A family of semidefinite relaxations for cones of positive polynomials

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Outline

- cones of positive polynomials
- Newton polytopes
- SOS relaxations
- generalized copositive cones
- relaxations based on copositive cones

Cones of positive polynomials

$\mathcal{L}_\mathcal{A}$ — linear space of polynomials

$$p(x_1, \ldots, x_n) = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha$$

$\mathcal{A} \subset \mathbb{N}^n$ — set of multi-indices

$\dim \mathcal{L}_\mathcal{A} = \#\mathcal{A}$

the polynomial is identified with the coefficient vector c_α

$\mathcal{P}_\mathcal{A}$ cone of positive polynomials in coefficient space

Example: Motzkin polynomial

$\mathcal{A} = \{(4,2,0), (2,4,0), (0,0,6), (2,2,2)\}$,

$p_M(x,y,z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2 \sim (1,1,1,-3)^T \in \mathcal{P}_\mathcal{A}$

Newton polytope

for $p \in \mathcal{L}_\mathcal{A}$

$$N(p) = \text{conv}\{\alpha \,|\, c_\alpha(p) \neq 0\}$$

Newton polytope associated with p

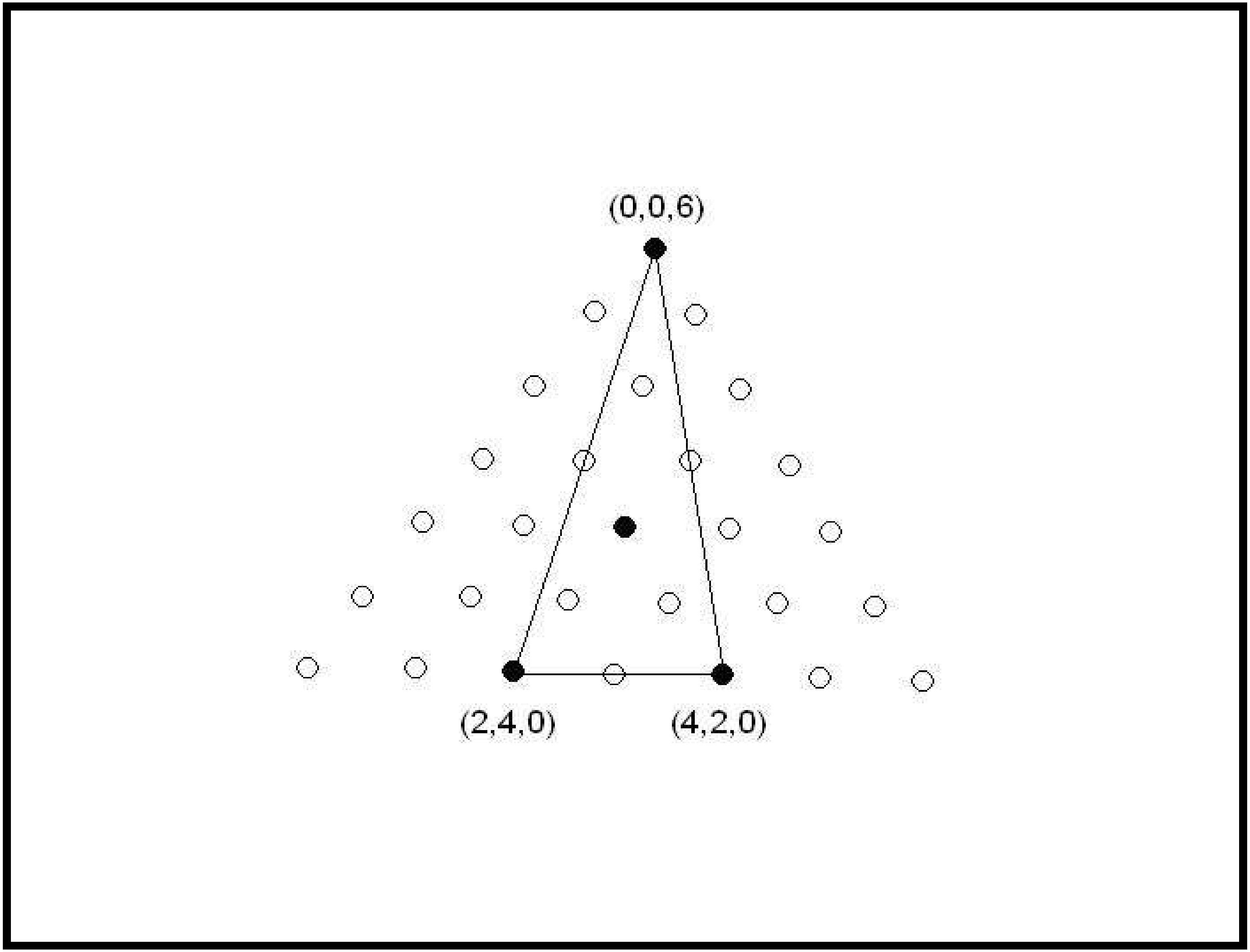
$$N_\mathcal{A} = \cup_{p \in \mathcal{L}_\mathcal{A}} N(p) = \text{conv}\mathcal{A}$$

