Lower bounds on the barrier parameter of convex cones

Roland Hildebrand

Université Grenoble 1 / CNRS

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   - Cross-ratio manifold
   - Boundary frames and Lagrangian submanifolds
3. Lower bound on the barrier parameter
   - General idea
   - Implementation
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Conic programs

Definition

A conic program over a regular (with non-empty interior, containing no lines) convex cone $K \subset \mathbb{R}^n$ is an optimization problem of the form

$$\min_{x \in K} \langle c, x \rangle : \quad Ax = b.$$

minimization of a linear function over the intersection of K with an affine subspace

Logarithmically homogeneous barriers

Definition (Nesterov, Nemirovski 1994)

Let $K \subset \mathbb{R}^n$ be a regular convex cone. A (self-concordant logarithmically homogeneous) barrier on K is a smooth function $F : K^o \to \mathbb{R}$ on the interior of K such that

- $F(\alpha x) = -\nu \log \alpha + F(x)$ (logarithmic homogeneity)
- $F''(x) \succ 0$ (convexity)
- $\lim_{x \to \partial K} F(x) = +\infty$ (boundary behaviour)
- $|F'''(x)[h,h,h]| \leq 2(F''(x)[h,h])^{3/2}$ (self-concordance)

for all tangent vectors h at x.
The homogeneity parameter ν is called the barrier parameter.

Duality

Theorem (Nesterov, Nemirovski 1994)

Let $K \subset \mathbb{R}^n$ be a regular convex cone and $F : K^o \to \mathbb{R}$ a barrier on K with parameter ν. Then the Legendre transform F^* is a barrier on $-K^*$ with parameter ν.

- the map $x \mapsto F'(x)$ takes the level surfaces of F to the level surfaces of F^*
- the map $x \mapsto -F'(x)$ is an isometry between K^o and $(K^*)^o$ with respect to the Hessian metrics defined by F'', $(F^*)''$

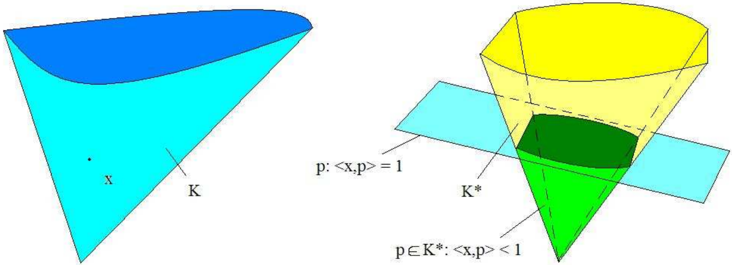
Barrier parameter

interior-point algorithms for solving conic programs make use of barriers

the smaller ν, the faster the algorithm

Question: For a given regular convex cone K, what is the smallest possible value ν^* of the barrier parameter for a barrier on K?

Volume function



volume function $V : K^o \ni x \mapsto Vol\{p \in K^* \mid \langle x, p \rangle < 1\}$

Universal barrier

Theorem (Nesterov, Nemirovski 1994)

There exists an absolute constant $c > 0$ such that

$$F(x) = c \log V(x)$$

is a $(c \cdot n)$-self-concordant barrier on $K \subset \mathbb{R}^n$.

Lemma (Güler 1996)

The universal barrier equals up to an additive constant $c \log \varphi(x)$, where

$$\varphi(x) = \int_{K^*} e^{-\langle x,p \rangle} dp$$

is the characteristic function of the cone.

Homogeneous cones

if the automorphism group of K acts transitively on K^o, then K is called homogeneous

homogeneous cones are related to T-algebras and a rank can be associated with them [Vinberg 1962]

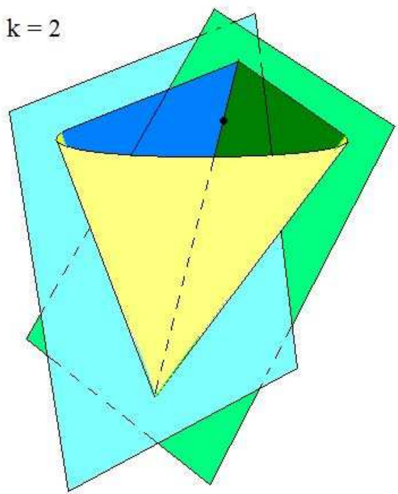
Lemma (Güler, Tunçel 1998)

Let K be a homogeneous convex cone with rank r.
Then the optimal barrier parameter on K equals r.

