Convex projective programming

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Outline

Conic optimization

- Conic programs
- Duality
- Radial transformations

Projective programming

- Conic programs in projective space
- Duality
- Feasibility
- Affine counterpart

Regular convex cones

Definition

A regular convex cone $K \subset \mathbb{R}^n$ is a closed convex cone having nonempty interior and containing no lines.

The dual cone

$$K^* = \{s \in \mathbb{R}_n \,|\, \langle x, s \rangle \geq 0 \quad \forall\, x \in K\}$$

of a regular convex cone K is also regular.

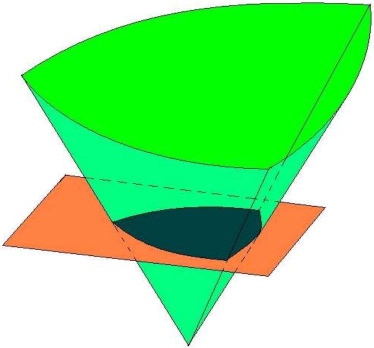
Conic programs

Definition

A conic program over a regular convex cone $K \subset \mathbb{R}^n$ is an optimization problem of the form

$$\min_{x \in K} \langle c, x \rangle : \quad Ax = b.$$

Geometric interpretation



the feasible set is the intersection of K with an affine subspace

$$\min_z \langle c', z \rangle : \; A'z + b' \in K$$

explicit parametrization

Duality

primal program

$$\min_{x \in K} \langle c, x \rangle : \; Ax = b$$

dualizing constraint $Ax = b$ gives

$$\min_{x \in K} \max_{z} \left(\langle c, x \rangle - \langle z, Ax - b \rangle \right)$$

Duality

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$$\max_{z} \min_{x \in K} \left(-\langle A^T z - c, x \rangle + \langle b, z \rangle \right)$$

Duality

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minimizing over x gives the dual program

$$\max_{s = -(A^T z - c) \in K^*} \langle b, z \rangle$$

Assumptions

suppose the conic program satisfies:

- the cost function is not constant on the feasible set ($c \notin \text{row}(A)$)
- the feasible set is properly affine ($b \neq 0$)

these conditions transform into each other under duality

Affine spaces

primal affine space

$$P_A = \{x \mid Ax = b\} \subset \mathbb{R}^n$$

$\dim P_A = k$, $n - k$ number of rows of A

dual affine space

$$D_A = \{s \mid \exists\, z : \; s = -(A^T z - c)\} \subset \mathbb{R}_n$$

$\dim D_A = n - k$

$\dim P_A + \dim D_A = n$

Linear spaces

$$P_A = \{x \mid Ax = b\},\ D_A = \{s \mid \exists\ z:\ \ s = -(A^T z - c)\}$$

primal displacements: $P_\Delta = \{\delta x \mid A\,\delta x = 0\}$
dual displacements: $D_\Delta = \{\delta s \mid \exists\ z : \delta s = -A^T \delta z\}$

$\langle \delta x, \delta s \rangle = 0,\ P_\Delta = D_\Delta^\perp$

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let P_L, D_L be the linear hulls of P_A, D_A

$P_\Delta \subset P_L$, $D_\Delta \subset D_L$ yields $P_L^\perp \subset D_\Delta$, $D_L^\perp \subset P_\Delta$

$$D_L^\perp \subset P_\Delta \subset P_L, \quad P_L^\perp \subset D_\Delta \subset D_L$$

codimensions equal 1

Level sets

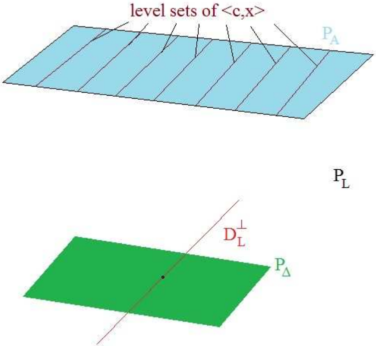
$$P_L = \{x \mid \exists\ \alpha :\ Ax = \alpha b\},\ D_L = \{s \mid \exists\ z, \beta :\ s = -(A^T z - \beta c)\}$$

for $\delta s = -A^T \delta z \in D_\Delta$ we have $\langle \delta s, x \rangle = -\delta z^T A x = -\alpha\, \delta z^T b$
hence $P_L^\perp = \{-A^T \delta z \mid b^T \delta z = 0\}$

$$D_L^\perp = \{\delta x \mid A\delta x = 0,\ \langle c, x \rangle = 0\}$$

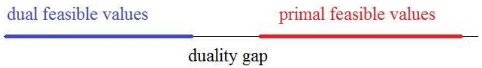
the orthogonal subspaces $P_L^\perp, D_L^\perp$ define precisely those directions in D_A, P_A where the dual and primal cost functions do not change