Newton polytope associated with $\mathcal{L}_\mathcal{A}$

Theorem Let $p \in \mathcal{P}_\mathcal{A}$ and let $\alpha^* \in \mathcal{A}$ be extremal in $N(p)$. Then $c_{\alpha^*} > 0$ and α^* is even.

hence assume w.r.o.g. that the extremal points of $\mathcal{A}$ are even

otherwise $\mathcal{P}_\mathcal{A}$ contained in proper subspace of $\mathcal{L}_\mathcal{A}$


(0,0,6)
(2,4,0)
(4,2,0)

Sums of squares

$$\Sigma_{\mathcal{A}} = \left\{ p \in \mathcal{L}_{\mathcal{A}} \,|\, \exists\ N, q_k :\ p = \sum_{k=1}^{N} q_k^2 \right\}$$

$$\Sigma_{h,\mathcal{A}} = \left\{ p \in \mathcal{L}_{\mathcal{A}} \,|\, \exists\ N, q_k :\ ph = \sum_{k=1}^{N} q_k^2 \right\}$$

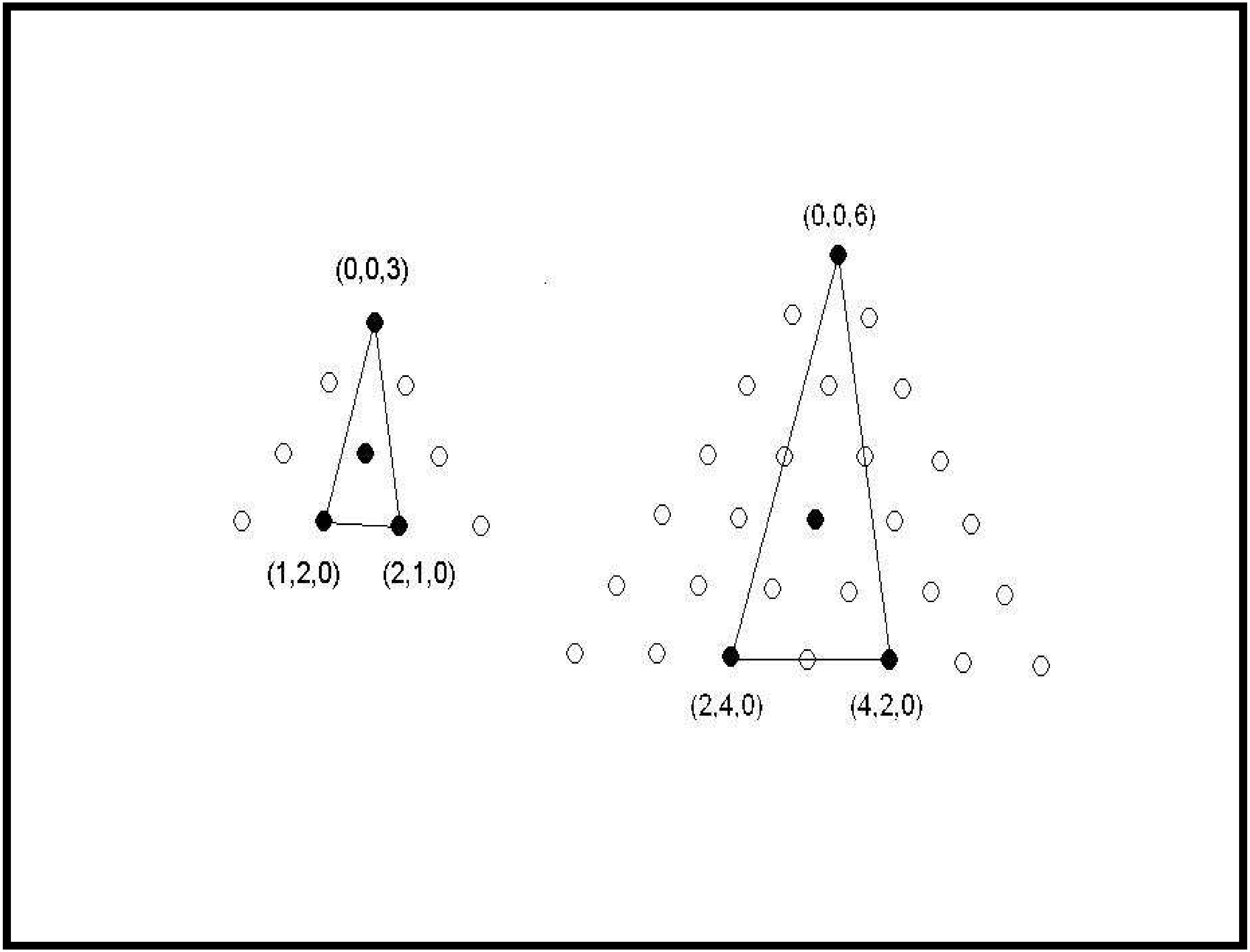
h nonzero positive polynomial

$\Sigma_{\mathcal{A}}, \Sigma_{h,\mathcal{A}}$ inner semidefinite relaxations of $\mathcal{P}_{\mathcal{A}}$

in general $\Sigma_{\mathcal{A}} \neq \mathcal{P}_{\mathcal{A}}$, not even $\dim \Sigma_{\mathcal{A}} = \dim \mathcal{P}_{\mathcal{A}}$, e.g. $p_M \notin \Sigma_{\mathcal{A}}$

Theorem Let $p = \sum_{k=1}^{N} q_k^2$. Then $N(q_k) \subset N(p)/2\ \forall\ k = 1, \ldots, N$.

$\Rightarrow$ if $p = \sum_{k=1}^{N} q_k^2 \in \mathcal{P}_{\mathcal{A}}$, then $q_k \in \mathcal{L}_{N_{\mathcal{A}}/2 \,\cap\, \mathbb{N}^n}$


(0,0,3)
(1,2,0)
(2,1,0)
(0,0,6)
(2,4,0)
(4,2,0)

Structure of $\Sigma_{\mathcal{A}}$

