2 Mathematical background

2.1 Norms

Optimization problems can rarely be solved exactly. Most often a solution algorithm delivers a sequence of iterates that converge to an optimal solution of the problem. In order to define convergence and to measure the quality of approximation, we shall need the notions of topology and norm.

Definition 2.1. Let V be a real vector space. A function $\cdot : V \times V \to \mathbb{R}$ is called a *scalar product* if

- $u \cdot v = v \cdot u$ for all $u, v \in V$,
- $(au + bv) \cdot w = a(u \cdot w) + b(v \cdot w)$ for all $a, b \in \mathbb{R}$, $u, v, w \in V$,
- $u \cdot u \geq 0$ for all $u \in V$ and $u \cdot u = 0$ if and only if $u = 0$.

Equivalently, a scalar product is a symmetric positive definite bilinear form on V.

Definition 2.2. Let V be a real vector space. A function $||\cdot|| : V \to \mathbb{R}_+$ is called a *norm* if

- $||u|| = 0$ if and only if $u = 0$,
- $||au|| = |a| \cdot ||u||$ for all $a \in \mathbb{R}$ and $u \in V$,
- $||u + v|| \leq ||u|| + ||v||$ for all $u, v \in V$.

Every scalar product on V defines a norm by $||u|| = \sqrt{u \cdot u}$, but not every norm can be represented in such a way. Norms defined by scalar products are called *Euclidean*.

Every norm on V defines a distance function, or *metric*, by $d(u, v) := ||u - v||$.

Two norms $||\cdot||, ||\cdot||'$ are called *strongly equivalent* if there exist constants $\alpha, \beta > 0$ such that

$$\alpha||u|| \leq ||u||' \leq \beta||u||$$

for all $u \in V$. In finite-dimensional vector spaces every two norms are strongly equivalent, because the continuous positive function $\frac{||u||}{||u||'}$ attains its minimum and its maximum on the unit sphere.

The *unit ball* and the *open unit ball* of a norm $||\cdot||$ are the sets

$$B_1 = \{u \in V \mid ||u|| \leq 1\}, \qquad B_1^o = \{u \in V \mid ||u|| < 1\},$$

respectively.

Examples:

- p-norms on $\mathbb{R}^n$: $||x||_p = \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}$, $p \in [1, +\infty)$; $||x||_\infty = \max_k |x_k|$;
- Euclidean norms on $\mathbb{R}^n$: $||x|| = \sqrt{x^T A x}$, A real symmetric with all eigenvalues positive;
- matrix (Schatten) p-norms on $\mathbb{R}^{n\times m}$: $||A||_p = \left(\sum_{k=1}^{\min(n,m)} \sigma_k(A)^p\right)^{1/p}$, $p \in [1, +\infty)$; $||A||_\infty = \max_k \sigma_k(A)$;
- matrix (Schatten) p-norms on $\mathcal{S}^n$: $||A||_p = \left(\sum_{k=1}^n |\lambda_k(A)|^p\right)^{1/p}$, $p \in [1, +\infty)$; $||A||_\infty = \max_k |\lambda_k(A)|$.

Here $\mathcal{S}^n$ is the $\frac{n(n+1)}{2}$-dimensional real vector space of real symmetric $n \times n$ matrices, $\sigma_k(A)$ are the singular values, and $\lambda_k(A)$ the eigenvalues of the matrix A.

The matrix 1-norms are also called *nuclear norms*, the matrix 2-norms *Frobenius norms*, and the matrix ∞-norms *spectral norms*. The Frobenius norm can be computed as $||A||_2 = \sum_{i,j} A_{ij}^2$ without resorting to the eigenvalues. On positive semi-definite matrices the nuclear norm equals the trace $\sum_i A_{ii}$.

Attention: There exist also other matrix p-norms defined by $||A||_p = \sup_{||x||_p=1} ||Ax||_p$ which are derived from the p-norms on the vector spaces the matrix is acting on as a linear operator. In this notion the spectral norm is, e.g., given by $||A||_2$, not $||A||_\infty$.

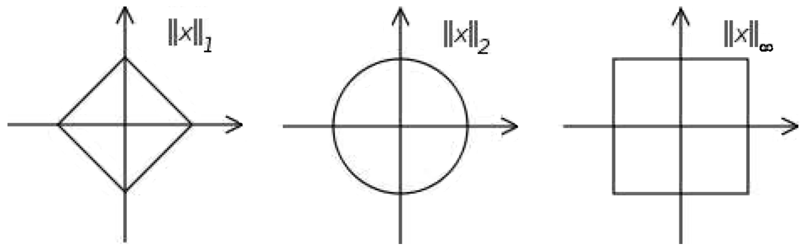

Figure 1: Unit balls of some vector p-norms in the plane

0-"norms" and sparsity: The expressions for the p-norms above are also defined for $p \in (0, 1)$, but these not anymore norms, because the unit balls cease to be convex, which is equivalent to a violation of the triangle inequality. Nevertheless, the expressions

$$\lim_{p\to 0} \sum_{k=1}^{n} |x_k|^p, \qquad \lim_{p\to 0} \sum_{k=1}^{\min(n,m)} \sigma_k(A)^p, \qquad \lim_{p\to 0} \sum_{k=1}^{n} |\lambda_k(A)|^p$$

are of interest for optimization, because the measure the number of non-zero components of a vector $x \in \mathbb{R}^n$ and the rank of a matrix $A \in \mathbb{R}^{n\times m}$ or $A \in \mathcal{S}^n$, respectively.

It is sometimes a desirable feature of the solution of an optimization problem that it be sparse (low number of non-zero components) or low rank in case of a matrix decision variable. In order to enforce such behaviour, one can add a penalty term $\mu||x||_1$ or $\mu||A||_1$ to the objective function of the problem, where $\mu > 0$ is a weight which emphasizes the importance of the sparsity / low rank property over the original objective function value of the solution.

2.2 Affine space

We have seen that the notion of convexity relies on the ability to define the segment $[x, y] = \{\lambda x + (1-\lambda)y \mid \lambda \in [0, 1]\}$ between points of the set under consideration. Obviously the notion of segment is invariant not only under automorphisms of the underlying vector space, but also under translations which do not preserve the zero vector and are hence not automorphisms. We therefore do not need the full structure of a vector space in order to work with convex sets. It is sufficient to keep the structure of an affine space, which can be obtained from a vector space by forgetting the location of the zero vector.

Definition 2.3. Let V be a real vector space. An *affine space* with associated vector space V is a set A together with a map $+ : A \times V \to A$ such that

- $x + 0 = x$ for all $x \in A$,
- $(x + u) + v = x + (u + v)$ for all $x \in A$, $u, v \in V$,
- $v \mapsto x + v$ is a bijection between V and A for all $x \in A$.

Example: An affine subspace A of a vector space W is an affine space in the sense above. The associated vector space V is the linear subspace of W spanned by the differences of the elements of A. In particular, the vector space W itself is also an affine space, with the associated vector space being again W.