Cones with corners

Lemma (Nesterov, Nemirovski 1994)

Let K be a regular convex cone, $z \in \partial K$, $U \subset \mathbb{R}^n$ a neighbourhood of z, $A_1, \dots, A_k \subset \mathbb{R}^n$ closed affine half-spaces with $z \in \partial A_i$ for all i such that the normals to the half-spaces at z are linearly independent and the intersection $U \cap K$ equals the intersection $U \cap A_1 \cap \dots \cap A_k$.
Then a lower bound on the barrier parameter of any barrier on K is given by $\nu_* = k$.

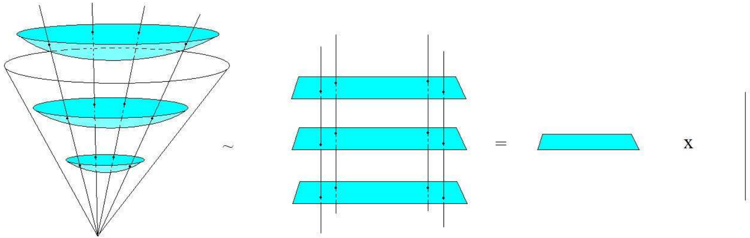

k = 2

Splitting theorem

Theorem (Tsuji 1982; Loftin 2002)

Let $K \subset \mathbb{R}^{n+1}$ be a regular convex cone, and $F : K^o \to \mathbb{R}$ a locally strongly convex logarithmically homogeneous function. Then the Hessian metric on K^o splits into a direct product of a radial 1-dimensional part and a transversal n-dimensional part. The submanifolds corresponding to the radial part are rays, the submanifolds corresponding to the transversal part are level surfaces of F.

all nontrivial information contained in the transversal part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Projective images of cones

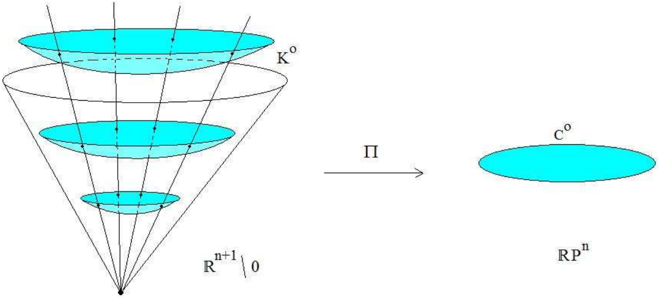
let $\mathbb{R}P^n, \mathbb{R}P_n$ be the primal and dual real projective space — lines and hyperplanes through the origin of $\mathbb{R}^{n+1}$

let $F : K^o \to \mathbb{R}$ be a barrier on a regular convex cone $K \subset \mathbb{R}^{n+1}$

the canonical projection $\Pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ maps $K \setminus \{0\}$ to a compact convex subset $C \subset \mathbb{R}P^n$

the canonical projection $\Pi^* : \mathbb{R}_{n+1} \setminus \{0\} \to \mathbb{R}P_n$ maps $K^* \setminus \{0\}$ to a compact convex subset $C^* \subset \mathbb{R}P_n$

the interiors of C, C^* are isomorphic to the transversal factors of $K^o, (K^*)^o$ and acquire the metric of these factors



passing to the projective space removes the radial factor

Self-concordance

Theorem

Let $K \subset \mathbb{R}^{n+1}$ a regular convex cone and $F : K^o \to \mathbb{R}$ a logarithmically homogeneous locally strongly convex function with homogeneity parameter ν. Then F is self-concordant if and only if

$$|F'''(x)[h,h,h]| \leq 2\frac{\gamma}{\sqrt{\nu}}\left(F''(x)[h,h]\right)^{3/2}$$

for all tangent vectors h which are parallel to the level surfaces of F. Here $\gamma = \frac{\nu-2}{\sqrt{\nu-1}}$.