$P_L^\perp, D_L^\perp$ define the displacements of the level sets



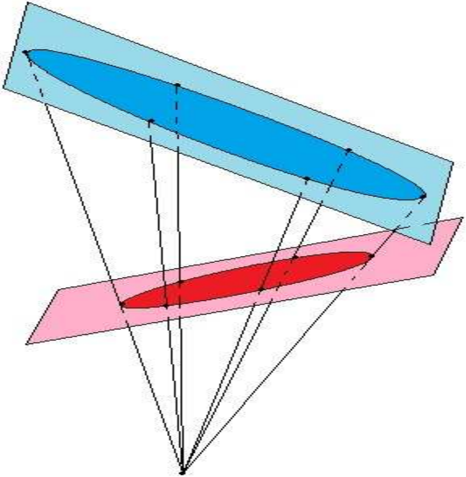
$P_A/D_L^\perp$ is canonically isomorphic to the (affine) line of cost function values

Primal and dual values



- primal and dual feasible values form intervals
- interiors of the intervals do not intersect
- there may or may not be a duality gap
- if the intervals intersect, the primal and dual feasible points corresponding to the intersection are orthogonal and optimal

Radial transformations

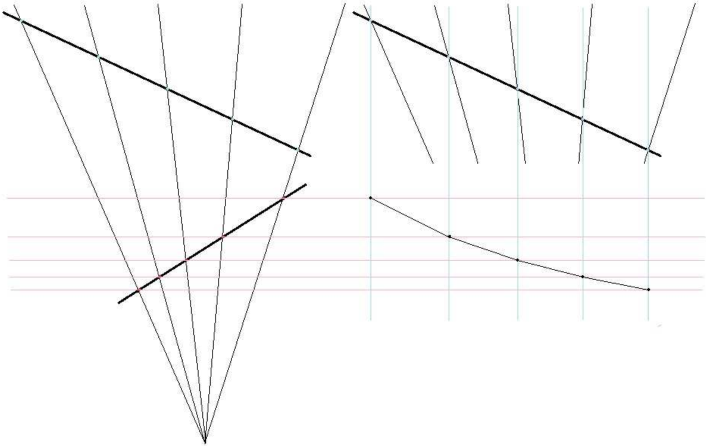


the image of the feasible set is not affinely equivalent to the original

it is projectively equivalent

P_L, D_L are invariant, but P_Δ, D_Δ are not

Transformation of the cost function



the image of the cost function is in general not affine
it is linear-fractional
the ensemble of the level sets is preserved

Equivalent conic programs

linear-fractional functions can be made affine by a monotonic transformation of the function value, $a = m \circ l$

such transformations preserve the ensemble of level sets

the minimum is mapped to the minimum

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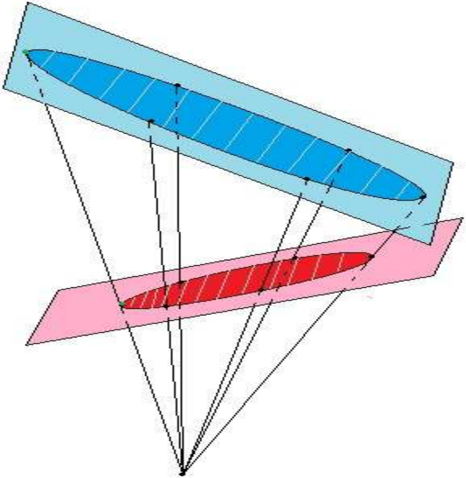
the minimum is mapped to the minimum

we get another conic program whose minimum is a multiple of the original one



neither the feasible set nor the cost function are affinely equivalent

but the solution of one conic program can easily be obtained from the solution of the other can we find a framework in which the two conic programs are the same?

Is there a possibility to build a theory of convex projective programming?

- have to optimize over subsets of projective space
- have to give up notion of the value of the cost function
- what remains are the level sets and their ordering

Cones in projective space

$\mathbb{P}^{n-1}$ — projective space, set of 1-dimensional subspaces of $\mathbb{R}^n$
$\mathbb{P}_{n-1}$ — dual projective space, set of 1-dimensional subspaces of $\mathbb{R}_n$

$\pi : \mathbb{R}^n \setminus \{0\} \to \mathbb{P}^{n-1}$, $\pi_* : \mathbb{R}_n \setminus \{0\} \to \mathbb{P}_{n-1}$ — projections

$K \subset \mathbb{R}^n$ regular convex cone, $C = \pi[K \setminus \{0\}] \subset \mathbb{P}^{n+1}$
$K^* \subset \mathbb{R}^n$ dual cone, $C^* = \pi_*[K^* \setminus \{0\}] \subset \mathbb{P}_{n-1}$

