Canonical barriers on convex cones

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Outline

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- statement and properties
- affine spheres
- Calabi theorem

Barriers

- self-concordant barriers
- duality
- canonical barrier

Geometry of barriers

- projectivization
- structures on products of projective spaces
- canonical barrier as minimal Lagrangian submanifold

Three-dimensional cones

- conformal type
- holomorphic differentials

Monge-Ampère equation

let $\Omega \subset \mathbb{R}^n$ be a convex domain containing no line

on the interior Ω^o we consider the Monge-Ampère equation

$$\log \det F'' = 2F$$

we look for a convex solution with boundary conditions

$$\lim_{x \to \partial\Omega} F(x) = +\infty$$

- exists and is unique (Cheng-Yau, Sasaki, Li, ...)
- real analytic
- equi-affinely invariant (w.r.t. unimodular affine maps)
- Hessian F'' turns Ω^o into a Riemannian manifold
- maximum principle: $\tilde{\Omega} \subset \Omega \Rightarrow \tilde{F} \geq F$

Regular convex cones

Definition

A regular convex cone $K \subset \mathbb{R}^n$ is a closed convex cone having nonempty interior and containing no lines.

The dual cone

$$K^* = \{s \in \mathbb{R}_n \,|\, \langle x, s \rangle \geq 0 \quad \forall\, x \in K\}$$

of a regular convex cone K is also regular.

here $\mathbb{R}_n$ is the dual space to $\mathbb{R}^n$

Solutions of MA equation on cones

- invariant w.r.t. unimodular linear maps
- logarithmically homogeneous: $F(\lambda x) = -\log\lambda + F(x)$ for all $x \in K^o$, $\lambda > 0$
- Legendre dual F^* of F is a solution for K^*

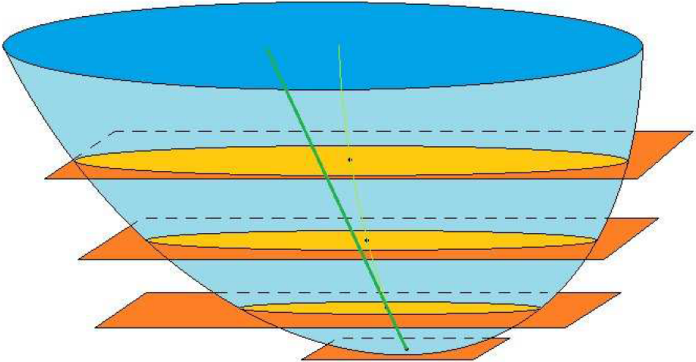
$$F^*(p) = \sup_{x \in K^o} \langle -p, x\rangle - F(x)$$

supremum attained at $p = -F'(x)$

- $(F^*)^* = F$
- $x \leftrightarrow p$ is an isometry between K^o and $(K^*)^o$
- level surfaces of F taken to level surfaces of F^*
- rays in K^o taken to rays in $(K^*)^o$ with inversion of the orientation

Affine hyperspheres

non-degenerate convex hypersurface in $\mathbb{R}^n$



the affine normal is the tangent to the curve made of the gravity centers of the sections

a hyperbolic proper affine sphere is a surface such that all affine normals meet at a point (the center) outside of the convex hull

Connection to Monge-Ampère equation

let $M \subset \mathbb{R}^n$ be a proper hyperbolic affine hypersphere

place the origin at the center of the affine hypersphere
the rays from the origin intersect M transversally

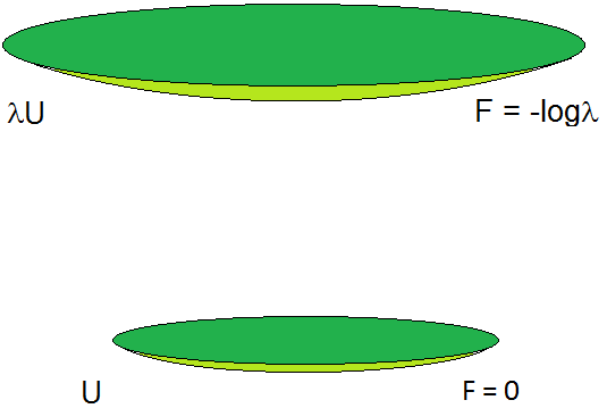
let $U \subset M$ be an open set such that each ray intersects U at most once