this is a condition on the transversal factor only

Product of projective spaces

between elements of $\mathbb{R}P^n, \mathbb{R}P_n$ there is no scalar product, but an orthogonality relation

the set

$$\mathcal{M} = \{(x, p) \in \mathbb{R}P^n \times \mathbb{R}P_n \,|\, x \not\perp p\}$$

is dense in $\mathbb{R}P^n \times \mathbb{R}P_n$

$$\partial\mathcal{M} = \{(x, p) \in \mathbb{R}P^n \times \mathbb{R}P_n \,|\, x \perp p\}$$

is a submanifold of $\mathbb{R}P^n \times \mathbb{R}P_n$ of codimension 1

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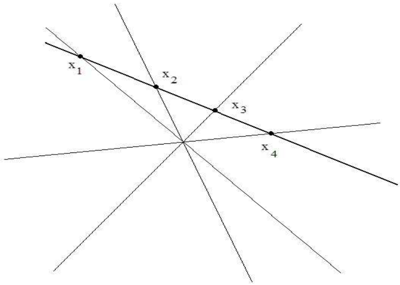
Cross-ratio

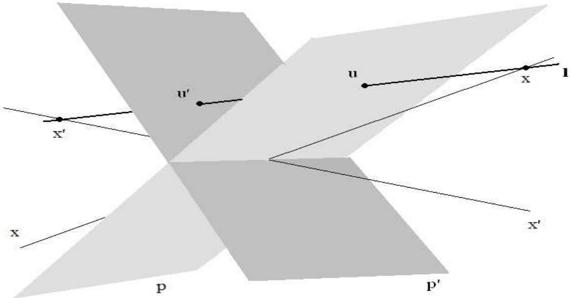


x_1, x_2, x_3, x_4 points on the projective line $\mathbb{R}P^1$

$$(x_1, x_2; x_3, x_4) = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_2 - x_3)(x_1 - x_4)}$$

Generalization to n dimensions

[Ariyawansa, Davidon, McKennon 1999]: instead of 4 collinear points use 2 points and 2 dual points



$(u, x'; u', x)$ — quadra-bracket of x, p, x', p'

Two-point function on $\mathcal{M}$

let $z = (x, p), z' = (x', p') \in \mathcal{M} \subset \mathbb{R}P^n \times \mathbb{R}P_n$

$$(z; z') = (z'; z) := (u, x'; u', x)$$

defines a symmetric function $(\cdot; \cdot) : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$

$$\lim_{z \to \partial\mathcal{M}} (z; z') = \pm\infty$$

Pseudo-metric

Theorem

There exists a complete pseudo-metric g of neutral signature on $\mathcal{M}$ such that

- *If $(z;z') > 0$, then the velocity vector of the geodesic linking z,z' has positive square and $d(z,z') = \arcsin\sqrt{(z;z')}$.*
- *If $(z;z') = 0$ the geodesic linking z,z' is light-like.*
- *If $(z;z') < 0$, then the velocity vector of the geodesic linking z,z' has negative square and $d(z,z') = arc\sinh\sqrt{-(z;z')}$.*

Symplectic structure

$\mathcal{M} \subset \mathbb{R}P^n \times \mathbb{R}P_n$ implies that the tangent space to $\mathcal{M}$ at $z = (x, p) \in \mathcal{M}$ is a direct product of two subspaces which are parallel to the factors: $h = (h_x, h_p)$

let $J : T\mathcal{M} \to T\mathcal{M}$ be a linear operator on the tangent bundle acting as $h = (h_x, h_p) \mapsto (h_x, -h_p)$

Theorem

The skew-symmetric form defined by

$$\omega(h, h') = g(Jh, h')$$

is a symplectic form which is parallel with respect to the pseudo-metric g.

Explicit representation

let $\{z = (x, p) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, 1 + p^T x \neq 0\}$ be a chart on $\mathcal{M}$ such that

x is the line going through $(1, x_1, \dots, x_n)^T \in \mathbb{R}^{n+1}$

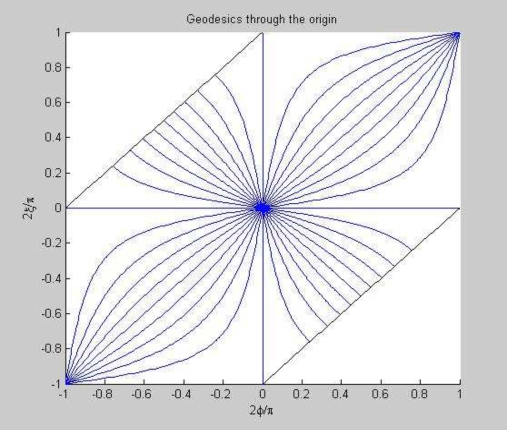
p is the hyperplane $\{y \in \mathbb{R}^{n+1} \,|\, y_0 + y_1 p_1 + \dots + y_n p_n = 0\}$