The third property of the definition allows to consider the elements of V as differences between points in A: for $x, y \in A$ we define $x - y$ to be the unique vector $v \in V$ such that $x = y + v$. The points of the segment $[x, y]$, where $x, y \in A$, can then be written as

$$\lambda x + (1-\lambda)y = y + \lambda(x - y).$$

Since $x - y$ is a vector, its multiple $\lambda(x - y)$ is also a vector, and hence the right-hand side of the relation is again a point of the affine space. Generally, every combination of points of affine space with coefficients summing to 0 can be interpreted as a vector in V, since it can be written as a linear combination of differences of elements

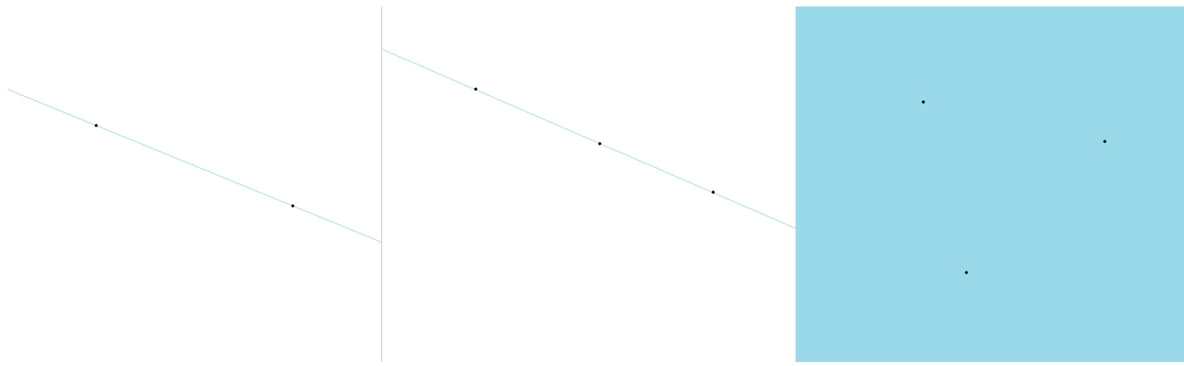

Figure 2: Affine hulls of sets of points

of A. Therefore every combination with coefficients summing to 1 can be seen again as a point in affine space, because it can be written as an element of affine space plus a combination of points with coefficients summing to 0.

Definition 2.4. Let $x_1, \dots, x_k$ be points in an affine space A. Then $\sum_{i=1}^k \lambda_i x_i$ is called an *affine combination* of the points $x_1, \dots, x_k$ if $\sum_{i=1}^k \lambda_i = 1$.

A finite set $\{x_1, \dots, x_k\}$ of points in an affine space A is called *affinely independent* if the relations $\sum_{i=1}^k \lambda_i u_i = 0$, $\sum_{i=1}^k \lambda_i = 0$ imply $\lambda_i = 0$ for all $i = 1, \dots, k$.

An *affine basis* of an affine space A is an affinely independent set $\{x_1, \dots, x_n\}$ such that every element of A is an affine combination of $x_1, \dots, x_n$.

If $\{x_1, \dots, x_n\}$ is an affine basis of A, then $\{x_2 - x_1, \dots, x_n - x_1\}$ is a basis of the vector space associated to A. Hence the dimension of A is equal to $n - 1$, i.e., the number of elements in any of its affine bases, minus 1.

Definition 2.5. The *affine hull* of a subset $X \subset A$ of an affine space is the set of all affine combinations of elements of X.

An *affine subspace* of A is a subset of A which equals its affine hull.

The affine hull of an arbitrary subset X is the smallest affine subspace of A which contains X, or equivalently, the intersection of all affine subspaces containing X.

Given an affine basis $\{x_1, \dots, x_n\}$ of A, we can represent every point $x \in A$ uniquely as an affine combination $x = \sum_{i=1}^n \lambda_i x_i$. The coefficients $\lambda_1, \dots, \lambda_n$ are called the *barycentric coordinates* of x with respect to the basis $\{x_1, \dots, x_n\}$.

Definition 2.6. A map $f : A \to B$ between affine spaces A, B with associated vector spaces U, V is called *affine* if it preserves affine combinations, i.e., $f(\sum_k \lambda_k x_k) = \sum_k \lambda_k f(x_k)$ for all $x_k \in A$ and all λ_k such that $\sum_k \lambda_k = 1$.

An *affine isomorphism* is a bijective affine map.

If we turn the affine spaces into vector spaces by arbitrarily designating a point x_0 as the origin and identifying an arbitrary point x of the affine space with the vector $x - x_0$, then the affine maps turn out to be linear maps plus translations by a constant vector.

2.3 Topology

Defining a topology on a set allows one to define the notion of convergence of sequences of points to points of the set. This is necessary since usually the output of an optimization algorithm will not consist of an optimal solution of the optimization problem, but merely of a sequence of iterates which converge to an optimal solution.

Definition 2.7. Let X be a set. A collection $\mathcal{T}$ of subsets $U \subset X$ is called a *topology* on X if

- $\emptyset, X \in \mathcal{T}$,

- finite intersections of sets in $\mathcal{T}$ are again in $\mathcal{T}$,
- arbitrary unions of sets in $\mathcal{T}$ are again in $\mathcal{T}$.

The set X equipped with a topology $\mathcal{T}$ is called a *topological space*.

The topology *defines* which subsets of X are *open* and which are *closed*: $U \subset X$ is open if $U \in \mathcal{T}$ and it is closed if $X \setminus U \in \mathcal{T}$. If $x \in X$ is a point, then any open set containing x is called a *neighbourhood* of x.

In this way, finite unions of closed sets and arbitrary intersections of closed sets are again closed.

Infinite unions of closed sets do not need to be closed, however: in $\mathbb{R}^n$ we have $\bigcup_{r<1} B_r = B_1^o$, where B_r is the closed ball of radius r, and B_1^o the open unit ball.

Definition 2.8. Let $U \subset X$ be a subset of a topological space. The *interior* of U, denoted $\mathrm{int}U$ or U^o, is the largest open set contained in U, or equivalently, the union of all open sets contained in U.

The *closure* of U, denoted $\mathrm{cl}\, U$ or $\bar{U}$, is the smallest closed set containing U, or equivalently, the intersection of all closed sets containing U.

The *boundary* of U, denoted $\mathrm{bd}\, U$ or ∂U, is the difference $\bar{U} \setminus U^o$.

Let U be a subset of an affine space. The *relative interior* of U, denoted $\mathrm{ri}\, U$, is the interior of U in the topology of the affine hull of U.

The *relative boundary* of U, denoted $\mathrm{rbd}\, U$, is the difference $\bar{U} \setminus \mathrm{ri}\, U$.

Example: Let $X = \mathbb{R}$ and $U = \mathbb{Q}$ the subset of rational numbers. Then $U^o = \emptyset$ and $\bar{U} = \mathbb{R}$.

Definition 2.9. Let $U, V \subset X$ be subsets of a topological space. The U is called *dense* in V if $U \subset V \subset \bar{U}$.

The topology allows to define the notion of convergence.

Definition 2.10. Let X be a topological space. A sequence $x_1, x_2, \dots$ of points *converges* to a point x^* if for every open set $U \subset X$ containing x^*, there exists a number N_U such that $x_k \in U$ for all $k \geq N_U$.

Thus $\{x_k\}$ converges to x^* if the sequence eventually enters and no more leaves arbitrarily small neighbourhoods of x^*.

Often it is convenient to define or describe topologies by the following simpler notion.

