Efficient Methods in Optimization: Simplex Method — Mixed Integer Linear Programs

Roland Hildebrand

MSIAM2, ENSIMAG

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Simplex pivot operation

the tableau
$$\begin{array}{c|c} -\gamma & \xi^T \\ \hline \mu & M \end{array}$$
with basic set B and non-basic set N evolves by the rules

$$\begin{bmatrix} i & \leftarrow & j \\ j & \leftarrow & i \\ \mu_{\tilde{B}} & \leftarrow & \mu_{\tilde{B}} - M_{ij}^{-1} M_{\tilde{B}j} \mu_i \\ \mu_i & \leftarrow & M_{ij}^{-1} \mu_i \\ \xi_{\tilde{N}} & \leftarrow & \xi_{\tilde{N}} - M_{ij}^{-1} \xi_j M_{i\tilde{N}}^T \\ \xi_j & \leftarrow & -M_{ij}^{-1} \xi_j \\ -\gamma & \leftarrow & -\gamma - M_{ij}^{-1} \xi_j \mu_i \\ M_{\tilde{B}\tilde{N}} & \leftarrow & M_{\tilde{B}\tilde{N}} - M_{ij}^{-1} M_{\tilde{B}j} M_{i\tilde{N}} \\ M_{\tilde{B}j} & \leftarrow & -M_{ij}^{-1} M_{\tilde{B}j} \\ M_{i\tilde{N}} & \leftarrow & M_{ij}^{-1} M_{i\tilde{N}} \\ M_{ij} & \leftarrow & M_{ij}^{-1} \end{bmatrix}$$

when pivoting at M_{ij}, where $\tilde{B} = B \setminus \{i\}$, $\tilde{N} = N \setminus \{j\}$

Primal simplex method

evolves the primal feasible ($\mu \geq 0$) simplex tableau until either unbounded-ness or optimality is detected

each step consists of the following stages:

- choose column $j \in N$ such that $\xi_j < 0$
- among those rows $k \in B$ such that $M_{kj} > 0$, let i be the index minimizing the ratio $M_{kj}^{-1} \mu_k$
- update the tableau by pivoting at element M_{ij}

algorithm stops if

- all ξ_j are nonnegative (optimality)
- all M_{kj} are non-positive (unbounded-ness)

Example

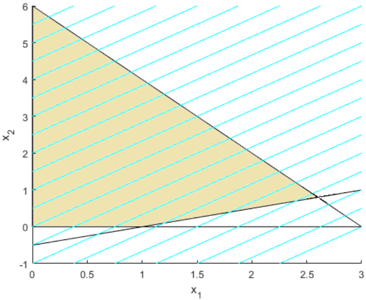
consider the LP

$$\min_{x \in \mathbb{R}^2_+} (3x_2 - 4x_1) :$$

$$x_1 - 2x_2 \le 1, \; 2x_1 + x_2 \le 6$$

introduce slacks

$$x_3 = 1 - x_1 + 2x_2, \; x_4 = 6 - 2x_1 - x_2$$

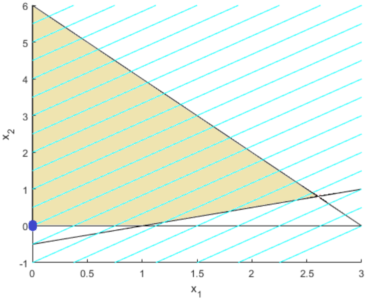


feasible set and objective level lines

vertex $(x_1, x_2) = (0, 0)$
corresponds to basic set
$B = (3, 4)$, non-basic set
$N = (1, 2)$, value 0 with tableau

0	-4	3
1	1	-2
6	2	1

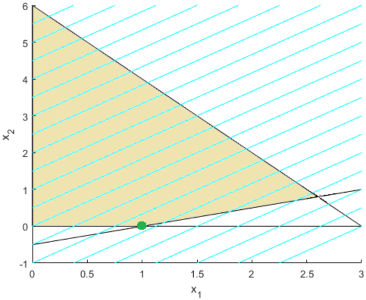
- choose pivot column 1 ($\xi_1 = -4 < 0$)
- choose pivot row 3 ($\frac{\mu_3}{M_{31}} = 1 < 3 = \frac{\mu_4}{M_{41}}$)
- pivot at element M_{31}



we arrive at the vertex $(x_1, x_2) = (1, 0)$ with basic set $B = (1, 4)$, non-basic set $N = (3, 2)$, value -4, and tableau