then

$$g = \frac{1}{2}\begin{pmatrix} 0 & \frac{(1+p^T x)I - px^T}{(1+p^T x)^2} \\ \frac{(1+p^T x)I - xp^T}{(1+p^T x)^2} & 0 \end{pmatrix}$$

$$\omega = \frac{1}{2}\begin{pmatrix} 0 & \frac{(1+p^T x)I - px^T}{(1+p^T x)^2} \\ -\frac{(1+p^T x)I - xp^T}{(1+p^T x)^2} & 0 \end{pmatrix}$$

2-dimensional case



$\mathbb{R}P^1 \sim S^1$
$\mathbb{R}P^1 \times \mathbb{R}P_1 \sim T^2$

$\mathbb{R}P^1$ parameterized by ϕ
$\mathbb{R}P_1$ parameterized by ξ
$(\phi, \xi) \in [-\frac{\pi}{2}, \frac{\pi}{2})^2$

$\partial\mathcal{M} = \{(\phi, \xi) \,|\, \xi = \phi \pm \frac{\pi}{2}\}$

$\partial\mathcal{M} \sim S^1$
$\mathcal{M} \sim S^1 \times \mathbb{R}$

$g = \cos^{-2}(\phi - \xi) d\phi d\xi$

Summary

the $2n$-dimensional manifold $\mathcal{M}$ consisting of pairs $z = (x, p)$ of lines $x \subset \mathbb{R}^{n+1}$ and hyperplanes $p \subset \mathbb{R}^{n+1}$ through the origin such that $x \cap p = \{0\}$ carries a natural

- pseudo-metric g whose geodesic distance is given by the cross-ratio
- symplectic structure ω which is compatible with g

call $\mathcal{M}$ equipped with g and ω the cross-ratio manifold

Images of conic boundaries

the canonical projection
$\Pi \times \Pi^* : (\mathbb{R}^{n+1} \setminus \{0\}) \times (\mathbb{R}_{n+1} \setminus \{0\}) \to \mathbb{R}P^n \times \mathbb{R}P_n$ maps the set

$$\Delta_K = \{(x,p) \in (\partial K \setminus \{0\}) \times (\partial K^* \setminus \{0\}) \,|\, x \perp p\}$$

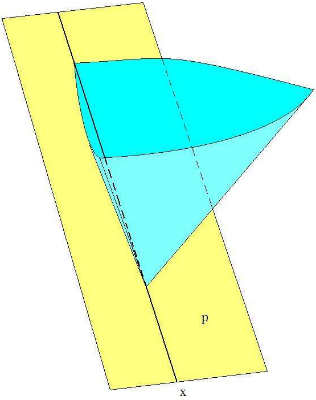
to a set $\delta_K \subset \partial\mathcal{M}$

Lemma

The projections π, π^* of $\mathbb{R}P^n \times \mathbb{R}P_n$ to the factors map δ_K onto ∂C and ∂C^*, respectively.
If K is smooth, then δ_K is homeomorphic to S^{n-1}.

call δ_K the boundary frame corresponding to the cone K

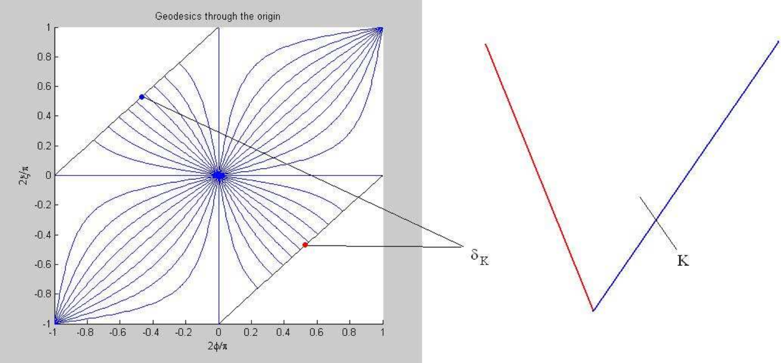
Geometric interpretation



the boundary frame δ_K consists of pairs $z = (x, p) \in \partial\mathcal{M}$ where