Definition 2.11. Let X be a set. A collection $\mathcal{B}$ of subsets $U \subset X$ is called a *base* if

- $\bigcup_{U \in \mathcal{B}} = X$,
- for every $U_1, U_2 \in \mathcal{B}$ and every $x \in U_1 \cap U_2$ there exists $U_3 \in \mathcal{B}$ such that $x \in U_3 \subset U_1 \cap U_2$.

Every base defines a topology by taking the open subsets of X to be arbitrary unions of elements of $\mathcal{B}$. On the other hand, every topology can be defined by a base, the largest such base being the topology itself.

A norm on a vector space V induces a *topology* on V by the base

$$\mathcal{B} = \{u + \varepsilon B_1^o \,|\, u \in V,\ \varepsilon > 0\}.$$

Thus a set $U \subset V$ is *open* if for every $u \in U$ there exists a constant $\varepsilon > 0$ such that $u + \varepsilon B_1 \subset U$.

The notion of convergence can then be reformulated as follows: a sequence $\{x_k\}$ converges to x^* if and only if $\lim_{k\to\infty} ||x_k - x^*|| = 0$.

Strongly equivalent norms define the same topology on V. In finite dimension all norms are strongly equivalent, hence every norm defines the same topology on V.

3 Convex sets

Convex sets are defined via affine combinations of two elements with nonnegative coefficients.

Definition 3.1. A subset $X \subset A$ of a real vector space or a real affine space is called *convex* if for all $x, y \in X$ and all $\lambda \in [0, 1]$ we have

$$\lambda x + (1 - \lambda) y \in X.$$

Examples:

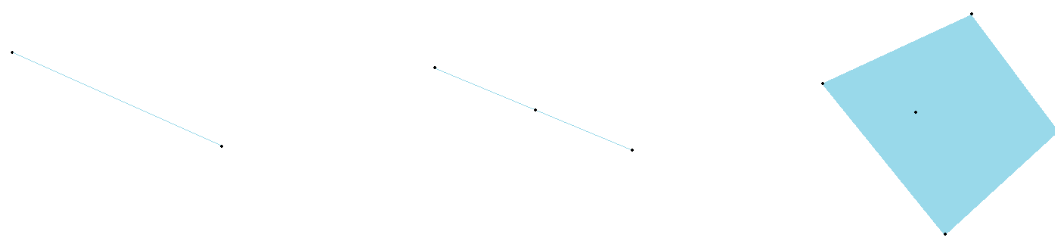

Figure 3: Convex hulls of sets of points

- the empty set $\emptyset$,
- the whole space A,
- singletons $\{x\}$,
- affine subspaces,
- open or closed affine half-spaces,
- open or closed norm balls $x + rB_1^o$, $x + rB_1$ around arbitrary points.

Here open and closed affine half-spaces are sets of the form $\{x \in A \,|\, a(x) < b\}$ and $\{x \in A \,|\, a(x) \leq b\}$, respectively, where a is a non-constant linear functional on A and $b \in \mathbb{R}$.

3.1 Convex hull

Definition 3.2. Let $x_1, \dots, x_k$ be points in an affine space A. Then $\sum_{i=1}^k \lambda_i x_i$ is called a *convex combination* of the points $x_1, \dots, x_k$ if $\sum_{i=1}^k \lambda_i = 1$ and $\lambda_i \geq 0$, $i = 1, \dots, k$.

The *convex hull* of a subset $X \subset A$ of an affine space is the set of all convex combinations of elements of X. It is denoted by $conv X$.

Lemma 3.3. *A set X is convex if and only if it equals its convex hull.*

Proof. Let $X = conv X$. Then, in particular, convex combinations of any two elements of X belong to X. Hence X is convex.

Let X be convex. We show by induction on k that a convex combination of k elements of X is in X. The definition of convexity yields the base of the induction for $k = 2$. Suppose we have proven that any convex combination of $k-1$ elements of X is in X. Let $x_1, \dots, x_k \in X$ and let $x = \sum_{i=1}^k \lambda_i x_i$ be a convex combination. If any of the coefficients λ_i vanishes, then x is actually a convex combination of strictly less than k elements and is in X by the induction hypothesis. Assume $\lambda_i > 0$ for all $i = 1, \dots, k$. Then we have

$$x = \sum_{i=1}^{k-1} \lambda_i x_i + \lambda_k x_k = \left(\sum_{i=1}^{k-1} \lambda_i \right) \sum_{i=1}^{k-1} \frac{\lambda_i}{\sum_{j=1}^{k-1} \lambda_j} x_i + \lambda_k x_k = (1 - \lambda_k) y + \lambda_k x_k.$$

Here $y = \sum_{i=1}^{k-1} \frac{\lambda_i}{\sum_{j=1}^{k-1} \lambda_j} x_i$ is a convex combination of $k-1$ elements of X and is hence in X. The point x has then been represented as convex combination of two elements of X and is hence also in X. $\square$

The following assertion follows immediately from Definition 3.1.

Lemma 3.4. *Arbitrary intersections of convex sets are convex.*

Corollary 3.5. *The convex hull of a set X is the smallest convex set which contains X, namely the intersection of all convex sets containing X.*

Proof. Since convex combinations of convex combinations are again convex combinations of the original points, the convex hull of X is equal to its own convex hull. By Lemma 3.3 it is hence convex. On the other hand, any convex set Y containing X must contain at least the convex hull of X, because $Y \supset X$ implies $Y = conv Y \supset conv X$. □

Further examples of convex sets:

- polytopes (convex hulls of a finite set of points),
- polyhedra (finite intersections of closed affine half-spaces),
- simplices (convex hull of an affinely independent set of points).

3.2 Operations preserving convexity

We now consider more operations which preserve convexity.

Definition 3.6. Let X, Y be subsets of a vector space. The set

$$X + Y := \{x + y \mid x \in X,\ y \in Y\}$$

is called *Minkowski sum* of X, Y.

This definition can be extended to the case where one of the sets X, Y is a subset of an affine space and the other a subset of the underlying vector space.

The following assertions follow easily from the definition of convexity.

- the Minkowski sum of convex sets is convex,
- images of convex sets under affine maps are convex,
- pre-images of convex sets under affine maps are convex,
- the interior X^o of a convex set X is convex,
- the relative interior $ri\, X$ of a convex set X is convex,
- the closure $cl\, X$ of a convex set X is convex.

We now come to the interplay between convexity and topology.

Lemma 3.7. *Let $X \neq \emptyset$ be convex. Then $ri\, X \neq \emptyset$.*

For non-convex sets this is in general not the case (consider $X = \mathbb{Q} \subset \mathbb{R}$, then $ri\, X = \emptyset$).

Proof. The affine hull $aff\, X$ possesses an affine basis of points in X. To construct such a basis, pick an arbitrary point $x_1 \in X$. If $aff\{x_1\} = aff\, X$, then $\{x_1\}$ is an affine basis of $aff\, X$. If $aff\{x_1\} \neq aff\, X$, then there exists a point $x_2 \in X \setminus aff\{x_1\}$. This point x_2 is affinely independent of x_1. We now repeat the process by comparing $aff\{x_1, x_2\}$ with $aff\, X$ and adjoin another affinely independent point $x_3 \in X$ if these affine hulls are not equal. Obviously the affine hulls become equal after $\dim aff\, X + 1$ steps.

