A partial differential equation characterizing determinants of symmetric cones

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Outline

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Regular convex cones

Definition

A regular convex cone $K \subset \mathbb{R}^n$ is a closed convex cone having nonempty interior and containing no lines.

let $\langle \cdot, \cdot \rangle$ be a scalar product on $\mathbb{R}^n$

$$K^* = \{p \in \mathbb{R}_n \,|\, \langle x, p \rangle \geq 0 \quad \forall\, x \in K\}$$

is called the dual cone

Symmetric cones

Definition

A regular convex cone $K \subset \mathbb{R}^n$ is called self-dual if there exists a scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ such that $K = K^*$.

Definition

A regular convex cone $K \subset \mathbb{R}^n$ is called homogeneous if the automorphism group $\mathrm{Aut}(K)$ acts transitively on K^o.

Definition

A self-dual, homogeneous regular convex cone is called symmetric.

Jordan algebras

an algebra A is a vector space V equipped with a bilinear operation $\bullet : V \times V \to V$

Definition

An algebra J is a Jordan algebra if

- $x \bullet y = y \bullet x$ for all $x, y \in J$ (commutativity)
- $x^2 \bullet (x \bullet y) = x \bullet (x^2 \bullet y)$ for all $x, y \in J$ (Jordan identity)

where $x^2 = x \bullet x$.

Definition

A Jordan algebra is formally real or Euclidean if $\sum_{k=1}^m x_k^2 = 0$ implies $x_k = 0$ for all k, m.

Examples

let Q be a real symmetric matrix and $e \in \mathbb{R}^n$ such that $e^T Q e = 1$

the quadratic factor $\mathcal{J}_n(Q)$ is the space $\mathbb{R}^n$ equipped with the multiplication

$$x \bullet y = e^T Qx \cdot y + e^T Qy \cdot x - x^T Qy \cdot e$$

let $\mathcal{H}$ be an algebra of Hermitian matrices over a real coordinate algebra $(\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O})$

then the corresponding Hermitian Jordan algebra is the vector space underlying $\mathcal{H}$ equipped with the multiplication

$$A \bullet B = \frac{AB + BA}{2}$$

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Classification of Euclidean Jordan algebras

Theorem (Jordan, von Neumann, Wigner 1934)

Every Euclidean Jordan algebra is a direct product of a finite number of Jordan algebras of the following types:

- *quadratic factor with matrix* Q *of signature* $+ - \cdots -$
- *real symmetric matrices*
- *complex Hermitian matrices*
- *quaternionic Hermitian matrices*
- *octonionic Hermitian* 3×3 *matrices*

Classification of symmetric cones

Theorem (Vinberg, 1960; Koecher, 1962)

The symmetric cones are exactly the cones of squares of Euclidean Jordan algebras, $K = \{x^2 \,|\, x \in J\}$.
Every symmetric cone can be hence represented as a direct product of a finite number of the following irreducible symmetric cones:

- *Lorentz (or second order) cone* $L_n = \left\{(x_0, \dots, x_{n-1}) \,|\, x_0 \geq \sqrt{x_1^2 + \dots + x_{n-1}^2}\right\}$
- *matrix cones* $S_+(n)$, $H_+(n)$, $Q_+(n)$ *of real, complex, or quaternionic hermitian positive semi-definite matrices*
- *Albert cone* $O_+(3)$ *of octonionic hermitian positive semi-definite* 3×3 *matrices*

Outline

Unital and simple Jordan algebras

Definition

A Jordan algebra is called unital if it possesses a unit element e, satisfying $u \bullet e = u$ for all $u \in J$.

Definition

A Jordan algebra is called simple if it is not nil and has no non-trivial ideal.

Theorem (Jordan, von Neumann, Wigner 1934)

Euclidean Jordan algebras are unital and decompose in a unique way into a direct product of simple Jordan algebras.

Exponential map

define recursively $u^{m+1} = u \bullet u^m$
with $u^0 = e$, define the exponential map

$$\exp(u) = \sum_{k=0}^{\infty} \frac{u^k}{k!}$$

Theorem (Köcher)

Let J be a Euclidean Jordan algebra and K its cone of squares. Then the exponential map is injective and its image is the interior of K,

$$\exp[J] = K^o.$$

Logarithm

let J be a Euclidean Jordan algebra with cone of squares K

then we can define the logarithm

$$\log : K^o \to J$$

as the inverse of the exponential map

Definition

Definition

Let J be a Jordan algebra. A symmetric bilinear form γ on J is called trace form if $\gamma(u, v \bullet w) = \gamma(u \bullet v, w)$ for all $u, v, w \in J$.