- the line x contains a ray in ∂K
- p is a supporting hyperplane at x

2-dimensional case



the boundary frame of a 2-dimensional cone consists of 2 points which can be linked by a (complete) geodesic with negative squared velocity

Second fundamental form

let $M \subset \mathcal{M}$ be a submanifold of a (pseudo-)Riemannian space

choose a point $x \in M$ and a tangent vector $h \in T_x M$

consider the geodesics $\gamma_M, \gamma_{\mathcal{M}}$ in M and in $\mathcal{M}$ through x with velocity h

there is a second-order deviation

$$\gamma_M(t) - \gamma_{\mathcal{M}}(t) = \frac{d^2}{dt^2}|_{t=0}(\gamma_M - \gamma_{\mathcal{M}})\frac{t^2}{2} + O(t^3)$$

whose main term depends quadratically on h

the acceleration is called the second fundamental form II of M

$II_x : T_x M \times T_x M \to (T_x M)^\perp$

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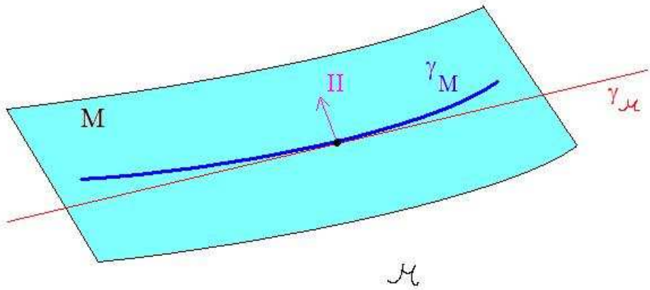


the second fundamental form measures the deviation of M from a geodesic submanifold

it is also called the extrinsic curvature

Isometry between the projective images

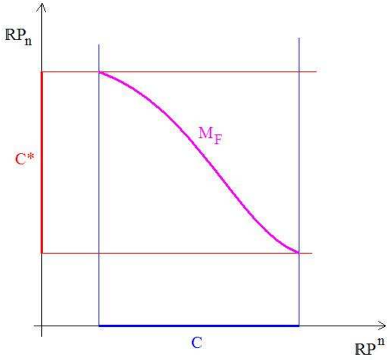
let $K \subset \mathbb{R}^{n+1}$ be a regular convex cone and $F : K^o \to \mathbb{R}$ a barrier on K

F defines a bijection between K^o and $(K^*)^o$ by $x \mapsto -F'(x)$

this factors through to an isometry between C^o and $(C^*)^o$

$$\begin{array}{ccc} K^o & \xrightarrow{-F'} & (K^*)^o \\ \Pi \downarrow & & \Pi^* \downarrow \\ C^o \sim K^o/\mathbb{R}_+ & \xrightarrow{\mathcal{I}_F} & (C^*)^o \sim (K^*)^o/\mathbb{R}_+ \end{array}$$

Images of barriers

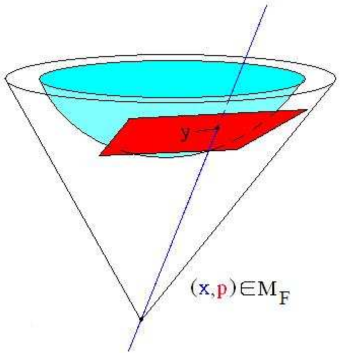


define M_F as the graph of the isometry $\mathcal{I}_F$

$$M_F = \Pi \times \Pi^* \left[\{ (x, -F'(x)) \,|\, x \in K^o \} \right] \subset \mathbb{R}P^n \times \mathbb{R}P_n$$

is a smooth submanifold

Geometric interpretation



the manifold M_F consists of pairs (x, p) where

- x is a line through a point $y \in K^o$
- p is parallel to the hyperplane which is tangent to the level surface of F at y

if $y \to \hat{y} \in \partial K$, then p tends to a supporting hyperplane at $\hat{y}$

Properties of M_F

Theorem (H., 2011)

Let $F : K^o \to \mathbb{R}$ be a barrier on a regular convex cone $K \subset \mathbb{R}^{n+1}$ with parameter ν. The manifold $M_F \subset \mathbb{R}P^n \times \mathbb{R}P_n$ is