define $F : \bigcup_{\lambda>0} \lambda U \to \mathbb{R}$ by

$$F(\lambda x) = -\log \lambda \quad \forall\ \lambda > 0,\ x \in U$$

then F is convex and satisfies the Monge-Ampère equation
$\log \det F = 2F$ (Sasaki 85)

the level surfaces of solutions of $\log \det F = 2F$ are affine hyperspheres



construction of F on $\bigcup_{\lambda>0} \lambda U$

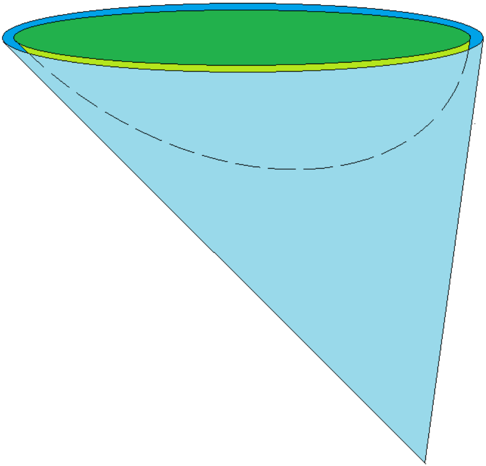
Calabi conjecture

the condition $\lim_{x \to \partial K} F(x) = +\infty$ implies that
$K^o = \bigcup_{\lambda > 0} \lambda M \simeq \mathbb{R}_+ \times M$ and M is asymptotic to ∂K

Theorem (Calabi conjecture; Fefferman 76, Cheng-Yau 86, Li 90, and others)

Let $K \subset \mathbb{R}^n$ be a regular convex cone. Then there exists a unique foliation of K^o by a homothetic family of affine complete and Euclidean complete hyperbolic affine hyperspheres which are asymptotic to ∂K.

Every affine complete, Euclidean complete hyperbolic affine hypersphere is asymptotic to the boundary of a regular convex cone.



the foliating hyperspheres are asymptotic to the boundary of K

Properties of affine spheres

- real-analytic
- trace of Hessian metric F'' is the Blaschke metric g of the affine sphere
- equi-affinely invariant (unimodular affine maps)
- Ricci curvature is non-positive and bounded from below by $-(n-2)g$ (Calabi 1972)
- duality realized by conormal map
- primal and dual affine spheres are isometric

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Self-concordant barriers

Definition (Nesterov, Nemirovski 1994)

Let $K \subset \mathbb{R}^n$ be a regular convex cone. A (self-concordant logarithmically homogeneous) barrier on K is a smooth function $F : K^o \to \mathbb{R}$ on the interior of K such that

- $F(\alpha x) = -\nu \log \alpha + F(x)$ (logarithmic homogeneity)
- $F''(x) \succ 0$ (convexity)
- $\lim_{x \to \partial K} F(x) = +\infty$ (boundary behaviour)
- $|F'''(x)[h, h, h]| \leq 2(F''(x)[h, h])^{3/2}$ (self-concordance)

for all tangent vectors h at x.

The homogeneity parameter ν is called the barrier parameter.

Application of barriers

used in interior-point methods for the solution of conic programs

$$\inf_{x \in K} \langle c, x \rangle \quad : \qquad Ax = b$$

define a family of convex optimization problems

$$\inf_{x} \tau \langle c, x \rangle + F(x) \quad : \qquad Ax = b$$

parameterized by $\tau > 0$

the solution $x^*(\tau)$ tends to the solution x^* of the original problem if $t \to +\infty$

the smaller the barrier parameter ν, the faster the IP method can increase τ and converge to x^*

Dual barrier

Theorem (Nesterov, Nemirovski 1994)

Let $K \subset \mathbb{R}^n$ be a regular convex cone and $F : K^o \to \mathbb{R}$ a barrier on K with parameter ν. Then the Legendre transform $F^*(p) = \sup_{x \in K^o} \langle -p, x \rangle + F(x)$ is a barrier on K^* with the same parameter ν.