Generic minimum polynomial

for every u in a unital Jordan algebra there exists m such that

- $u^0, u^1, \dots, u^{m-1}$ are linearly independent
- $u^m = \sigma_1 u^{m-1} - \sigma_2 u^{m-2} + \cdots - (-1)^m \sigma_m u^0$

$p_u(\lambda) = \lambda^m - \sigma_1 \lambda^{m-1} + \cdots + (-1)^m \sigma_m$ is the minimum polynomial of u

Theorem (Jacobson, 1963)

There exists a unique minimal polynomial $p(\lambda) = \lambda^m - \sigma_1(u)\lambda^{m-1} + \cdots + (-1)^m \sigma_m(u)$, the generic minimum polynomial, such that $p_u | p$ for all u. The coefficient $\sigma_k(u)$ is homogeneous of degree k in u. The coefficient $t(u) = \sigma_1(u)$ is called generic trace and the coefficient $n(u) = \sigma_m(u)$ the generic norm.

Generic bilinear trace form

Theorem (Jacobson)

Let J be a unital Jordan algebra. The symmetric bilinear form

$$\tau(u,v) = t(u \bullet v)$$

is a trace form, called the generic bilinear trace form.

for Euclidean Jordan algebras with cone of squares K we have

$$\log n(x) = t(\log x) = \tau(e, \log x)$$

for all $x \in K^o$

Euclidean Jordan algebras

Theorem (Köcher)

Let J be a unital real Jordan algebra. Then the following conditions are equivalent.

- *J is Euclidean*
- *there exists a positive definite trace form γ on J.*

if J is a simple Euclidean Jordan algebra, then any non-degenerate trace form γ on J is proportional to the generic bilinear trace form τ

hence $\gamma(e, \log x)$ is proportional to $\log n(x)$

Outline

1. Symmetric cones
   - Geometric characterization
   - Algebraic characterization
2. Jordan algebras
   - Exponential and logarithm
   - Trace forms and determinant
3. The partial differential equation
   - Hessian metrics
   - The PDE
   - Connection with Jordan algebras

Notation for derivatives

let $F : U \to \mathbb{R}$ be a smooth function on $U \subset \mathbb{A}^n$, where $\mathbb{A}^n$ is the n-dimensional affine real space

we note $\frac{\partial F}{\partial x^\alpha} = F_{,\alpha}$, $\frac{\partial^2 F}{\partial x^\alpha \partial x^\beta} = F_{,\alpha\beta}$ etc.

note $F^{,\alpha\beta}$ for the inverse of the Hessian

Hessian metrics

Definition

Let $U \subset \mathbb{A}^n$ be a domain equipped with a pseudo-metric g. Then g is called Hessian if there locally exists a smooth function F such that $g = F''$. The function F is called Hessian potential.

the geodesics of a pseudo-metric obey the equation

$$\ddot{x}^\alpha + \sum_{\beta\gamma} \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma$$

with $\Gamma^\alpha_{\beta\gamma}$ the Christoffel symbols

for a Hessian metric we have

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} \sum_\delta F^{,\alpha\delta} F_{,\beta\gamma\delta}$$

Parallelism condition

the third derivative F''' of the Hessian potential is parallel with respect to the Hessian metric F'' if

$$\frac{\partial}{\partial x^\delta} F_{,\alpha\beta\gamma} + \sum_\eta \left(\Gamma^\eta_{\alpha\delta} F_{,\beta\gamma\eta} + \Gamma^\eta_{\beta\delta} F_{,\alpha\gamma\eta} + \Gamma^\eta_{\gamma\delta} F_{,\alpha\beta\eta} \right) = 0$$

in short notation $\hat{D} D^3 F = 0$, with D the flat connection of $\mathbb{A}^n$ and $\hat{D}$ the Levi-Civita connection of the Hessian metric

$$F_{,\alpha\beta\gamma\delta} = \frac{1}{2} \sum_{\rho\sigma} F^{,\rho\sigma} \left(F_{,\alpha\beta\rho} F_{,\gamma\delta\sigma} + F_{,\alpha\gamma\rho} F_{,\beta\delta\sigma} + F_{,\alpha\delta\rho} F_{,\beta\gamma\sigma} \right)$$