- *a nondegenerate Lagrangian submanifold of $\mathcal{M}$*
- *bounded by δ_K in $\mathbb{R}P^n \times \mathbb{R}P_n$*
- *its metric is $-\nu$ times the metric generated by $C^o, (C^*)^o$*
- *its second fundamental form II satisfies*

$$F'''[h,h,h'] = 2\omega(II(\tilde{h},\tilde{h}),\tilde{h}')$$

for all vectors h, h' tangent to the level surfaces of F and their images $\tilde{h}, \tilde{h}'$ on the tangent bundle TM_F.

"Lagrangian" means $\omega|_{M_F} = 0$

Interpretation of self-concordance

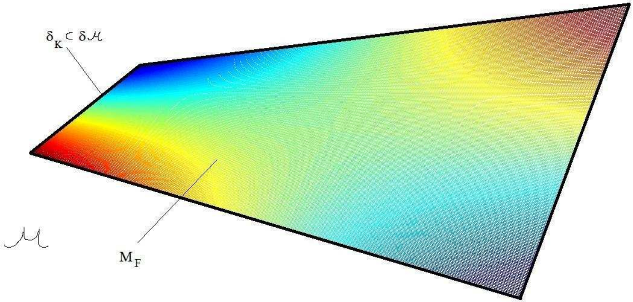
recall: F''' bounded by 2 on the whole tangent space $\Leftrightarrow$
F''' bounded by $\frac{2\gamma}{\sqrt{\nu}}$ with $\gamma = \frac{\nu-2}{\sqrt{\nu-1}}$ on the transversal factor

Corollary

Let $K \subset \mathbb{R}^{n+1}$ be a regular convex cone and $F : K^o \to \mathbb{R}$ a locally strongly convex logarithmically homogeneous function with parameter ν.
Then F is self-concordant if and only if the Lagrangian submanifold $M_F \subset \mathcal{M}$ has its second fundamental form bounded by $\gamma = \frac{\nu-2}{\sqrt{\nu-1}}$.

the barrier parameter determines how close M_F is to a geodesic submanifold

Geometric interpretation of a barrier



- complete negative definite Lagrangian submanifold
- bounded by δ_K
- second fundamental form bounded by $\gamma = \frac{\nu-2}{\sqrt{\nu-1}}$

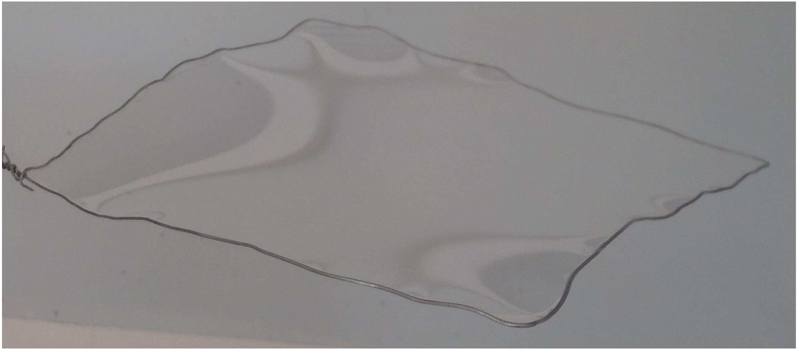
Geometric reformulation

Question: For a given boundary frame $\delta_K \subset \partial\mathcal{M}$, how to inscribe a Lagrangian submanifold $M_F \subset \mathcal{M}$ which is as flat as possible.

the ambient manifold $\mathcal{M}$ is not flat — above we mean "flat" in the sense of close to a geodesic submanifold

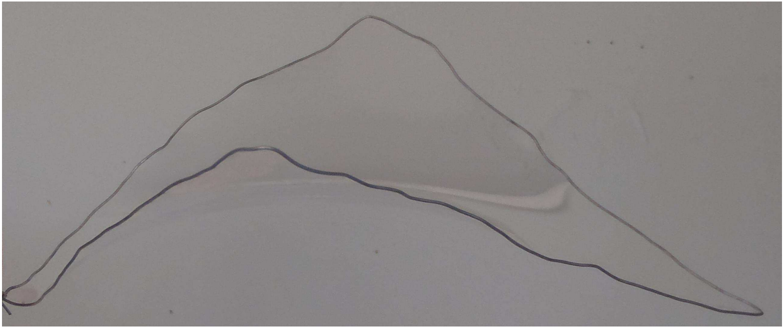
... let us first consider the problem in a flat Riemannian manifold

Flat example



this is more or less flat ...