- the map $\mathcal{I} : x \mapsto p = -F'(x)$ takes the level surfaces of F to the level surfaces of F^*
- $\mathcal{I}$ takes rays in K^o to rays in $(K^*)^o$ with inversion of the orientation
- $\mathcal{I}$ is an isometry between K^o and $(K^*)^o$ with respect to the Hessian metrics defined by F'', $(F^*)''$

Canonical barrier

Theorem (H., 2014; independently D. Fox, 2015)

Let $K \subset \mathbb{R}^n$ be a regular convex cone. Then the convex solution of the Monge-Ampère equation $\log\det F'' = 2F$ *with boundary condition* $F|_{\partial K} = +\infty$ *is a logarithmically homogeneous self-concordant barrier (the canonical barrier) on* K *with parameter* $\nu = n$.

main idea of proof: use non-positivity of the Ricci curvature

already conjectured by O. Güler

- invariant under the action of $SL(\mathbb{R}, n)$
- fixed under unimodular automorphisms of K
- additive under the operation of taking products
- invariant under duality

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Primal-dual view on barriers

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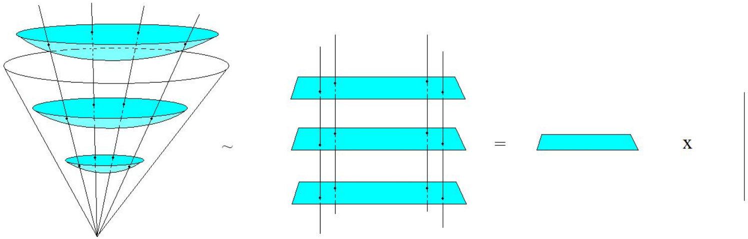
Splitting theorem

Theorem (Tsuji 1982; Loftin 2002)

Let $K \subset \mathbb{R}^{n+1}$ be a regular convex cone, and $F : K^o \to \mathbb{R}$ a locally strongly convex logarithmically homogeneous function.
Then the Hessian metric on K^o splits into a direct product of a radial 1-dimensional part and a transversal n-dimensional part. The submanifolds corresponding to the radial part are rays, the submanifolds corresponding to the transversal part are level surfaces of F.

the isometry defined by the Legendre duality respects the splitting but inverts the direction of the rays all nontrivial information is contained in the transversal part



Projective images of cones

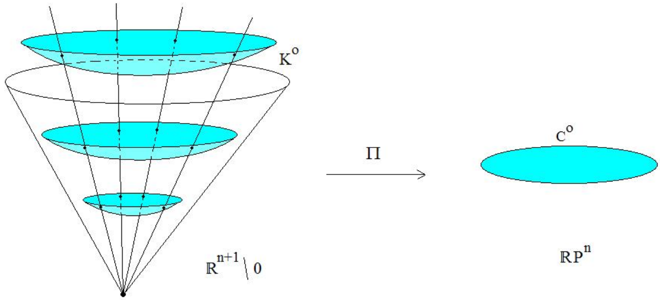
let $\mathbb{R}P^n, \mathbb{R}P_n$ be the primal and dual real projective space — lines and hyperplanes through the origin of $\mathbb{R}^{n+1}$

let $F : K^o \to \mathbb{R}$ be a barrier on a regular convex cone $K \subset \mathbb{R}^{n+1}$

the canonical projection $\Pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ maps $K \setminus \{0\}$ to a compact convex subset $C \subset \mathbb{R}P^n$

the canonical projection $\Pi^* : \mathbb{R}_{n+1} \setminus \{0\} \to \mathbb{R}P_n$ maps $K^* \setminus \{0\}$ to a compact convex subset $C^* \subset \mathbb{R}P_n$

the interiors of C, C^* are isomorphic to the mutually isometric transversal factors of $K^o, (K^*)^o$ and acquire the metric of these factors



passing to the projective space removes the radial factor

Product of linear spaces

neither the vector space $\mathbb{R}^n$ nor its dual $\mathbb{R}_n$ carry a canonical metric only a family of equivalent metrics which all lead to the same flat affine connection

the product $\mathbb{R}^n \times \mathbb{R}_n$ has a lot more structure

- flat pseudo-Riemannian metric $G((x,p);(y,q)) = \frac{1}{2}(\langle x,q\rangle + \langle y,p\rangle)$
- symplectic form $\omega((x,p);(y,q)) = \frac{1}{2}(\langle x,q\rangle - \langle y,p\rangle)$
- inversion $J : (x,p) \mapsto (x,-p)$ of the tangent bundle with integrable eigenspace distributions
- compatibility conditions $\hat{\nabla}\omega = 0, \; G = \omega J$