Integrability condition

differentiating with respect to x^η and substituting the fourth order derivatives by the right-hand side, we get

$$\begin{aligned} F_{,\alpha\beta\gamma\delta\eta} = \frac{1}{4} \sum_{\rho,\sigma,\mu,\nu} F^{,\rho\sigma} F^{,\mu\nu} (& F_{,\beta\eta\nu} F_{,\alpha\rho\mu} F_{,\gamma\delta\sigma} + F_{,\alpha\eta\mu} F_{,\rho\beta\nu} F_{,\gamma\delta\sigma} \\ + & F_{,\gamma\eta\nu} F_{,\alpha\rho\mu} F_{,\beta\delta\sigma} + F_{,\alpha\eta\mu} F_{,\rho\gamma\nu} F_{,\beta\delta\sigma} + F_{,\beta\eta\nu} F_{,\gamma\rho\mu} F_{,\alpha\delta\sigma} \\ + & F_{,\gamma\eta\mu} F_{,\rho\beta\nu} F_{,\alpha\delta\sigma} + F_{,\beta\eta\nu} F_{,\delta\rho\mu} F_{,\alpha\gamma\sigma} + F_{,\delta\eta\mu} F_{,\rho\beta\nu} F_{,\alpha\gamma\sigma} \\ + & F_{,\delta\eta\nu} F_{,\alpha\rho\mu} F_{,\beta\gamma\sigma} + F_{,\alpha\eta\mu} F_{,\rho\delta\nu} F_{,\beta\gamma\sigma} + F_{,\delta\eta\nu} F_{,\gamma\rho\mu} F_{,\alpha\beta\sigma} \\ + & F_{,\gamma\eta\mu} F_{,\rho\delta\nu} F_{,\alpha\beta\sigma}) \end{aligned}$$

anti-commuting δ, η gives the integrability condition

$$\begin{aligned} F^{,\rho\sigma} F^{,\mu\nu} (& F_{,\beta\eta\nu} F_{,\delta\rho\mu} F_{,\alpha\gamma\sigma} + F_{,\alpha\eta\mu} F_{,\rho\delta\nu} F_{,\beta\gamma\sigma} + F_{,\gamma\eta\mu} F_{,\rho\delta\nu} F_{,\alpha\beta\sigma} \\ & - F_{,\beta\delta\nu} F_{,\eta\rho\mu} F_{,\alpha\gamma\sigma} - F_{,\alpha\delta\mu} F_{,\rho\eta\nu} F_{,\beta\gamma\sigma} - F_{,\gamma\delta\mu} F_{,\rho\eta\nu} F_{,\alpha\beta\sigma}) = 0. \end{aligned}$$

let $K^{\alpha}_{\beta\gamma} = -\Gamma^{\alpha}_{\beta\gamma} = -\frac{1}{2}\sum_{\delta} F^{,\alpha\delta}F_{,\beta\gamma\delta}$, then $K^{\alpha}_{\beta\gamma} = K^{\alpha}_{\gamma\beta}$

contracting the integrability condition with $F^{,\eta\zeta}$, we get

$$\sum_{\mu,\rho}\Big(K^{\zeta}_{\alpha\mu}K^{\mu}_{\delta\rho}K^{\rho}_{\beta\gamma} + K^{\zeta}_{\beta\mu}K^{\mu}_{\delta\rho}K^{\rho}_{\alpha\gamma} + K^{\zeta}_{\gamma\mu}K^{\mu}_{\delta\rho}K^{\rho}_{\alpha\beta} - K^{\mu}_{\alpha\delta}K^{\zeta}_{\rho\mu}K^{\rho}_{\beta\gamma} - K^{\mu}_{\beta\delta}K^{\zeta}_{\rho\mu}K^{\rho}_{\alpha\gamma} - K^{\mu}_{\gamma\delta}K^{\zeta}_{\rho\mu}K^{\rho}_{\alpha\beta}\Big) = 0$$

this is satisfied if and only if

$$\sum_{\alpha,\beta,\gamma,\delta,\mu,\rho}\Big(K^{\zeta}_{\alpha\mu}K^{\mu}_{\delta\rho}K^{\rho}_{\beta\gamma}u^{\alpha}u^{\beta}u^{\gamma}v^{\delta} - K^{\mu}_{\alpha\delta}K^{\zeta}_{\rho\mu}K^{\rho}_{\beta\gamma}u^{\alpha}u^{\beta}u^{\gamma}v^{\delta}\Big) = 0$$

for all tangent vectors u, v

Jordan algebra defined by F

choose a point $e \in U$ and define a multiplication on $T_e U$ by $u \bullet v = K(u, v)$,

$$(u \bullet v)^\alpha = \sum_{\beta,\gamma} K^\alpha_{\beta\gamma} u^\beta v^\gamma$$