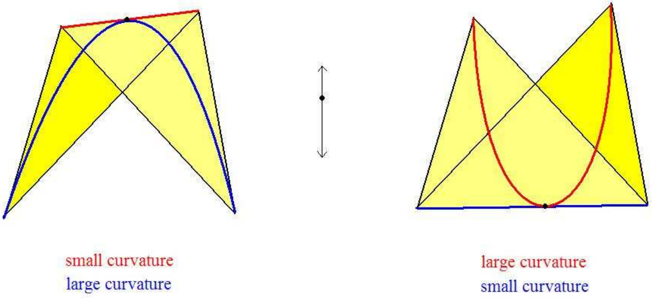
Curved example



... this is not flat at all

how to bound the curvature from below?

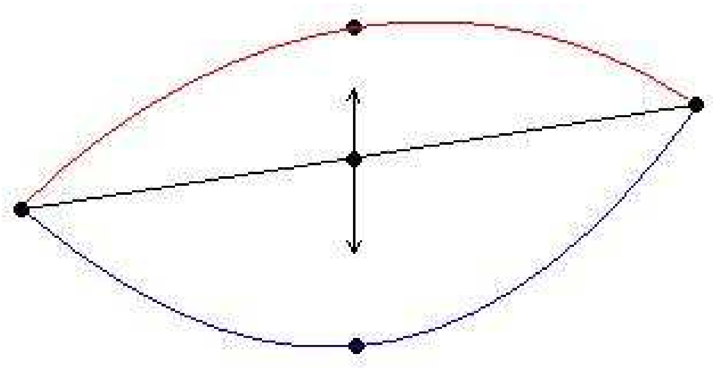
Minimax problem



a lower bound on the curvature is given by best trade-off between the two directions

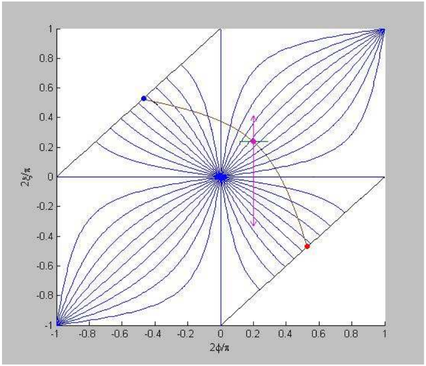
have to solve a minimax problem on the height of the position of the central point

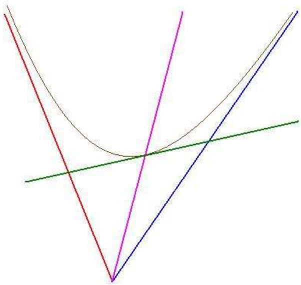
1-dimensional subproblem



determine a lower bound on the curvature under a constraint on the height of the central point

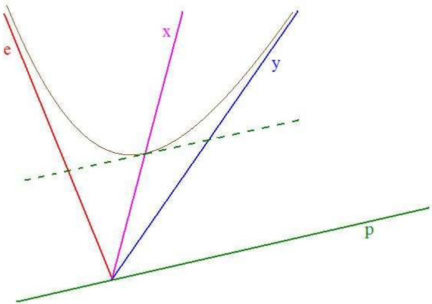
1-dimensional subproblem





- primal position of central point $\leftrightarrow$ ray in K^o
- dual position of central point $\leftrightarrow$ slope of level curve

Solution of subproblem



let $r = (e, y; x, p) \in (-\infty, 0)$ be the cross-ratio

then the barrier parameter is not smaller than $\nu_* = \frac{2}{1+\frac{|r+1|}{r-1}}$

Minimax problem

let $K \subset \mathbb{R}^n$ be a regular convex cone

let (e_k, y_k), $k = 1, \dots, n$, be opposite points on ∂K such that all segments $[e_k, y_k]$ intersect in a common point x

have to solve minimax problem on the direction p of $-F'(x)$

$$\nu_* = \min_p \max_k \frac{2}{1 + \frac{|(e_k, y_k; x, p) + 1|}{(e_k, y_k; x, p) - 1}}$$