these all together define a flat para-Kähler space form

Product of projective spaces

between elements of $\mathbb{R}P^n, \mathbb{R}P_n$ there is an orthogonality relation

the set

$$\mathcal{M} = \{(x,p) \in \mathbb{R}P^n \times \mathbb{R}P_n \,|\, x \not\perp p\}$$

is dense in $\mathbb{R}P^n \times \mathbb{R}P_n$

its complement

$$\begin{aligned}\partial\mathcal{M} &= \{(x,p) \in \mathbb{R}P^n \times \mathbb{R}P_n \,|\, x \perp p\} \\ &\simeq O(n+1)/(O(1) \times O(1) \times O(n-1))\end{aligned}$$

is a submanifold of $\mathbb{R}P^n \times \mathbb{R}P_n$ of codimension 1

Para-Kähler structure on $\mathcal{M}$

Theorem (Gadea, Montesinos Amilibia 1989)

The space $\mathcal{M}$ is a hyperbolic para-Kähler space form, it carries a natural para-Kähler structure with constant negative sectional curvature.

para-Kähler manifold:

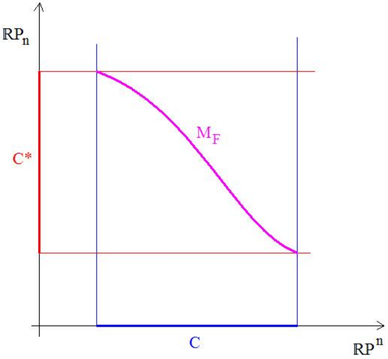
- even dimension
- pseudo-metric of neutral signature
- symplectic structure satisfying $\nabla\omega = 0$
- para-complex structure J satisfying $g(X, Y) = \omega(JX, Y)$

J is an involution of $T_x\mathcal{M}$ with the ± 1 eigenspaces forming n-dimensional integrable distributions

Representation of barriers

the bijection $x \mapsto -F'(x)$ factors through to an isometry between C^o and $(C^*)^o$

$$\begin{array}{ccc} K^o & \xrightarrow{-F'} & (K^*)^o \\ \Pi \downarrow & & \Pi^* \downarrow \\ C^o \sim K^o/\mathbb{R}_+ & \xrightarrow{\mathcal{I}_F} & (C^*)^o \sim (K^*)^o/\mathbb{R}_+ \end{array}$$

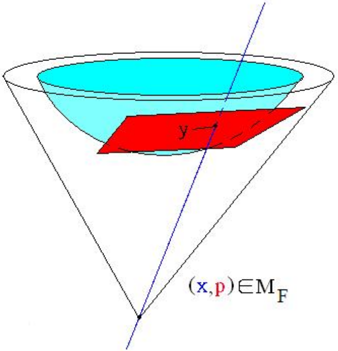


define the smooth submanifold M_F as the graph of the isometry $\mathcal{I}_F$

$$M_F = \Pi \times \Pi^* \left[\{ (x, -F'(x)) \,|\, x \in K^o \} \right.$$

$$\dim M_F = n = \frac{1}{2} \dim \mathcal{M}$$

Geometric interpretation



the manifold M_F consists of pairs (x, p) where

- x is a line through a point $y \in K^o$
- p is parallel to the hyperplane which is tangent to the level surface of F at y

if $y \to \hat{y} \in \partial K$, then p tends to a supporting hyperplane at $\hat{y}$

Properties of M_F

Theorem

Let $F : K^o \to \mathbb{R}$ be a barrier on a regular convex cone $K \subset \mathbb{R}^{n+1}$ with parameter ν. The manifold $M_F \subset \mathbb{R}P^n \times \mathbb{R}P_n$ is

- *a complete nondegenerate hyperbolic Lagrangian submanifold of $\mathcal{M}$*
- *its submanifold metric is $-\nu^{-1}$ times the metric induced by the isometry $\mathcal{I}_F$*
- *its second fundamental form II satisfies*

$$C = \nu^{-1} F'''[h, h, h'] = 2\omega(II(\tilde{h}, \tilde{h}), \tilde{h}')$$

for all vectors h, h' tangent to the level surfaces of F and their images $\tilde{h}, \tilde{h}'$ on the tangent bundle TM_F.