Let hence $x_1, \ldots, x_k \in X$ form an affine basis of the affine hull of X. Then the simplex $\Sigma = conv\{x_1, \ldots, x_k\}$ is a subset of X, and the relative interior of Σ is given by the set

$$ri\, \Sigma = \left\{ \sum_{i=1}^{k} \lambda_i x_i \mid \lambda_i > 0,\ \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

Since $aff\, \Sigma = aff\, X$, any point in $ri\, \Sigma$ is also in $ri\, X$. □

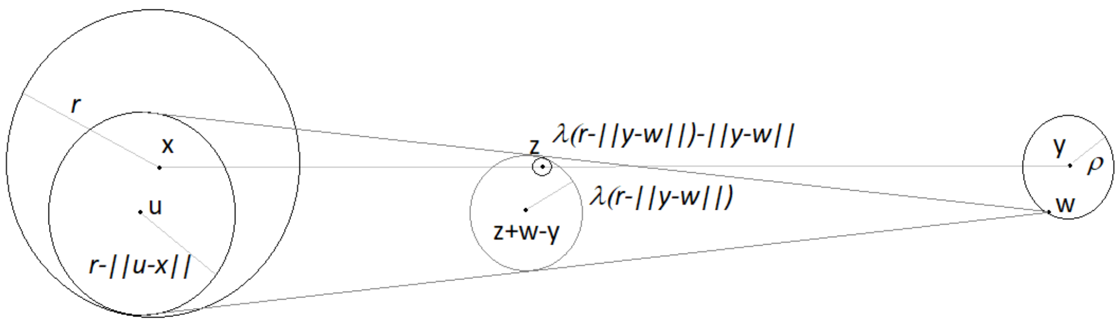

Figure 4: Proof of Lemma 3.8. Radii are shown in *italic*.

We now need an auxiliary lemma.

Lemma 3.8. *Let X be a convex set, let $x \in ri\,X$ and $y \in clX$. Then the half-open segment $[x, y) = \{\lambda x + (1-\lambda)y \,|\, \lambda \in (0, 1]\}$ is a subset of $ri\,X$.*

Proof. By definition there exists $r > 0$ such that $(x + rB_1) \cap aff\,X \subset X$. Let $\lambda \in (0, 1]$ and $z = \lambda x + (1-\lambda)y$. Set $\rho = \frac{\lambda r}{1+\lambda}$. Since $y \in clX$, there exists $w \in X$ such that $||y - w|| < \rho$.

Set $u = x+w-y$. Then $u \in aff\,X$ as an affine combination of points in $aff\,X$. Moreover, $||u-x|| = ||w-y|| < r$. Hence $(u + (r - ||u - x||)B_1) \cap aff\,X \subset (x + rB_1) \cap aff\,X \subset X$. We then get

$$\lambda[(u + (r - ||u - x||)B_1) \cap aff\,X] + (1-\lambda)w = [z + w - y + \lambda(r - ||y - w||)B_1] \cap aff\,X \subset X$$

by the convexity of X. But

$$z + w - y + \lambda(r - ||y - w||)B_1 \supset z + (\lambda(r - ||y - w||) - ||y - w||)B_1$$

and $\lambda(r - ||y - w||) - ||y - w|| = (1 + \lambda)(\rho - ||y - w||) > 0$. Therefore $(z + (1 + \lambda)(\rho - ||y - w||)B_1) \cap aff\,X \subset X$, and $z \in ri\,X$. □

This will allow us to show that for convex sets the relative interior and the closure can be obtained from each other.

Lemma 3.9. *Let X be a convex set. Then $cl\,ri\,X = clX$ and $ri\,clX = ri\,X$.*

Proof. Clearly $cl\,ri\,X \subset clX$ and $ri\,clX \supset ri\,X$.

Let $y \in clX$. Then $X \neq \emptyset$ and there exists a point $x \in ri\,X$. It follows that $[x, y) \subset ri\,X$, and hence $y \in cl\,ri\,X$.

Let now $z \in ri\,clX$. Then $X \neq \emptyset$ and there exists $x \in ri\,X$. Further there exists $\varepsilon > 0$ such that $(z + \varepsilon B_1) \cap aff\,X \subset clX$. We have $[x, z] \subset aff\,X$, and there exists $y \in (z + \varepsilon B_1) \cap aff\,X$ such that y lies on the line through x and z and such that $z \in [x, y)$. But then $z \in ri\,X$ by Lemma 3.8. □

3.3 Distance from a convex set

Let $X \subset \mathbb{R}^n$ be a non-empty closed convex set, and let $d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$ be the Euclidean distance function on $\mathbb{R}^n$. Then the distance function

$$D(x) = \inf_{y \in X} d(x, y)$$

is everywhere defined. Moreover, the infimum is attained for every $x \in \mathbb{R}^n$.

Indeed, let $z \in X$ be a point and set $r = ||x - z||$. Then

$$\inf_{y \in X} d(x, y) = \inf_{y \in X \cap (x + rB_1)} d(x, y).$$

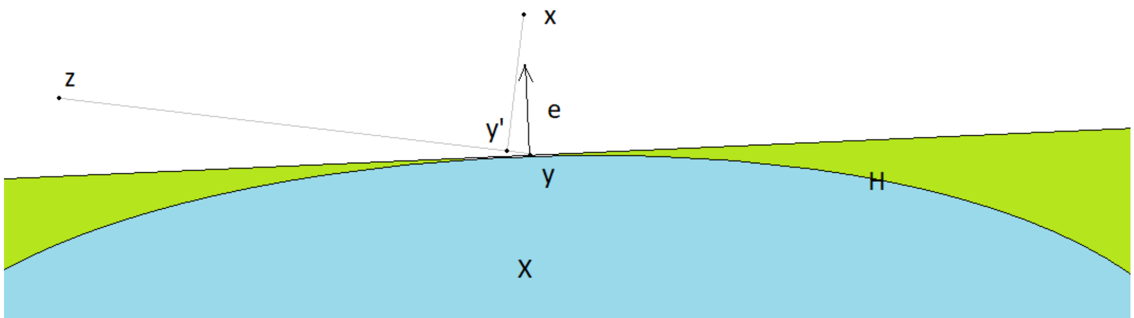

Figure 5: Were $z \in X$, then y', which is closer to x than y, would also be in X.

But $X \cap (x + rB_1)$ is compact, and hence the infimum on this intersection is attained. But then also the infimum over X is attained at the same point.

Moreover, the infimum is unique. Assume there exist two points $y, y' \in X$ such that $d(x, y) = d(x, y') = D(x)$. Then $D(x) \le d(x, \frac{y+y'}{2}) = \sqrt{D^2(x) - \frac{||y'-y||^2}{4}}$, which gives $||y' - y||^2 \le 0$ and hence $y' = y$.

Definition 3.10. Let X be a closed convex set and x be a point. The unique point $y \in X$ which minimizes the distance to x is called the *projection* of x on X.

Fix $x \in \mathbb{R}^n \setminus X$ and let $y \in X$ be such $D(x) = d(x, y) > 0$. Set $u = x - y$ and $e = \frac{u}{||u||}$ and define the closed half-space

$$H = \{z \in \mathbb{R}^n \,|\, e^T z \le e^T y\}.$$

The boundary to this half-space is the hyperplane through y which is perpendicular to e, and H is the half-space opposite to x.