C, C^* compact convex sets containing no projective lines

Feasible sets in projective space

$P_A = \{x \mid Ax = b\}$, $D_A = \{s \mid \exists\, z : \; s = -(A^T z - c)\}$

$P_P = \pi[P_A] = \pi[P_L \setminus \{0\}]$, $D_P = \pi[D_A] = \pi[D_L \setminus \{0\}]$ are projective subspaces of dimensions $k, n-k$

the primal and dual feasible sets project to $C \cap P_P \subset \mathbb{P}^{n-1}$, $C^* \cap D_P \subset \mathbb{P}_{n-1}$

Orthogonal subspaces

affine subspaces $A \subset \mathbb{R}^n$ do not have an orthogonal affine space $A^\perp \subset \mathbb{R}_n$

but projective subspaces $P \subset \mathbb{P}^{n-1}$ have an orthogonal projective space $P^\perp \subset \mathbb{P}_{n-1}$

let $L = \pi^{-1}[P] \cup \{0\} \subset \mathbb{R}^n$, and $L^\perp \subset \mathbb{R}_n$ its orthogonal subspace

define $P^\perp = \pi^*[L^\perp \setminus \{0\}]$

if $\dim P = k$, then $\dim L = k + 1$, $\dim L^\perp = n - k - 1$,
$\dim P^\perp = n - k - 2$, $(P^\perp)^\perp = P$

$\dim P + \dim P^\perp = n - 2$

let $P \subset \mathbb{P}^{n-1}$, $D \subset P_{n-1}$ be projective subspaces, then

$$P^\perp \subset D \qquad \Leftrightarrow \qquad D^\perp \subset P$$

Orthogonality and duality

we have $D_L^\perp \subset P_L$, $P_L^\perp \subset D_L$ with codimension 2

apply projections π, π_*

we get $D_P^\perp \subset P_P$, $P_P^\perp \subset D_P$ with codimension 2

Lemma Let $P_1, P_2 \subset \mathbb{P}^{n-1}$ be projective subspaces of dimensions k_1, k_2 such that $P_1 \subset P_2$ and $k_2 - k_1 = 2$. Then the set of projective subspaces P of dimension k, $k_1 < k < k_2$, such that $P_1 \subset P \subset P_2$, is isomorphic to the projective line $\mathbb{P}^1$.

the affine line $P_A/D_L^\perp$ of cost function values is replaced by the projective line

Values of feasible points

$C \subset \mathbb{P}^{n-1}$, $C^* \subset \mathbb{P}_{n-1}$ — dual pair of closed convex sets containing no lines

let $x \in C \cap P_P$ be a primal feasible point

Lemma If $x \notin D_P^\perp$, then there exists a unique projective subspace P of dimension $k-1$ such that $x \in P$ and $D_P^\perp \subset P \subset P_P$.

call this the value of the point x

the map $\perp$ is an isomorphism between the projective line of primal values and the set of subspaces D such that $P_P^\perp \subset D \subset D_P$, i.e., the projective line of dual values

we shall identify them in the sequel

Values and orthogonality

Lemma Let $P_P \subset \mathbb{P}^{n-1}$, $D_P \subset \mathbb{P}_{n-1}$ be projective subspaces of dimension $k, n-k$ such that $D_P^\perp \subset P_P$. Let P, D be such that $D_P^\perp \subset P \subset P_P$, $P_P^\perp \subset D \subset D_P$, each inclusion being proper. Let $x \in P \setminus D_P^\perp$, $s \in D \setminus P_P^\perp$ be points.
Then $D = P^\perp$ if and only if $x \perp s$.

if $D = P^\perp$, then $x \in P$ yields $s \in D = P^\perp \subset x^\perp$

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Then $D = P^\perp$ if and only if $x \perp s$.

if $D = P^\perp$, then $x \in P$ yields $s \in D = P^\perp \subset x^\perp$

- we have $s \in D_P$ and $D_P^\perp \subset s^\perp$
- let now $x \perp s$, then $x \cup D_P^\perp \subset s^\perp$
- this gives $P \subset s^\perp$, $s \in P^\perp \cap D$, and $D = P^\perp$

Primal and dual values

by convexity of C, C^* the sets of primal feasible values and of dual feasible values are intervals on $\mathbb{P}^1$

let $x \in C \cap P_P$, $s \in C^* \cap D_P$ be points in the interior of the feasible sets, with values P, D

they correspond to feasible points $\bar{x}, \bar{s}$ in the interior of $K \cap P_A$, $K^* \cap D_A$

but $\langle \bar{x}, \bar{s} \rangle > 0$, hence $x \not\perp s$, and $P^\perp \neq D$

the interiors of the intervals of primal and dual feasible values do not intersect

