by $C = \{x \,|\, (x, 0) \in C'\}$. In case $C \neq \emptyset$ the optimal value of the auxiliary problem equals zero, and every optimal vertex of C' corresponds to a vertex of C. In the case $C = \emptyset$ the optimal value of the auxiliary problem is strictly positive. We then launch the simplex algorithm on the auxiliary problem with the set of basic variables z. This set corresponds to the vertex $(0, b)$ of C' (recall that $b \geq 0$).

Note that the objective function of the auxiliary problem can be written as $\mathbf{1}^T(b - Ax)$. Hence we start the so-called *phase 1* with the simplex table

$-\mathbf{1}^T b$	$-\mathbf{1}^T A$
0	c^T
b	A

The second row is included to keep track of the evolution of the original objective as aa function of the non-basic variables. After reaching optimality, the value of the table is inspected. If it is strictly positive, then the original problem is infeasible. If it is zero, then all auxiliary variables z can be made non-basic and the corresponding columns removed from the table, along with the auxiliary first row.

Although the simplex method works very well in practice, its worst-case performance is exponential in the number of variables in standard form, i.e., in the number of inequality constraints.

The simplex algorithm is a representative of the class of *active set methods*. Such methods keep track of a subset of inequality constraints which are active at the current iterate. These constraints then yield additional equalities which help to determine the current point.

Quadratic programs with convex objective can be solved by a similar method, the *Beale algorithm*. Here the number of indices corresponding to active constraints can vary, and the algorithm jumps from one face of the feasible polyhedron to an adjacent one.

4.4 Simplex method and duality

Let us write the dual program to the linear program (6). We obtain

$$\max_{p\geq 0, s} \mu^T s + \gamma : \qquad p + \begin{pmatrix} I \\ M^T \end{pmatrix} s = \begin{pmatrix} 0 \\ \xi \end{pmatrix}.$$

Dividing the vector p of dual variables into subvectors $p_B, p_{\bar{B}}$ according to the division of x, we get

$$\max_{p\geq 0, s} \mu^T s + \gamma : \qquad p_B + s = 0, \quad p_{\bar{B}} + M^T s = \xi.$$

Finally, eliminating $s = -p_B$ and turning the maximization problem into a minimization problem we get

$$\min_{p\geq 0} \mu^T p_B - \gamma : \qquad p_{\bar{B}} - M^T p_B = \xi.$$

Interpreting B as the set of *non-basic* dual variables and $\bar{B}$ as the set of *basic* dual variables, we see that this LP corresponds to the simplex table

γ	μ^T
ξ	$-M^T$

Thus the dual table equals the primal table, which corresponds to LP (6), transposed with the diagonal blocks multiplied by -1. Every transformation (7) of the primal table can hence be interpreted also as a transformation of the dual table.

Note that the dual table is feasible if and only if $\xi \geq 0$. In this case we shall say that the primal table is *dual feasible*. A simplex transformation on a feasible dual table can be interpreted as a *dual simplex* step on the primal table. An algorithm performing dual simplex steps is called *dual simplex method*.

The (dual) simplex method is particularly useful if a sequence of slightly differing LPs has to be solved. When changing some elements of the objective function, an optimal simplex table stays feasible, and has to be returned to optimality by primal simplex steps. On the contrary, if some elements of the right-hand side of the constraints are changed, an optimal table stays dual feasible and has to be returned to primal feasibility (and hence to optimality) by dual simplex steps.

4.5 Application: mixed integer linear programs

A mixed integer linear program (MILP) is an LP with additional integrality constraints on a part of the decision variables:

$$\min_{x \geq 0} \langle c, x \rangle : \qquad Ax = b, \quad x_i \in \mathbb{Z} \quad \forall\, i \in I,$$

where I is a subset of indices. Such a problem is non-convex and actually in general NP-hard.

By removing the integrality constraints we obtain the *linear relaxation* of the program, namely the LP

$$\min_{x \geq 0} \langle c, x \rangle : \qquad Ax = b. \tag{8}$$

Since the feasible set of the LP is larger than that of the original MILP, its optimal value is a lower bound on the value of the MILP. Let x^* be the solution of this LP.

If the subvector x_I^* of the solution happens to be integral, then x^* is feasible for the MILP and hence yields its optimal solution. However, in general there exists an index $i \in I$ such that x_i^* is fractional. Consider the two linear programs

$$\min_{x \geq 0} \langle c, x \rangle : \qquad Ax = b, \quad x_i \leq \lfloor x_i^* \rfloor, \tag{9}$$

$$\min_{x \geq 0} \langle c, x \rangle : \qquad Ax = b, \quad x_i \geq \lceil x_i^* \rceil. \tag{10}$$

The feasible sets of LPs (9),(10) are disjoint, but their union contains the feasible set of the original MILP. On the other hand, neither of the feasible sets of LPs (9),(10) contains the solution x^* of the original linear relaxation (8). Hence the minimum of the two values of LPs (9),(10) is a better lower bound on the optimal value of the MILP than the optimal value of LP (8).