Claim: $X \subset H$. Indeed, let $z \in X \setminus H$. Then on the line segment $[z, y]$ there exists a point y' which is strictly closer to x than y. But $z, y \in X$, hence $y' \in X$, contradicting the optimality of y (see Fig. 5).

Therefore we have the inclusion $\{y\} \subset X \subset H$. Now consider $D(x')$ at an arbitrary point $x' \notin H$. We have the bounds

$$d(x', y) \ge \min_{y' \in X} d(x', y') = D(x') \ge \min_{y' \in H} d(x', y').$$

Set $u' = x' - x$ and decompose this vector into a component v which is orthogonal to e and a component αe which is collinear with e, $u' = \alpha e + v$ (see Fig. 6). We get

$$d(x', y) = ||(\alpha + ||u||)e + v|| = \sqrt{(\alpha + ||u||)^2 + ||v||^2}, \qquad \min_{y' \in H} d(x', y') = \alpha + ||u||,$$

in the second case the minimizer $y' \in H$ of the distance to x' being given by $y' = y + v$. Therefore we obtain the explicit bounds

$$\alpha + ||u|| \le D(x') \le (\alpha + ||u||)\sqrt{1 + \frac{||v||^2}{(\alpha + ||u||)^2}} = \alpha + ||u|| + O(||v||^2).$$

Since $D(x) = ||u||$ and $\alpha = e^T u' = e^T (x' - x)$, we get

$$D(x') = D(x) + e^T (x' - x) + O(||x' - x||^2).$$

This decomposition can be seen as the first order Taylor approximation of D around x. It follows that $D(x)$ is differentiable at x and its gradient is given by e.

We get the following result.

Theorem 3.11. *Let $X \subset \mathbb{R}^n$ be a closed convex set, and let $D(x)$ be the Euclidean distance from x to X. Let further $u(x) \in \mathbb{R}^n$ be the difference between x and the projection of x on X. Then at any point $x \notin X$ the function $D(x)$ is differentiable, and its gradient is given by $\frac{u(x)}{||u(x)||}$.*

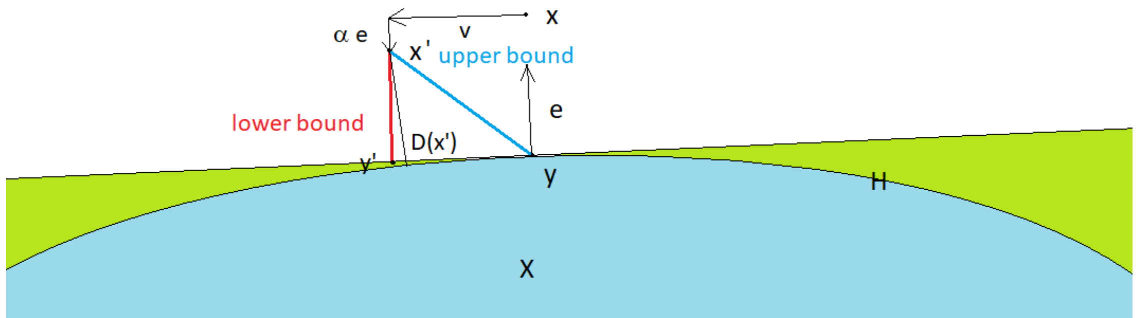

Figure 6: Upper and lower bound on $D(x')$.

3.4 Separation

In this subsection we consider a fundamental property of convex sets, namely that convex sets can be separated or supported by hyperplanes. The latter property is closely linked to duality which we shall consider later.

Definition 3.12. Let X, Y be non-empty convex sets. The distinct parallel affine hyperplanes

$$H_1 = \{x \mid a^T x = b_1\}, \qquad H_2 = \{x \mid a^T x = b_2\}$$

separate X and Y *strongly* if

$$\sup_{x \in X} a^T x \leq b_1 < b_2 \leq \inf_{y \in Y} a^T y.$$

We say that X and Y can be separated strongly if there exist affine hyperplanes which separate them strongly.

Here $a(x) := a^T x$ is a non-zero linear functional, represented by a non-zero vector in any given coordinate system.

We may also say that the linear functional a separates X, Y strongly.

Definition 3.13. Let X, Y be non-empty convex sets. The affine hyperplane

$$H = \{x \mid a^T x = b\}$$

separates X and Y *properly* if

$$\sup_{x \in X} a^T x \leq b \leq \inf_{y \in Y} a^T y, \qquad \inf_{x \in X} a^T x < \sup_{y \in Y} a^T y.$$

We say that X and Y can be separated properly if there exists an affine hyperplane which separates them properly.

Thus H separates X and Y properly if X and Y lie in opposite closed half-spaces with respect to H and at least one of the sets X, Y is not contained in H.

We may also say that the linear functional a separates X, Y properly.

Clearly strong separation implies proper separation, but there exist non-intersecting closed convex sets which can be separated properly but not strongly.

Example: Consider the sets $X = \{(x_1, x_2) \mid x_1 > 0,\ x_1 x_2 \geq 1\}$, $Y = \{(y_1, y_2) \mid y_1 < 0,\ y_1 y_2 \leq -1\}$. Then the hyperplane $H = \{x \mid e_1^T x = 0\}$ separates X and Y properly, where $e_1 = (1, 0)^T$. However, X and Y cannot be separated strongly, because the distance between these sets is zero.

We shall now consider the special case when one of the sets is a singleton.

Lemma 3.14. *Let V be a real vector space equipped with a Euclidean distance function. Let X be a non-empty closed convex set and y a point. Then the singleton $\{y\}$ can be separated strongly from X if and only if $d(y, X) = \min_{x \in X} d(x, y) > 0$.*

Proof. If $y \in X$, then $\{y\}$ cannot be separated strongly from X. Hence assume that $y \notin X$. Then $d(y, X) > 0$. Let $y^* \in X$ be the projection of y on X. Set $a = y - y^* \neq 0$. Then the hyperplanes $H_1 = \{x \mid a^T x = a^T y\}$ and $H_2 = \{x \mid a^T x = a^T y^*\}$ separate X and $\{y\}$ strongly. $\square$

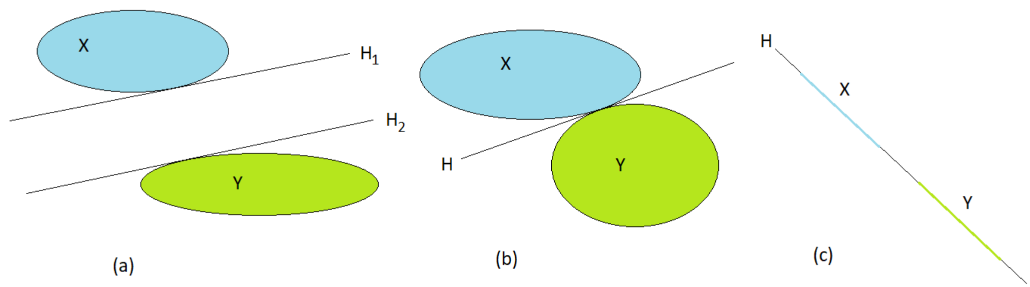

Figure 7: (a) H_1, H_2 separate X, Y strongly; (b) H separates X, Y properly; (c) H does not separate X, Y.