C is called the cubic form and is totally symmetric
$C \mapsto -C$ if $K \mapsto K^*$

Self-concordance and curvature

Corollary

Let $K \subset \mathbb{R}^{n+1}$ be a regular convex cone and $F : K^o \to \mathbb{R}$ a locally strongly convex logarithmically homogeneous function with parameter ν.
Then F is self-concordant if and only if the Lagrangian submanifold $M_F \subset \mathcal{M}$ has its second fundamental form bounded by $\gamma = \frac{\nu-2}{\sqrt{\nu-1}}$.

the barrier parameter determines how close M_F is to a geodesic submanifold of $\mathcal{M}$

Images of conic boundaries

the canonical projection
$\Pi \times \Pi^* : (\mathbb{R}^{n+1} \setminus \{0\}) \times (\mathbb{R}_{n+1} \setminus \{0\}) \to \mathbb{R}P^n \times \mathbb{R}P_n$ maps the set

$$\Delta_K = \{(x, p) \in (\partial K \setminus \{0\}) \times (\partial K^* \setminus \{0\}) \,|\, x \perp p\}$$

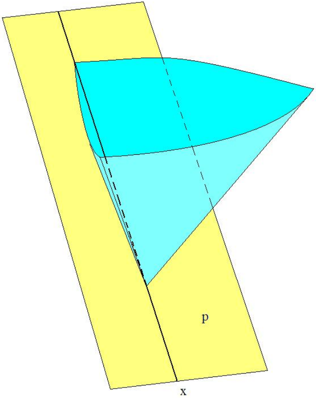
to a set $\delta_K \subset \partial \mathcal{M}$

Lemma

The set δ_K is is homeomorphic to S^{n-1}.
The projections π, π^* of $\mathbb{R}P^n \times \mathbb{R}P_n$ to the factors map δ_K onto ∂C and ∂C^*, respectively.
For every barrier F on K, $\partial M_F = \delta_K$.

call δ_K the boundary frame corresponding to the cone K

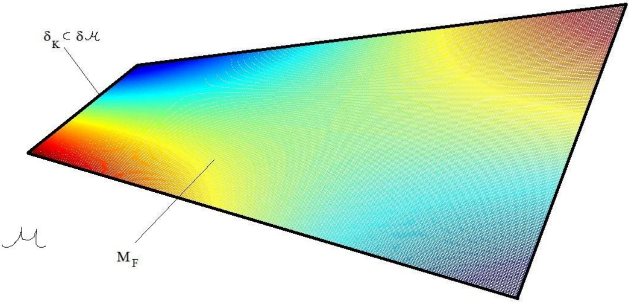
Geometric interpretation



the boundary frame δ_K consists of pairs $z = (x, p) \in \partial\mathcal{M}$ where

- the line x contains a ray in ∂K
- p is a supporting hyperplane at x

Primal-dual representation of a barrier



- complete negative definite Lagrangian submanifold, $\simeq \mathbb{R}^n$
- bounded by $\delta_K \simeq S^{n-1}$
- second fundamental form bounded by $\gamma = \frac{\nu-2}{\sqrt{\nu-1}}$

Canonical barrier and minimal submanifolds

Definition

Let $\mathcal{M}$ be a pseudo-Riemannian manifold. Then $M \subset \mathcal{M}$ is a minimal submanifold if M is a stationary point of the volume functional with respect to variations with compact support.

a submanifold is minimal if and only if its *mean curvature* vanishes identically

Theorem (H., 2011)

Let $K \subset \mathbb{R}^n$ be a regular convex cone and $F : K^o \to \mathbb{R}$ be a barrier on K.
Then the submanifold $M_F \subset \mathcal{M}$ is minimal if and only if the level surfaces of F are affine hyperspheres.

the canonical barrier is given by the unique minimal complete negative definite Lagrangian submanifold of $\mathcal{M}$ which can be inscribed in the boundary frame $\delta_K \subset \partial\mathcal{M}$

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Conformal type of cones

for 3-dimensional cones K the submanifolds M_F are 2-dimensional
M_F is a complete non-compact simply connected Riemann surface

Uniformization theorem: Every simply connected Riemann surface is conformally equivalent to either the unit disc $\mathbb{D}$, or the complex plane $\mathbb{C}$, or the Riemann sphere S, equipped with either the hyperbolic metric, or the flat (parabolic) metric, or the spherical (elliptic) metric, respectively.