then $T_e U$ becomes a commutative algebra J

the integrability condition becomes

$$K(K(K(u,u),v),u) = K(K(u,v),K(u,u))$$

or

$$(u^2 \bullet v) \bullet u = (u \bullet v) \bullet u^2$$

hence J is a Jordan algebra

Trace form

the pseudo-metric $g = F''(e)$ satisfies

$$\begin{aligned} g(u \bullet v, w) &= \sum_{\beta,\gamma,\delta,\rho} F_{,\beta\gamma} K^{\beta}_{\delta\rho} u^{\delta} v^{\rho} w^{\gamma} \\ &= -\frac{1}{2} \sum_{\beta,\gamma,\delta,\rho,\sigma} F_{,\beta\gamma} F_{,\delta\rho\sigma} F^{,\sigma\beta} u^{\delta} v^{\rho} w^{\gamma} = -\frac{1}{2} \sum_{\gamma,\delta,\rho} F_{,\delta\rho\gamma} u^{\delta} v^{\rho} w^{\gamma} \\ &= -\frac{1}{2} \sum_{\beta,\gamma,\delta,\rho,\sigma} F_{,\beta\delta} u^{\delta} F_{,\rho\gamma\sigma} F^{,\sigma\beta} v^{\rho} w^{\gamma} \\ &= \sum_{\beta,\gamma,\delta,\rho} F_{,\delta\beta} u^{\delta} K^{\beta}_{\rho\gamma} v^{\rho} w^{\gamma} = g(u, v \bullet w). \end{aligned}$$

hence g is a trace form

Algebra defined by F

Theorem (H., 2012)

Let $F : U \to \mathbb{R}$ be a solution of the equation $\hat{D}D^3F = 0$. Let $e \in U$ and let J be the algebra defined on T_eU by the structure coefficients $K^{\alpha}_{\beta\gamma} = -\frac{1}{2}\sum_{\delta} F^{,\alpha\delta}F_{,\beta\gamma\delta}$ at e.
Then J is a Jordan algebra, and the Hessian metric $g = F''(e)$ is a non-degenerate trace form on J.

- if F is convex and log-homogeneous, then J is Euclidean
- if in addition J is simple, then g is proportional to the generic bilinear trace τ

Logarithmically homogeneous functions

Definition

Let $U \subset \mathbb{R}^n$ be an open conic set. A logarithmically homogeneous function on U is a smooth function $F : U \to \mathbb{R}$ such that

$$F(\alpha x) = -\nu \log \alpha + F(x)$$

for all $\alpha > 0$, $x \in U$.
The scalar ν is called the homogeneity parameter.

F defined by algebra

Theorem (H., 2012)

Let J be a Euclidean Jordan algebra and K its cone of squares. Let γ be a non-degenerate trace form on J. Then $F : K^o \to \mathbb{R}$ defined by

$$F(x) = -\gamma(e, \log x)$$

is a solution of the equation $\hat{D}D^3F = 0$ such that $F''(e) = \gamma$ and, under identification of T_eK^o and J, the multiplication in J is given by $K^{\alpha}_{\beta\gamma} = -\frac{1}{2}\sum_{\delta} F^{,\alpha\delta}F_{,\beta\gamma\delta}$ at e.

Main results

Theorem (H., 2012)

Let $K = K_1 \times \cdots \times K_m$ be a symmetric cone and $K_1, \dots, K_m$ its irreducible factors.
Then for every set of non-zero reals $\alpha_1, \dots, \alpha_m$, the function $F : K^o \to \mathbb{R}$ given by

$$F(A_1, \dots, A_m) = -\sum_{k=1}^{m} \alpha_k \log n(A_k)$$

is log-homogeneous and satisfies the equation $\hat{D}D^3F = 0$. The function F is convex if and only if $\alpha_k > 0$ for all k.

Theorem (H., 2012)

Let $U \subset \mathbb{A}^n$ be a subset of affine space and let $F : U \to \mathbb{R}$ be a log-homogeneous convex solution of the equation $\hat{D}D^3F = 0$. Then there exists a symmetric cone $K = K_1 \times \cdots \times K_m \subset \mathbb{A}^n$, positive reals $\alpha_1, \dots, \alpha_m$, and a constant c such that F can be extended to a solution $\tilde{F} : K^o \to \mathbb{R}$ given by

$$\tilde{F}(A_1, \dots, A_m) = -\sum_{k=1}^{m} \alpha_k \log n(A_k) + c.$$

when dropping convexity assumption, generalization beyond Euclidean Jordan algebras possible:

Hildebrand R. *Centro-affine hypersurface immersions with parallel cubic form*. arXiv preprint math.DG:1208.1155

Thank you