Lemma 3.15. *Let $X \subset V$ be a convex set and let $y \notin ri\, X$. Then $\{y\}$ and X can be properly separated.*

Proof. Introduce a Euclidean distance in the ambient vector space V.

If $y \notin clX$, then $d(y, clX) > 0$ and $\{y\}$ can be strongly separated from clX. The same hyperplanes then separate $\{y\}$ from X strongly, and thus $\{y\}$ and X can be separated properly.

Hence assume that $y \in clX \setminus ri\, X = rbdX$, where $rbdX$ denotes the relative boundary of X. Let A be the affine hull of X. Then there exists a sequence of points $\{y_k\}_{k\geq 1}$ such that $y_k \in A \setminus clX$ and $\lim_{k\to\infty} y_k = y$. Let y_k^* be the projection of y_k on clX, and let $a_k = y_k - y_k^*$, $e_k = \frac{a_k}{||a_k||}$. Let e^* be an accumulation point of the sequence $\{e_k\}$ which must exist because all e_k are elements of the unit sphere. Without loss of generality we may assume that $\{e_k\}$ converges to e.

For every fixed $z \in clX$ we have $a_k^T y_k > a_k^T y_k^* \geq a_k^T z$ and hence $e_k^T y_k > e_k^T z$. Passing to the limit $k \to \infty$, we get that $e^T y \geq e^T z$. However, the linear functional $x \mapsto e^T x$ is not constant on A and thus not constant on X by construction. Therefore the hyperplane $H = \{x \,|\, e^T x = e^T y\}$ separates $\{y\}$ and clX properly. Thus it separates $\{y\}$ from X properly. □

The main result of this subsection is the following *Separation Theorem.*

Theorem 3.16. *The non-empty convex sets X and Y can be separated properly if and only if their relative interiors do not intersect.*

Proof. For the sake of contradiction, suppose there exists $z \in ri\, X \cap ri\, Y$ and a hyperplane $H = \{x \,|\, a^T x = b\}$ which separates X and Y properly. Then

$$a^T z \leq \sup_{x\in X} a^T x \leq b \leq \inf_{y\in Y} a^T y \leq a^T z,$$

and hence all inequalities are satisfied with equality.

Let A be the affine hull of X. Then there exists $\varepsilon > 0$ such that $(z + \varepsilon B_1) \cap A \subset X$, and hence

$$\sup_{x\in(z+\varepsilon B_1)\cap A} a^T x \leq a^T z.$$

It follows that the linear functional $x \mapsto a^T x$ is constant on A and hence also on X. Likewise, this functional is constant on the affine hull of Y and hence also on Y. But this contradicts the second condition $\inf_{x\in X} a^T x < \sup_{y\in Y} a^T y$ implied by proper separation of X and Y by H.

Therefore, if X and Y can be separated properly, then $ri\, X \cap ri\, Y = \emptyset$. Let us prove the converse implication.

Assume that $ri\, X \cap ri\, Y = \emptyset$. Then $0 \notin ri\, X - ri\, Y = \{x - y \,|\, x \in ri\, X,\ y \in ri\, Y\}$. By Lemma 3.15 the singleton $\{0\}$ can then be properly separated from $ri\, X - ri\, Y$. This implies that there exists a non-zero a such that

$$0 \leq \inf_{x\in ri\, X,\, y\in ri\, Y} a^T(x - y), \qquad 0 < \sup_{x\in ri\, X,\, y\in ri\, Y} a^T(x - y).$$

Dropping the relative interior will not change the infimum or the supremum, which yields

$$\sup_{x\in X} a^T x \leq \inf_{y\in Y} a^T y, \qquad \inf_{x\in X} a^T x < \sup_{y\in Y} a^T y.$$

Therefore X and Y can be separated properly. $\square$

A similar reasoning allows to prove the following necessary and sufficient condition for strong separation.

Theorem 3.17. *Let $X, Y \subset V$ be non-empty convex sets and let V be equipped with a Euclidean distance. Then X and Y can be separated strongly if and only if $d(X,Y) = \inf_{x\in X, y\in Y} d(x,y) > 0$.*

Proof. If the distinct parallel affine hyperplanes H_1, H_2 separate X and Y strongly, then $d(X,Y) \geq d(H_1,H_2) > 0$.

Let on the other hand $d(X,Y) > 0$. Then also $d(\{0\}, X-Y) = d(\{0\}, cl(X-Y)) > 0$. Let $a \neq 0$ be the projection of $\{0\}$ on $cl(X-Y)$. Then

$$\inf_{x\in X} a^T x - \sup_{y\in Y} a^T y = \inf_{z\in X-Y} a^T z = \inf_{z\in cl(X-Y)} a^T z = ||a||^2$$

and hence $\inf_{x\in X} a^T x > \sup_{y\in Y} a^T y$. It is now easily seen that X and Y can be strongly separated. $\square$

In particular, a compact convex set can always be strongly separated from a closed convex set if the sets do not intersect. This follows from the fact that the distance function reaches its minimum on the compact set.

Definition 3.18. Let X be a non-empty convex set and $x \in rbdX$. A hyperplane H which separates $\{x\}$ from X properly is called a *supporting hyperplane* to X at x.

By Lemma 3.15 supporting hyperplanes exist at every relative boundary point. Moreover, the intersection of X with a supporting hyperplane has strictly lower dimension than X, because X is not contained in the hyperplane by definition of proper separation.

Lemma 3.19. *Let $X \neq V$ be a closed convex set. Then X equals the intersection of all closed half-spaces which contain X.*

Proof. Let C be the intersection of all closed half-spaces which contain X. By assumption there exists a point $x \notin X$. Then $\{x\}$ can be strongly separated from X, and there exists a closed half-space containing X. Thus C is non-empty and $X \subset C$.

Let now $z \in C \setminus X$. Then $\{z\}$ can be strongly separated from X, and there exists a closed half-space which contains X but not z. This leads to a contradiction, and such z cannot exist. $\square$

3.5 Theorem on the alternative

Polyhedral sets are particularly simple convex sets. If a hyperplane is supporting to a polyhedral set P, then this relation can be certified in the form of a convex combination of the inequalities defining the polyhedral set.

Lemma 3.20. *Let $P = \{x \mid Ax \leq b\}$ be a non-empty polyhedral set, and let $H = \{x \mid u^T x = b_0\}$ be a hyperplane containing a point $x^* \in P$ and such that the open half-space $C = \{x \mid u^T x > b_0\}$ has an empty intersection with P, i.e., $u^T x^* = b_0$, $u^T x \leq b_0$ for all P. Then there exists a nonnegative vector $\mu \geq 0$ such that $u = A^T \mu$, $b_0 = b^T \mu$.*

Proof. Let I be the set of indices of rows for which the inequality $Ax^* \leq b$ is an equality, i.e., the index set of active constraints at x^*. Then there exists $\epsilon > 0$ such that for all $x \in x^* + B_\epsilon$ we have $(Ax)_j < b_j$ for all $j \notin I$. Define another polyhedral set by $P' = \{x \mid (Ax)_i \leq b_i \quad \forall\, i \in I\}$. Then $P \subset P'$. We claim that C has also an empty intersection with P', i.e., $u^T x \leq b_0$ for all $x \in P'$.