due to Klein, Riemann, Schwarz, Koebe, Poincaré, Hilbert, Weyl, Radó ... 1880–1920

there exists a global (isothermal) chart on M_F such that
$g = e^{2\phi}(dx_1^2 + dx_2^2)$
here $z = x_1 + ix_2$, $z \in \mathbb{D}$ or $z \in \mathbb{C}$

cones can be classified with respect to conformal type of their canonical barrier

Holomorphic cubic differential

let $K \subset \mathbb{R}^3$ be a regular convex cone and F its canonical barrier
let M_F be equipped with a complex isothermal coordinate z
the cubic form C can be decomposed as

$$C = \left[\begin{pmatrix} U_1 & -U_2 \\ -U_2 & -U_1 \end{pmatrix}, \begin{pmatrix} -U_2 & -U_1 \\ -U_1 & U_2 \end{pmatrix} \right]$$

$U = U_1 + iU_2$ is a cubic differential, $U(w) = U(z)(\frac{dz}{dw})^3$ under coordinate changes

compatibility requirements on ϕ, U [Liu, Wang 1997]:

$$\frac{\partial U}{\partial \bar{z}} = 0, \quad |U|^2 = 2e^{6\phi} - 8e^{4\phi} \frac{\partial^2 \phi}{\partial z \partial \bar{z}}$$

here $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$

U is holomorphic

Canonical barrier on 3D cones

Theorem (follows from (Simon, Wang 1993))

- *Let $K \subset \mathbb{R}^3$ be a regular convex cone, let M_F be the Riemann surface defined by the canonical barrier, with an isothermal global coordinate z and metric $g = e^{2\phi}|dz|^2$. Then the associated holomorphic cubic differential U satisfies*

$$|U|^2 = 2e^{6\phi} - 2e^{4\phi}\Delta\phi = 2e^{6\phi}(1 + \mathsf{K}),$$

  where Δ is the ordinary Laplacian and K the Gaussian curvature.

- *Every simply connected non-compact Riemann surface with complete metric $g = e^{2\phi}|dz|^2$ and holomorphic cubic differential U satisfying above relation defines a regular convex cone $K \subset \mathbb{R}^3$ with its canonical barrier, up to linear isomorphisms.*

Correspondence $K \leftrightarrow (\phi, U)$

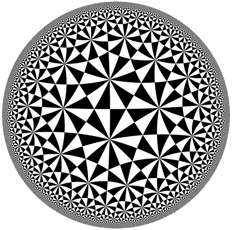
$$|U|^2 = 2e^{6\phi} - 2e^{4\phi}\Delta\phi = 2e^{6\phi}(1 + \mathrm{K})$$

- level surfaces of F can be recovered from (ϕ, U) by solving a Cauchy initial value problem of a PDE
- [Simon, Wang 1993] gives a necessary and sufficient integrability condition on ϕ
- for given ϕ, U is determined up to a constant factor $e^{i\varphi}$
- the isomorphism classes of cones with isometric canonical barrier form a S^1 family
- K^* is on the opposite side w.r.t. K
- for given U, there exists at most one solution ϕ (maximum principle)

Known results (selection)

[Dumas, Wolf 2015] polynomials U of degree k correspond to polyhedral cones K with $k+3$ extreme rays
$U = z^k$ corresponds to the cone over the regular $(k+3)$-gon
M_F conformally equivalent to $\mathbb{C}$

[Wang 1997; Loftin 2001; Labourie 2007]
holomorphic functions on compact Riemann surface of genus ≥ 2 form a finite-dimensional space
each such function U determines a unique metric g on the surface and its universal cover
the corresponding cone K has an automorphism group with cocompact action on the level surfaces on F
∂K is C^1, but in general nowhere C^2
M_F conformally equivalent to $\mathbb{D}$



Open questions

Which cones allow barriers such that the corresponding Riemann surface is conformally equivalent to $\mathbb{C}$?

Which entire functions are cubic forms of an affine hypersphere? Are there functions other than polynomials?

Which holomorphic functions on $\mathbb{D}$ are cubic forms of an affine hypersphere?

(All functions U which are bounded in the hyperbolic metric on $\mathbb{D}$ will work [Benoist, Hulin 14].)

Hildebrand R. Canonical barriers on convex cones. *Math. Oper. Res.* **39**(3):841–850, 2014.

Thank you!