$\mathcal{F} \subset \mathbb{N}^n$, $\quad x_{\mathcal{F}}$ the vector of monomials $\{x^\beta\}_{\beta \in \mathcal{F}}$

$$\begin{aligned} \Sigma_{\mathcal{F},\mathcal{A}} &= \{p \in \mathcal{L}_{\mathcal{A}} \,|\, \exists\, N, q_k \in \mathcal{L}_{\mathcal{F}} : p(x) = \sum_{k=1}^{N} q_k^2\} \\ &= \{p \in \mathcal{L}_{\mathcal{A}} \,|\, \exists\, C \succeq 0 : p(x) = x_{\mathcal{F}}^T C x_{\mathcal{F}}\} \end{aligned}$$

is an inner semidefinite relaxation for $\mathcal{P}_{\mathcal{A}}$

w.r.o.g. $\mathcal{F} \subset N_{\mathcal{A}}/2 \cap \mathbb{N}^n$

$\mathcal{F}$ smaller $\Rightarrow$ relaxation weaker

$\Sigma_{\mathcal{A}} = \Sigma_{N_{\mathcal{A}}/2 \cap \mathbb{N}^n, \mathcal{A}}$ is the strongest of this type, taking $\mathcal{F} \supset N_{\mathcal{A}}/2 \cap \mathbb{N}^n$ yields the same relaxation

$\mathcal{P}_\mathcal{A}$ as generalized copositive cone

suppose $\mathcal{F} + \mathcal{F} \supset \mathcal{A}$

$$\mathcal{X}_\mathcal{F} = \{x_\mathcal{F} \,|\, x \in \mathbb{R}^n\}, \quad \mathcal{C}_\mathcal{F} = \{C \,|\, X^T C X \geq 0 \quad \forall\ X \in \mathcal{X}_\mathcal{F}\}$$

$$\mathcal{P}_\mathcal{A} = \{p \in \mathcal{L}_\mathcal{A} \,|\, \exists\ C \in \mathcal{C}_\mathcal{F} : p(x) = x_\mathcal{F}^T C x_\mathcal{F}\}$$

Definition [Luo,Sturm,Zhang 2003] Let $\mathcal{X} \subset \mathbb{R}^m$. A quadratic form C is called *copositive w.r.t. to the domain* $\mathcal{X}$ if $x^T C x \geq 0$ for all $x \in \mathcal{X}$.