Indeed, suppose there exists $z \in P'$ such that $u^T z > u^T x^*$. Then for all $\lambda \in (0,1]$ we have $z_\lambda = \lambda z + (1-\lambda)x^* \in P'$ and $u^T z_\lambda > u^T x^*$. But for λ small enough we have $z_\lambda \in x^* + B_\epsilon$, and hence $z_\lambda \in P$, a contradiction.

We now define the polyhedral cone $K = \{A^T \mu \mid \mu \geq 0,\ \mu_j = 0\ \forall\, j \notin I\}$. We claim that $u \in K$.

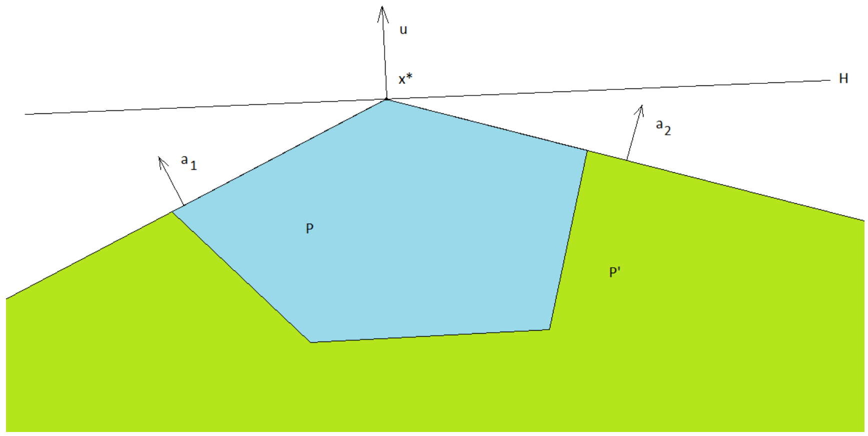

Figure 8: Lemma 3.20: The supporting linear functional u can be represented as a combination of the functionals a_1, a_2 corresponding to the active constraints with nonnegative coefficients.

Indeed, suppose $u \notin K$. Then u can be separated from K, and there exists δ such that $u^T\delta > 0$, $v^T\delta \leq 0$ for all $v \in K$. In particular, $(A\delta)_i \leq 0$ for all $i \in I$. Hence $x^* + \delta \in P'$. But then $u^T(x^* + \delta) \leq u^T x^*$ and $u^T\delta \leq 0$, a contradiction.

Hence there exists $\mu \geq 0$, $\mu_j = 0$ for all $j \notin I$, such that $u = A^T\mu$. It follows that $\mu^T(Ax^* - b) = 0$ and $b_0 = \mu^T Ax^* = \mu^T b$, which yields the desired assertion. □

As a consequence, we obtain the following Theorem on the Alternative.

Theorem 3.21. *(Farkas) Let $P = \{x \,|\, Ax \leq b\}$ be a polyhedral set. Then either $P \neq \emptyset$, or there exists $\mu \geq 0$ such that $\mu^T A = 0$, $\mu^T b = -1$.*

Proof. Clearly if $P \neq \emptyset$, then such a μ cannot exist.

Let $P = \emptyset$. Then the non-empty polyhedral set

$$P' = \{(x,t) \,|\, Ax - bt = (A, -b)(x^T, t)^T \leq 0\}$$

has an empty intersection with the open half-space $C = \{(x,t) \,|\, t = (0,1)(x^T,t)^T > 0\}$. By the Lemma 3.20 there exists $\mu \geq 0$ such that $(0,1) = \mu^T(A, -b)$. □

We provide also a version of the theorem with equalities.

Corollary 3.22. *Let $P = \{x \mid Ax \leq b,\ Cx = d\}$ be a polyhedral set. Then either $P \neq \emptyset$, or there exists $\mu \geq 0$, ν such that $\mu^T A + \nu^T C = 0$, $\mu^T b + \nu^T d = -1$.*

Proof. Represent $P = \{x \mid Ax \leq b,\ Cx \leq d,\ -Cx \leq -d\}$. By the previous theorem, either $P \neq \emptyset$, or there exist $\mu, \nu_+, \nu_- \geq 0$ such that $\mu^T A + \nu_+^T C - \nu_-^T C = 0$, $\mu^T b + \nu_+^T d - \nu_-^T d = -1$. In the latter case, set $\nu = \nu_+ - \nu_-$. □

3.6 Faces and extremal points

We shall now investigate the structure of the boundary of a convex set.

Definition 3.23. A convex subset F of a convex set X is called a *face* of X if for every line segment $l \subset X$ such that $F \cap ri\, l \neq \emptyset$ we have $l \subset F$.

A face F of X is called *proper* if $F \neq \emptyset$ and $F \neq X$.

A face F of a convex set X is called *exposed* if there exists a hyperplane H such that $F = X \cap H$ and $X \not\subset H$.

The hyperplane H in the last definition necessarily separates F and X properly.

Definition 3.24. Let X be a convex set. A point $x \in X$ is called *extremal* if the singleton $\{x\}$ is a face of X. It is called *exposed* if $\{x\}$ is an exposed face of X.

From the definition it follows that $x \in X$ is an extremal point of X if and only if $X \setminus \{x\}$ is convex.

Theorem 3.25. *(Straszievicz, [1, Theorem 18.6]) Let X be closed convex. Then the set of exposed points of X is dense in the set of its extreme points.*

Example: Let $\mathbb{R}^2 \supset X = ([-1,0] \times [-1,1]) \cup B_1$ be the union of a rectangle and a half-disc. The faces of X are

- X,
- $\{-1\} \times [-1,1]$, $[-1,0] \times \{-1\}$, $[-1,0] \times \{1\}$,
- $\{(x,y)\}$, where $x \geq 0$ and $x^2 + y^2 = 1$,
- $\emptyset$.

Of these, the zero-dimensional faces $\{(0,-1)\}$ and $\{(0,1)\}$ are not exposed. They can be represented as limits of exposed points, however.

Lemma 3.26. *Intersections of faces and faces of faces are again faces.*

Proof. Let $F \subset X$ be a face of X and $F' \subset F$ a face of F. Let $l \subset X$ be a line segment such that $ri\, l \cap F' \neq \emptyset$. Then $ri\, l \cap F \neq \emptyset$, because $F' \subset F$. Hence $l \subset F$, because F is a face of X. But then $l \subset F'$, because F' is a face of F. Hence F' is a face of X.

Let F_1, F_2 be faces of X, and let $F = F_1 \cap F_2$. Let $l \subset X$ be a line segment such that $ri\, l \cap F \neq \emptyset$. Then $ri\, l \cap F_i \neq \emptyset$ for $i = 1, 2$, because $F \subset F_i$. Hence $l \subset F_i$, because F_i are faces of X. But then $l \subset F$, and F is also a face of X. □

Definition 3.27. Let X be a closed convex set and $x \in X$. The *minimal face* of x in X is the smallest face of X containing x.

It is given by the intersection of all faces containing x. A face F is the minimal face of x if and only if $x \in ri\, F$. The set X then decomposes into a disjoint union of the relative interiors of its faces.

We shall now investigate when a closed convex set possesses extremal points. We need the following auxiliary result.

