Efficient methods in optimization

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all documents allowed

1. Affine geometry

Consider the sets

$$X_1 = \{(-1,0),(0,1),(1,0)\} \subset \mathbb{R}^2,$$
$$X_2 = \{(0,1),(2,-1),(-1,2)\} \subset \mathbb{R}^2.$$

- Which of these sets are affinely independent?
- Find the affine hulls of these sets.
- Does there exist an affine map from X_1 onto X_2? Does there exist an affine map from X_2 onto X_1?

2. Which of the following sets are convex / open / closed?

- $L_3 = \{(x_0,x_1,x_2) \in \mathbb{R}^3 \mid x_0 \geq \sqrt{x_1^2+x_2^2}\}$;
- $\{(x,y,z) \in \mathbb{R}^3 \,|\, (x^p+y^p+z^p)^{1/p} \leq 1\}$ for $p = \frac{1}{2}, 1, 3$;
- $\{(x,y) \in \mathbb{R}^2 \,|\, y > e^x\}$;
- $\{(x,y) \in \mathbb{R}^2 \,|\, 0 < y \leq x^2\}$;

3. Projections.

Let $X = [-1,1]^2$ be the unit ball of the ∞-norm in $\mathbb{R}^2$. Let $\mathbb{R}^2$ be equipped with the standard Euclidean distance.

Compute the distance from and the projection on the convex set X for the following points:

- $a = (\frac{1}{2}, \frac{1}{2})$;
- $b = (2, \frac{1}{2})$;
- $c = (-\frac{3}{2}, 2)$.

4. Compute supporting hyperplanes to the following convex sets:

- to $\{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 \leq 1\}$ at $(0,-1)$ and at $(1,0)$;
- to $\{(x,y) \in \mathbb{R}^2 \,|\, e^x \leq y \leq 2\}$ at $(0,1)$ and at $(0,2)$.

5. Which pairs of convex sets are properly / strongly separated?

- $X = \{(x,y) \in \mathbb{R}^2 \,|\, x \geq 1\}$, $Y = \{(x,y) \in \mathbb{R}^2 \mid |x|+|y| \leq 1\}$.
- $X = \{(x,y) \in \mathbb{R}^2 \,|\, (x+1)^2+y^2 \leq 1\}$, $Y = \{(x,y) \in \mathbb{R}^2 \mid x \geq 1,\ y \geq 1\}$.
- $X = \{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 \leq 1\}$, $Y = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0\}$.

Provide a separating linear functional where this is possible.