$\mathcal{C}_\mathcal{F}$ cone of copositive forms w.r.t. to $\mathcal{X}_\mathcal{F}$

in general $\mathcal{C}_\mathcal{F}$ contains a linear subspace, induced by linear dependencies between the elements of XX^T, $X \in \mathcal{X}_\mathcal{F}$

condition $p \in \mathcal{L}_\mathcal{A}$ translates into linear constraints on C

Structure of $\mathcal{P}_{\mathcal{A}}$ (cont.)

$\mathcal{P}_{\mathcal{A}}$ is a projection of a section of the copositive cone $\mathcal{C}_{\mathcal{F}}$

projection: along the linear subspace

section: sets coefficients with indices in $(\mathcal{F} + \mathcal{F}) \setminus \mathcal{A}$ to zero

Structure of relaxation $\Sigma_{\mathcal{F},\mathcal{A}}$

relaxation $\Sigma_{\mathcal{F},\mathcal{A}}$: copositive cone w.r. to $\mathcal{X}_{\mathcal{F}}$ replaced by PSD cone (copositive w.r. to the whole space $\mathbb{R}^{\#\mathcal{F}}$)

larger domain $\Rightarrow$ smaller copositive cone

$\Sigma_{\mathcal{F},\mathcal{A}}$ projection of a section of the PSD cone

LMI representable copositive cones

examples of domains $\mathcal{X}$ with LMI representable copositive cone:

$$\mathcal{X} = \{x \mid B(x) \geq 0\}, \quad \mathcal{X} = \{x \mid B(x) = 0\}$$

B quadratic form ($\mathcal{S}$-lemma)

$$\mathcal{X} = \mathbb{R}_+^k, \quad k = 1, \ldots, 4$$

(classical copositive cones)

$$\mathcal{X} = E \cap H$$

E ellipsoid, H affine half-space ([Sturm,Zhang 2001])

$\mathcal{X}$ set of rank 1 matrices of size $2 \times n$ (matrices PSD $2n \times 2n$ block-Hankel)

$\mathcal{X}_{\mathcal{F}}$ for different $\mathcal{F}$

Theorem (Main observation) Let $\mathcal{F} = \{\beta^1, \ldots, \beta^N\} \subset \mathbb{Z}^n$, $\mathcal{F}' = \{{\beta'}^1, \ldots, {\beta'}^N\} \subset \mathbb{Z}^n$ s.t.

- $\beta^k \equiv {\beta'}^k \mathrm{mod} 2$ for all $k = 1, \ldots, N$
- the images of the matrices $(\beta^1, \ldots, \beta^N)^T$ and $({\beta'}^1, \ldots, {\beta'}^N)^T$ coincide

Then the closures of $\mathcal{X}_{\mathcal{F}}$ and $\mathcal{X}_{\mathcal{F}'}$ coincide.

$\Leftrightarrow$ $\exists$ invertible linear map $\mathcal{H} : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $\mathcal{F}' = \mathcal{H}[\mathcal{F}]$ and parity is preserved on $\mathcal{F}$

Consequence: $\mathcal{C}_{\mathcal{F}} = \mathcal{C}_{\mathcal{F}'}$

$\mathcal{A} \to \mathcal{A}' = \mathcal{H}[\mathcal{A}]$ corresponds to a nonlinear change of variables x in $\mathbb{R}^n$