Lemma 3.28. *Let X be a closed convex set. Suppose there exists a point $z \in X$ and a non-zero vector v such that the ray $\{z + \alpha v \mid \alpha \geq 0\}$ is contained in X. Then for every $x \in X$, the ray $\{x + \alpha v \mid \alpha \geq 0\}$ is contained in X.*

Proof. Let $\mu > 0$ be arbitrary, and set $\alpha = \frac{\beta}{1-\lambda}$ for $\lambda \in (0,1)$. Then we have

$$\lambda x + (1-\lambda)(z + \alpha v) = \lambda x + (1-\lambda)z + \beta v \in X$$

for all $\lambda \in (0,1)$. Let $\lambda \to 1$, then by closedness of X we also get $x + \beta v \in X$. □

Lemma 3.29. *Let X be a non-empty closed convex set. Then X has extremal points if and only if X does not contain a line (a 1-dimensional affine subspace).*

Proof. Suppose X contains a line l. Let $z, w \in l$ be two distinct points and set $v = z - w$. By the previous lemma, for every $x \in X$ we have $\{x + \alpha v \mid \alpha \in \mathbb{R}\} \subset X$. Hence through every point of X there runs a line which belongs to X. Therefore X cannot have extreme points.

Suppose now that X does not contain a line. We proceed by induction over the dimension of X. If $\dim X = 0$, then X is a singleton and contains an extremal point. Suppose we have shown the assertion of the lemma for all convex sets X with $\dim X < d$. Let $\dim X = d$. Choose a point $x \in X$ and consider a line l through x which lies in the affine hull of X. Since $l \not\subset X$, there exists a point $y \in l$ which lies on the relative boundary of X. Let H be an affine hyperplane separating y from X properly. Then $X' = H \cap X$ is a non-empty closed convex set of dimension $\dim X' < d$, and it does not contain a line. By the induction hypothesis, X' has extremal points. But these points have then also to be extremal points of X. □

Lemma 3.30. *A compact convex set equals the convex hull of its extreme points.*

Proof. Clearly the convex hull of the extremal points is a subset of the set itself. We then have to show that every point is a convex combination of extremal points.

We also proceed by induction on the dimension. If $X = \emptyset$ or $\dim X = 0$, then the assertion is evident. Assume the assertion is proven for $\dim X < d$, and let X be a compact convex set of dimension d. Let $x \in X$ be a point, and let l be a line through x which lies in the affine hull of X. Then $l \cap X$ must be a closed line segment, because X is compact. The end-points y, z of this segment lie on the relative boundary of X and x is a convex combination of y, z. Let H_y, H_z be affine hyperplanes which separate y, z from X properly, and let $X_y = H_y \cap X$, $X_z = H_z \cap X$. The sets X_y, X_z are convex, compact, and have dimension strictly smaller than d. Hence y, z are convex combinations of extremal points of X_y, X_z, respectively, by the induction hypothesis. But these extremal points are also extremal points of X. Hence y, z are convex combinations of extremal points of X, and x is also such a combination. □

Now we shall consider the extremal points of polyhedral sets.

Lemma 3.31. *A polyhedral set has a finite number of extremal points. In particular, a bounded polyhedral set is a polytope.*

Proof. Let $X = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a polyhedral set, where $b \in \mathbb{R}^m$, and let $x \in X$ be an extremal point of X. Let $a_1, \dots, a_m$ be the rows of A. Let $I = \{i \in \{1, \dots, m\} \mid a_i x = b_i\}$ be the index set of the active inequality constraints. In particular, we have $a_i x < b_i$ for all $i \notin I$. Then among the row vectors a_i, $i \in I$, there are n linearly independent vectors. Indeed, suppose there exists a non-zero vector $v \in \mathbb{R}^n$ such that $a_i v = 0$ for all $i \in I$. Then there exists $\varepsilon > 0$ such that $x \pm \varepsilon v \in X$. This contradicts the extremality of x.

Thus there exist n linearly independent row vectors $a_{i_1}, \dots, a_{i_n}$ of A such that $a_{i_j} x = b_{i_j}$ for all $j = 1, \dots, n$. On the other hand, for any linearly independent set of n row vectors $a_{i_1}, \dots, a_{i_n}$ of A the system $a_{i_j} x = b_{i_j}$, $j = 1, \dots, n$, uniquely determines a solution x. If this point x is in X, then it is an extremal point of X.

Since there exists only a finite number of sets of n linearly independent row vectors of A, the number of extremal points of X must also be finite. □

In principle one can therefore compute all extremal points of a polyhedral set by checking for each set of n linearly independent row vectors $a_{i_1}, \dots, a_{i_n}$ of A whether the solution of the system $a_{i_j} x = b_{i_j}$, $j = 1, \dots, n$ is feasible.

Finally, we consider extremal points of intersections of convex sets with affine subspaces.

Lemma 3.32. *Let $X \subset \mathbb{R}^n$ be a closed convex set and let A be an affine subspace of dimension m. Then every extreme point of $C = X \cap A$ is located in a face of X of dimension at most $n - m$.*

Proof. Let $x^* \in C$ be an extreme point, and let $F \subset X$ be the minimal face of X containing x^*, and let $k = \dim F$. Then $x^* \in ri\, F$. We have $\dim F \cap A \geq m + k - n$. Since $x^* \in ri\,(F \cap A) \subset C$ and x^* is extremal in C we get $\dim F \cap A = 0$, and therefore $k \leq n - m$. □

By a similar reasoning, if the minimal face of x^* in C has dimension m', then $\dim F \cap A = m'$ and $\dim F = k \leq n + m' - m$.

3.7 Polar

Definition 3.33. Let $X \subset \mathbb{R}^n$ be a closed convex set containing the origin. The *polar* X° of X is defined as the set

$$\{p \in (\mathbb{R}^n)^* \mid \langle x, p \rangle \leq 1 \ \forall\, x \in X\}.$$

It follows that $(X^\circ)^\circ$.

Clearly if $X \subset X'$ are closed convex sets containing the origin, then $(X')^\circ \subset X^\circ$.

Lemma 3.34. *Let $X \subset \mathbb{R}^n$ be a closed convex set containing the origin. Then X° is bounded if and only if the origin is an interior point of X.*

Proof. We have $0 \in \operatorname{int} X$ if and only if there exists $r > 0$ such that the ball of radius r around the origin is contained in X. But this is equivalent to the condition that X° is contained in the ball of radius r^{-1} around the origin. $\square$

Definition 3.35. Let X be a closed convex set containing the origin, and let F be a face of X. The *complementary face* $F^* \subset X^\circ$ is defined to be the set of points

$$\{p \in X^\circ \mid \langle x, p \rangle = 1 \ \forall\ x \in F\}.$$

Let $X \subset \mathbb{R}^n$ be a closed convex set containing the origin in its interior. Then the sets ∂X, ∂X°, $\{(x,p) \in X \times X^\circ \mid \langle x, p \rangle = 1\}$ are homeomorphic to the sphere S^{n-1}. While this is evident for the first two sets due to their compactness, the last one is in a continuous bijection with the 1-level set of the distance function to X.

References

[1] R. Tyrrell Rockafellar. *Convex Analysis*, volume 28 of *Princeton Mathematical Series*. Princeton University Press, Princeton, 1970.