4	4	-5
1	1	-2
4	-2	5

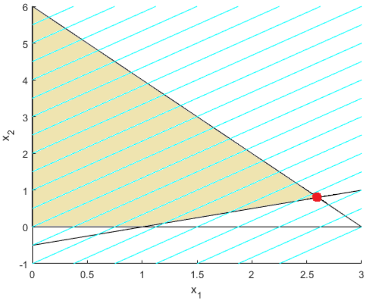
- choose pivot column 2 ($\xi_2 = -5 < 0$)
- choose pivot row 4 ($M_{42} = 5 > 0$)
- pivot at element M_{42}



we arrive at the vertex $(x_1, x_2) = (\frac{13}{5}, \frac{4}{5})$ with basic set $B = (1, 2)$, non-basic set $N = (3, 4)$, value -8, and tableau

8	2	1
$\frac{13}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
$\frac{4}{5}$	$-\frac{2}{5}$	$\frac{1}{5}$

tableau is optimal, $\xi \geq 0$



Dual simplex method

evolves the dual feasible ($\xi \geq 0$) simplex tableau until either infeasibility or optimality is detected

each step consists of the following stages:

- choose column $i \in B$ such that $\mu_i < 0$
- among those columns $k \in N$ such that $M_{ik} < 0$, let j be the index minimizing the ratio $-M_{ik}^{-1}\xi_k$
- update the tableau by pivoting at element M_{ij}

algorithm stops if

- all μ_i are nonnegative (optimality)
- all M_{ik} are nonnegative (infeasibility)

Example

consider again the LP

$$\min_{x \in \mathbb{R}^2_+} (3x_2 - 4x_1) :$$

$$x_1 - 2x_2 \leq 1, \; 2x_1 + x_2 \leq 6$$

with optimal solution
$(x_1, x_2) = (\frac{13}{5}, \frac{4}{5})$

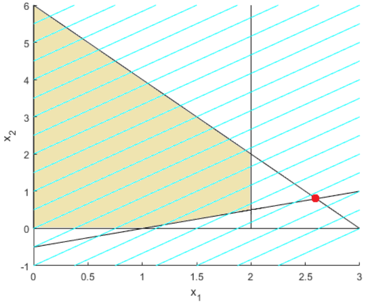
suppose we add a *new constraint*

$$x_1 \leq 2$$

introduce a new slack variable

$$x_5 = 2 - x_1$$

optimal point becomes infeasible



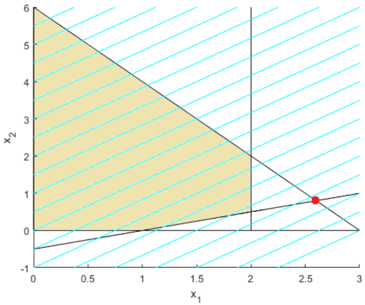
the new slack is basic, and we have to add a *new row* to the tableau

now $B = (1, 2, 5)$, $N = (3, 4)$

we have

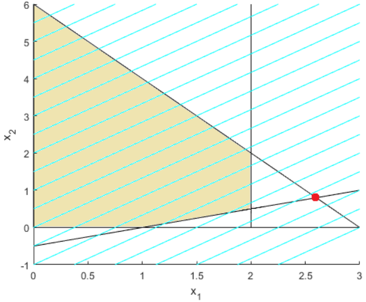
$$\begin{aligned} x_5 &= 2 - (\frac{13}{5} - \frac{1}{5}x_3 - \frac{2}{5}x_4) \\ &= -\frac{3}{5} - (-\frac{1}{5}x_3 - \frac{2}{5}x_4) \end{aligned}$$

8	2	1
$\frac{13}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
$\frac{4}{5}$	$-\frac{2}{5}$	$\frac{1}{5}$
$-\frac{3}{5}$	$-\frac{1}{5}$	$-\frac{2}{5}$



$B = (1, 2, 5)$, $N = (3, 4)$

8	2	1
$\frac{13}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
$\frac{4}{5}$	$-\frac{2}{5}$	$\frac{1}{5}$
$-\frac{3}{5}$	$-\frac{1}{5}$	$-\frac{2}{5}$



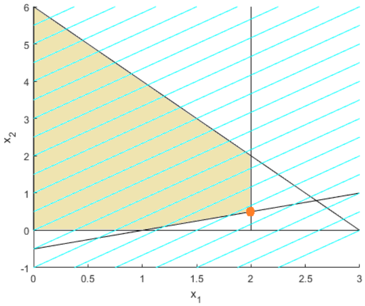
dual simplex step:

- choose pivot row 5 $(\mu_5 = -\frac{3}{5} < 0)$
- choose pivot column 4 $(-\frac{\xi_4}{M_{54}} = \frac{5}{2} < 10 = -\frac{\xi_3}{M_{53}})$
- pivot at element M_{54}

we arrive at the vertex $(x_1, x_2) = (2, \frac{1}{2})$ with basic set $B = (1, 2, 4)$, non-basic set $N = (3, 5)$, value $-\frac{13}{2}$, and tableau