$n - 1$ parameters to adjust in order to minimize the maximum over n quantities

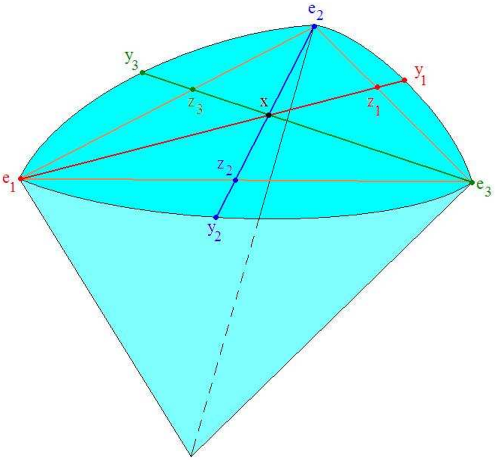
Solution of minimax problem

- let z_k be the point on $[e_k, y_k]$ that can be expressed as a linear combination of $e_1, \dots, e_{k-1}, e_{k+1}, \dots, e_n$
- let $q_k = (e_k, x; y_k, z_k)$ be the cross-ratios of the collinear quadruples

Theorem

A lower bound on the barrier parameter of K is given by

$$\nu_* = \frac{2}{1 - \frac{|(\sum_{k=1}^n q_k) - 2|}{\sum_{k=1}^n q_k}}.$$



x is required to lie in K^o but not necessarily in the simplex generated by e_k

Upper bound on the bound

Lemma (H., 2011)

Let $K \subset \mathbb{R}^n$ be a regular convex cone and ν_* a lower bound on the barrier parameter of barriers on K constructed as above. Then

- $\nu_* \geq 2$*, with equality if and only if there exists a second-order cone fitting the data*
- $\nu_* \leq n$*, with equality if and only if K is a simplicial cone generated either by the e_k or by the y_k*

Two "round" cones

let $p, q > 2$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $\gamma \geq 0$ be the root of the equation

$$q(1+\gamma^{-1/q}) = p(1+\gamma^{-1/p})$$

consider the power cone

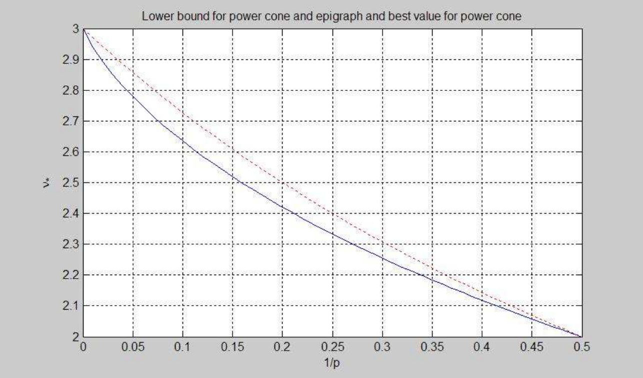
$$K_p = \left\{(x,y,z) \in \mathbb{R}^3 \,|\, x \geq 0,\ y \geq 0,\ z^2 \leq x^{1/p}y^{1/q}\right\}$$

and the epigraph of the p-norm in $\mathbb{R}^2$

$$K_e = \left\{(x,y,z) \in \mathbb{R}^3 \,|\, z \leq (|x|^p + |y|^p)^{1/p}\right\}$$

Lemma (H., 2011)

A lower bound on the barrier parameter for K_p, K_e is given by $\nu_* = 1 + \frac{1+\gamma^{1/p}}{1+\gamma^{1/q}}$.



solid blue: lower bound for the power cone and the epigraph
dotted red: best known value for the power cone

Epigraph of $||\cdot||_\infty$-norm

consider the epigraph of the $||\cdot||_\infty$-norm in $\mathbb{R}^{n-1}$

$$K = \{x \in \mathbb{R}^n \,|\, x_0 \geq |x_k| \quad \forall\, k = 1, \dots, n-1\}$$

Lemma

An optimal barrier for K is given by

$$F(x_0, \dots, x_{n-1}) = -\sum_{k=1}^{n-1} \log(x_0^2 - x_k^2) + (n-2)\log x_0.$$

It has barrier parameter n.

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Thank you