The process of splitting LP (8) into two stronger LPs (9),(10) by constraining one of the integer variables is called *branching*. MILP solvers proceed by recursively splitting the feasible set of the MILP into smaller parts by branching on integer variables whose value happened to be fractional in the solutions of the relaxations. Since the LP relaxations yield bounds on the value of the original MILP, the whole algorithm is called *branch-and-bound*.

Modern MILP solvers usually bring forward additional features strengthening the LP relaxations, such as presolve algorithms tightening the bounds on the integer variables or cuts separating fractional solutions from the feasible set of the MILP.

We shall stress one property which makes the dual simplex method a particularly useful method for solving the LP relaxations appearing in the course of the branch-and-bound algorithm.

Suppose we use the simplex method to solve LP (8) and obtained an optimal simplex table. LPs (9),(10) differ from (8) by the addition of one constraint. Introducing corresponding slack variables, we see that at the optimal point x^* of LP (8) these variables are negative, because x^* is not feasible for LPs (9),(10). Hence these slacks enter the basic set of variables. By adding a new row to the table corresponding to the slack, the table remains dual feasible, but loses primal feasibility at just one entry, which will be contained in the interval $(-1, 0)$. It will therefore in general take only a few dual simplex iterations to return the table to optimality and thus to solve LPs (9),(10).

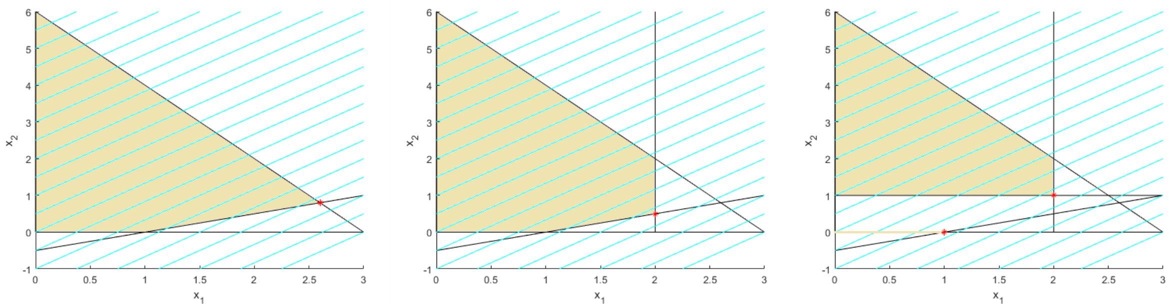

Figure 2: Feasible region and optimal solution for LP (11) (left), LP (12) (center), and LPs (13),(14) (right).

During a general branching step, the constraint on the integer variable defining the branching will not be added, but merely tightened. This corresponds to changing the value of one entry in the vector b of the table, after which again the dual simplex method can be started to return the table to optimality.

Example: Consider the MILP

$$\min_{x\in\mathbb{R}^2_+} (3x_2 - 4x_1): \qquad x_1 - 2x_2 \le 1,\ 2x_1 + x_2 \le 6,\ x \in \mathbb{Z}^2.$$

Its first LP relaxation is given by

$$\min_{x\in\mathbb{R}^2_+} (3x_2 - 4x_1): \qquad x_1 - 2x_2 \le 1,\ 2x_1 + x_2 \le 6, \tag{11}$$

with optimal solution $x^* = (\frac{13}{5}, \frac{4}{5})$ and value -8.

This solution is not integer, and both integer variables have fractional values. We may thus branch on either variable. Choosing the variable x_1, we obtain the infeasible LP

$$\min_{x\in\mathbb{R}^2_+} (3x_2 - 4x_1): \qquad x_1 - 2x_2 \le 1,\ 2x_1 + x_2 \le 6,\ x_1 \ge 3$$

and the LP

$$\min_{x\in\mathbb{R}^2_+} (3x_2 - 4x_1): \qquad x_1 - 2x_2 \le 1,\ 2x_1 + x_2 \le 6,\ x_1 \le 2, \tag{12}$$

whose solution is given by $x^* = (2, \frac{1}{2})$ with value $-\frac{13}{2}$. Thus the lower bound on the optimal value of the MILP has improved from -8 to $\min(+\infty, -\frac{13}{2}) = -\frac{13}{2}$.

We need to pursue only the branch defined by the second LP (12). Its solution has only one fractional variable x_2, with value $\frac{1}{2}$, branching on which yields the two LPs

$$\min_{x\in\mathbb{R}^2_+} (3x_2 - 4x_1): \qquad x_1 - 2x_2 \le 1,\ 2x_1 + x_2 \le 6,\ x_1 \le 2,\ x_2 \ge 1, \tag{13}$$

$$\min_{x\in\mathbb{R}^2_+} (3x_2 - 4x_1): \qquad x_1 - 2x_2 \le 1,\ 2x_1 + x_2 \le 6,\ x_1 \le 2,\ x_2 \le 0. \tag{14}$$

Both LPs yields integer solutions, namely $(2, 1)$ and $(1, 0)$, with values -5 and -4, respectively.

Thus the optimal value of the MILP is the lower of these values, namely -5, and the corresponding solution is $(2, 1)$.

Exercise: Construct the simplex tables for these LPs and solve them by the dual simplex method.