Infeasibility and feasibility

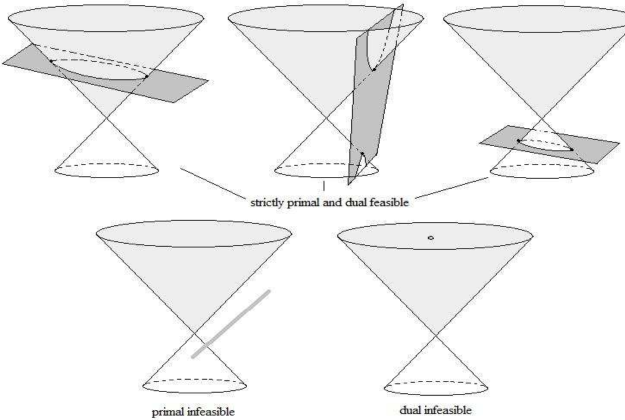
Theorem The following are equivalent (primal infeasibility):

- $P_P^{\perp} \cap (C^*)^o \neq \emptyset$
- all values are dual feasible
- $P_P \cap C = \emptyset$

Theorem The following are equivalent (primal strict feasibility):

- $P_P^{\perp} \cap C^* = \emptyset$
- the interval of primal feasible values is solid
- $P_P \cap C^o \neq \emptyset$

depends on the relation between the singular set $P_P^{\perp}$ and the dual convex set C^*

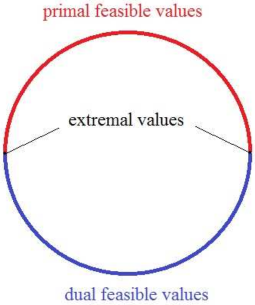


duality gaps can occur only if $P_P^\perp$ touches C^* or $D_P^\perp$ touches C

Regular case

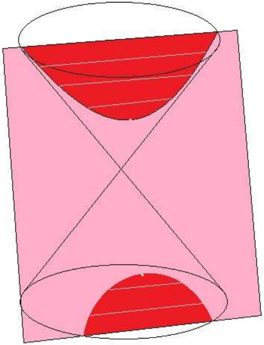
let $P_P^\perp \cap C^* = \emptyset$, $D_P^\perp \cap C = \emptyset$, then

- both the primal and dual feasible values form a proper closed interval
- the interior of one is the complement of the other (no duality gap)



extremal values correspond to minimization and maximization of the cost function

Corresponding conic program



- optimization over intersection of P_A with $K \cup (-K)$
- value $\pm\infty$ becomes an ordinary point

Other objects

distances:

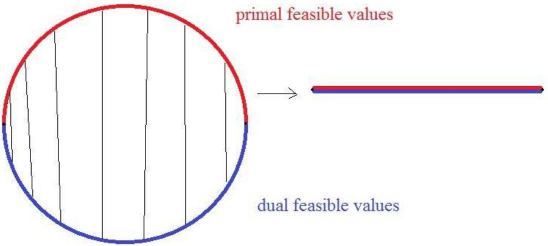
- product $\mathbb{P}^{n-1} \times \mathbb{P}_{n-1}$ is a pseudo-Riemannian space
- distance between pairs (x, s) measured by the projective cross-ratio

barriers:

- can be constructed from log-homogeneous barriers
- represented as Riemannian submanifolds $M \subset \mathbb{P}^{n-1} \times \mathbb{P}_{n-1}$
- self-concordance parameter ν and curvature γ related by $\gamma = \frac{\nu-2}{\sqrt{\nu-1}}$
- links to affine differential geometry

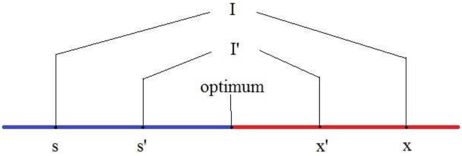
Central path

- set of primal-dual feasible points (x, s) on the barrier: $(P_P \times D_P) \cap M$
- identifies the interval of primal values with the interval of dual values
- links the two extremal points
- distance on the central path bounds the progress of affine IPM from below



Interior-point methods

let (x, s), (x', s') be two primal-dual feasible pairs



$|I'|/|I| \leq (\cosh d - \sinh d)^2$, where d is the distance between (x, s) and (x', s')

Conclusion

features of the projective theory

- set of objective values is compact
- primal and dual programs are always bounded
- duality gap and feasibility determined by position of singular subspaces $P_P^{\perp}, D_P^{\perp}$
- simple geometric interpretation of barriers and central paths
- fusion of primal and dual setup
- mathematical basis is affine differential geometry

outlook

- fully projective interior-point methods
- additional structure when cone is symmetric

Thank you