$\frac{13}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
2	0	1
$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{5}{2}$

tableau is optimal, $\mu \geq 0$



Mixed integer linear programs

linear program with additional integrality constraints on a part of the decision variables

$$\min_{x \geq 0} \langle c, x \rangle : \qquad Ax = b, \quad x_i \in \mathbb{Z} \quad \forall\ i \in I$$

in general NP-hard

removing the integrality constraints yields the *linear relaxation*

$$\min_{x \geq 0} \langle c, x \rangle : \qquad Ax = b$$

feasible set of LP larger than that of MILP
optimal value of LP is a *lower* bound on the value of the MILP

Branching

let x^* be the solution of the LP relaxation

if x_I^* happens to be integral, then x^* is *optimal* also for the MILP

in general there exists an index $i \in I$ such that x_i^* is fractional

branching on x_i means constructing the two linear programs

$$\min_{x \geq 0} \langle c, x \rangle : \qquad Ax = b, \quad x_i \leq \lfloor x_i^* \rfloor \tag{1}$$

$$\min_{x \geq 0} \langle c, x \rangle : \qquad Ax = b, \quad x_i \geq \lceil x_i^* \rceil \tag{2}$$

- feasible sets of LPs (1),(2) are disjoint
- their union contains the feasible set of the original MILP
- but does not contain the former LP solution x^*

the minimum of the values of LPs (1),(2) is a better lower bound

Branch-and-bound

MILP solvers

- recursively split the feasible set of the MILP into smaller parts (branch)
- and solve the corresponding LP relaxations (bound)

in addition there may be modules strengthening the LP relaxations:

- presolve algorithms tightening the bounds on the integer variables
- cuts separating fractional solutions from the feasible set of the MILP

Use of dual simplex

the LPs obtained by branching differs from the original LP by one constraint, namely an integer bound on the branching variable x_i

in the optimal simplex table of the original LP the slack corresponding to the constraint is basic (the constraint is not active)

changing the constraint amounts to changing the corresponding value in the vector b to a negative value in $(-1, 0)$
this modification turns the table infeasible, but it remains dual feasible

hence we may return the table to optimality by the dual simplex method

Example

consider the MILP

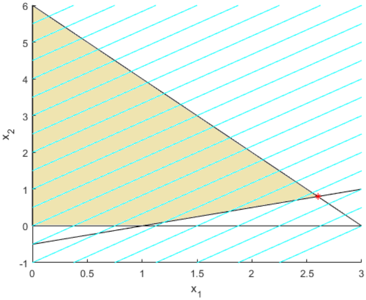
$$\min_{x \in \mathbb{R}^2_+} (3x_2 - 4x_1) : \quad x_1 - 2x_2 \leq 1,\ 2x_1 + x_2 \leq 6,\ x \in \mathbb{Z}^2$$

linear relaxation:

$$\min_{x \in \mathbb{R}^2_+} (3x_2 - 4x_1) :$$

$$x_1 - 2x_2 \leq 1,\ 2x_1 + x_2 \leq 6$$

solution $(\frac{13}{5}, \frac{4}{5})$, value -8



Example

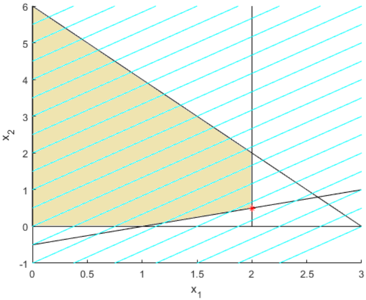
branch on x_1
value at optimal solution $x_1^* = \frac{13}{5}$

next LP:

$$\min_{x \in \mathbb{R}^2_+} (3x_2 - 4x_1) :$$

$$x_1 - 2x_2 \le 1,\ 2x_1 + x_2 \le 6,\ x_1 \le 2$$

solution $(2, \frac{1}{2})$, value $-\frac{13}{2}$



the second LP is infeasible with constraint $x_1 \ge 3$

Example

branch on x_2, value at optimal solution $x_2^* = \frac{1}{2}$

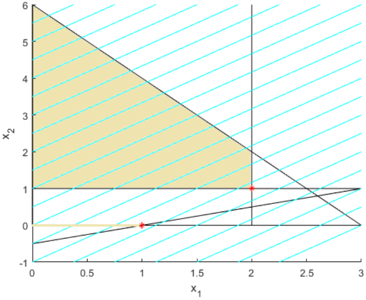
next LPs:

$$\min_{x \in \mathbb{R}^2_+} (3x_2 - 4x_1):$$

$$x_1 - 2x_2 \leq 1, 2x_1 + x_2 \leq 6, x_1 \leq 2, x_2 \geq 1$$

or

$$x_1 - 2x_2 \leq 1, 2x_1 + x_2 \leq 6, x_1 \leq 2, x_2 \leq 0$$



solutions $(2, 1)$ and $(1, 0)$
values -5 and -4

both LPs have integer solutions
optimal value of the MILP is the lower one -5, solution is $(2, 1)$