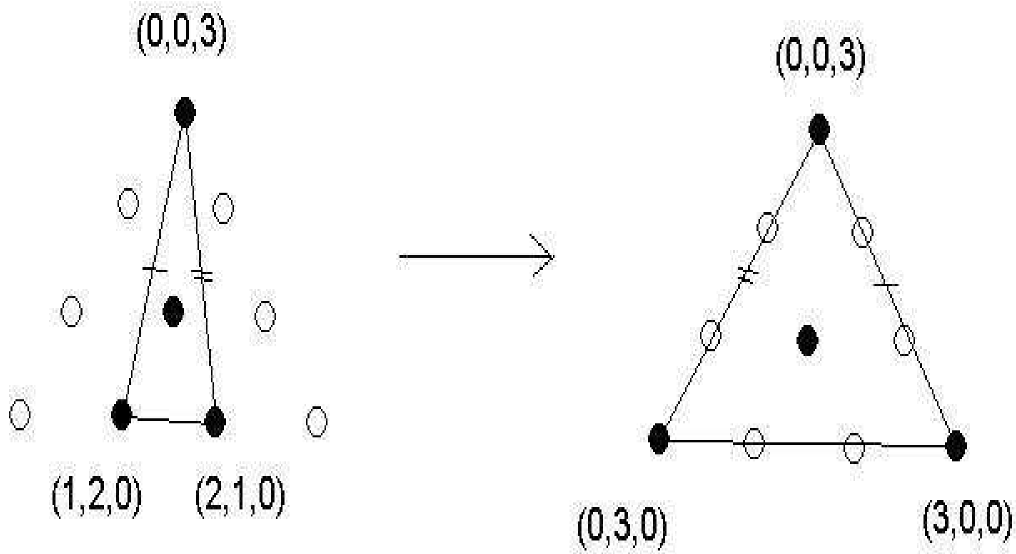
$\mathcal{F} + \mathcal{F} \supset \mathcal{A} \Rightarrow \mathcal{F}' + \mathcal{F}' \supset \mathcal{A}'$

$$\mathcal{P}_{\mathcal{A}} = \mathcal{P}_{\mathcal{A}'}, \qquad \Sigma_{\mathcal{F},\mathcal{A}} = \Sigma_{\mathcal{F}',\mathcal{A}'}$$

Motzkin polynomial: $\mathcal{F} = \{(0,0,3),(1,2,0),(2,1,0),(1,1,1)\}$

$$\begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 3 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 3 & 0 & 0 & 1 \end{pmatrix}$$

$\mathcal{F}' = \{(0,0,3),(3,0,0),(0,3,0),(1,1,1)\}$



Motzkin polynomial

$$p'_M(x,y,z) = x^6+y^6+z^6-3x^2y^2z^2 = (x^2+y^2+z^2)(x^4+y^4+z^4-x^2y^2-y^2z^2-z^2x^2)$$

$$\begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix} \succeq 0$$

p'_M is SOS

moreover: $\mathcal{P}_{\mathcal{A}} = \mathcal{P}_{\mathcal{A}'} = \Sigma_{\mathcal{A}'}$

$$p_1(c) = c_1x^4y^2+c_2x^2y^4+c_3z^6-c_4x^2y^2z^2, \qquad p_2(c) = c_2x^6+c_1y^6+c_3z^6-c_4x^2y^2z^2$$

$$p_1(c) \geq 0 \iff p_2(c) \geq 0 \iff p_2(c) \text{ is SOS}$$

New family of relaxations

let again $\mathcal{F} = N_{\mathcal{A}}/2 \cap \mathbb{N}^n$

$\Sigma_{\mathcal{F},\mathcal{A}} = \Sigma_{\mathcal{F}',\mathcal{A}'}$ but we can have

$$\mathcal{F}'' \supset \mathcal{F}' \quad \text{strictly}$$

s.t.

$$N_{\mathcal{F}''} = N_{\mathcal{F}'} = N_{\mathcal{A}'}/2$$

relaxation $\Sigma_{\mathcal{A}'} = \Sigma_{\mathcal{F}'',\mathcal{A}'}$ with $\mathcal{F}'' = N_{\mathcal{A}'}/2 \cap \mathbb{Z}^n$ can be sharper than $\Sigma_{\mathcal{A}}$

we can have $\dim \Sigma_{\mathcal{A}'} > \dim \Sigma_{\mathcal{A}}$

the relaxation can even be exact for some $\mathcal{A}'$

$\mathcal{H}$ isomorphism of $\mathbb{Z}^n$ ($\det \mathcal{H} = \pm 1$), then $\Sigma_{\mathcal{A}} = \Sigma_{\mathcal{A}'}$

$\Rightarrow$ we can consider equivalence classes