A Geometric Theory of Barriers in Conic Optimization

Roland Hildebrand

Laboratoire Jean Kuntzmann
Université Grenoble 1 / CNRS

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Outline

Affine Differential Geometry

- Affine connections and affine metrics
- Riemannian metrics
- Hessian and Codazzi structures

Self-concordant barriers

- Barriers on convex sets — Hessian structures
- Conic barriers as centro-affine hypersurface immersions
- Barriers on convex cones — Codazzi structures

Cross-ratio manifold

- Conic barriers as Lagrangian submanifolds of the CRM
- Local approximations
- Distance function
- Minimal submanifolds and affine spheres

Affine connections

M — n-dimensional manifold, X, Y — vector fields on M

∇_X — operator of covariant differentiation along vector field X

$$(\nabla_X Y)^i = \left(\frac{\partial Y^i}{\partial x^k} + \Gamma^i_{jk} Y^j \right) X^k$$

Einstein summation convention: summation over repeating indices

Γ^i_{jk} — Christoffel symbols of ∇

$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ — curvature of ∇

Flat connections

Definition A connection ∇ is called *flat* if its curvature is zero.

∇ flat $\Leftrightarrow$ locally there exists a coordinate system s.t. $\Gamma^i_{jk} = 0$

Definition A connection ∇ is called *projectively flat* if there exists a flat connection ∇' such that the geodesics of ∇ and ∇' coincide as sets.

Connections on hypersurfaces

$M^{n-1} \subset M^n$ — hypersurface, D — affine connection on M^n

How D can induce a connection ∇ on M^{n-1}?

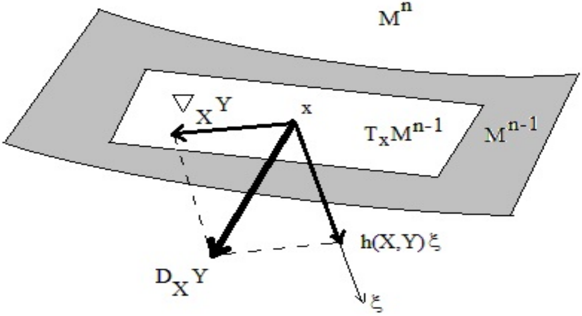
ξ — transversal vector field

$$D_X Y = \nabla_X Y + h(X, Y)\xi, \qquad X, Y \in TM^{n-1}$$

- *affine connection* ∇: projection of D along ξ
- *affine metric* h: transversal component of D
- *cubic form* $C = \nabla h$ — 3-tensor

K. Nomizu, T. Sasaki. Affine differential geometry: geometry of affine immersions. Vol. 111 of Cambridge Tracts in Math. Cambridge University Press, 1994.

Affine connection and affine metric

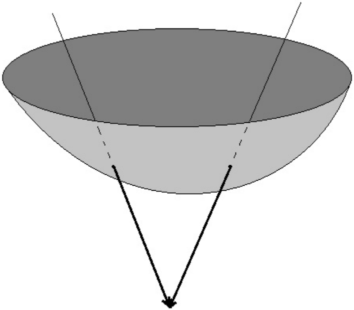


Centro-affine immersions

$M \subset \mathbb{R}^n$ centro-affine hypersurface, D flat connection on $\mathbb{R}^n$
$\xi(x) = -x$, $x \in M$

$\Rightarrow$ affine connection ∇ projectively flat, cubic form C symmetric

∇ *centro-affine connection*, h *centro-affine metric*
invariance under homothety



(Pseudo)-Riemannian metrics

$g : T_xM \times T_xM \to \mathbb{R}$ (positive definite) nondegenerate quadratic form

$$g(X, Y) = g_{ij} X^i Y^j$$

gives rise to *Levi-Civita connection* $\hat{\nabla}$

$$g_{il} \Gamma^l_{jk} = \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right)$$

metric parallel w.r. to its Levi-Civita connection

$$\hat{\nabla} g = 0$$

Hessian and Codazzi structures

Definition Let ∇ be an affine connection and g a pseudo-metric. If ∇g is totally symmetric, then (∇, g) is called *Codazzi structure*.

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(Z, X) = (\nabla_Z g)(X, Y)$$

$\bar{\nabla} = 2\hat{\nabla} - \nabla$ — dual connection
$(\bar{\nabla}, g)$ — dual Codazzi structure

Definition A Codazzi structure (∇, g) with ∇ flat is called *Hessian structure*.

locally $g = f''$, $\nabla g = f'''$ for some scalar function $f : M \to \mathbb{R}$

H. Shima. The geometry of Hessian structures. World Scientific, 2007.

Barriers on convex sets

$C \subset \mathbb{R}^n$ closed convex set

barrier: $F : C^o \to \mathbb{R}$ smooth function

- $F(x) \to \infty$ as $x \to \partial C$
- Hessian $F'' \succ 0$
- self-concordance: $|F'''(x)[h,h,h]| \leq 2(F''(x)[h,h])^{3/2}$ for all $h \in T_x C^o$
- $F''(x)[h,h] \geq \nu^{-1}(F'(x)[h])^2$ for all $h \in T_x C^o$

ν — self-concordance parameter

Hessian structure

on C^o

- D — flat affine connection from $\mathbb{R}^n$
- metric $g = F''$
- symmetric 3-tensor $T = F'''$
- $T = Dg$

(D, g) *Hessian structure*

Dual barrier

$\mathbb{R}_n = (\mathbb{R}^n)^*$ dual space

Legendre transform $F^* : \mathbb{R}_n \to \mathbb{R}$, $F^*(p) = \sup_{x \in C^o} \langle p, x \rangle - F(x)$

F^* is a self-concordant barrier on its domain $(C^*)^o$ with the same self-concordance parameter as F

Let (D^*, g^*) be the Hessian structure induced by F^* on $(C^*)^o$.

Primal-dual symmetry

$x \mapsto F'(x)$ defines a bijection $C^o \to (C^*)^o$

Under this bijection the Hessian structures (D, g) and (D^*, g^*) are dual to each other.

Barriers on convex cones

logarithmically homogeneous barrier

$$F(\lambda x) = -\nu \log \lambda + F(x) \quad \forall\, x \in K^o,\ \lambda > 0$$

ν — parameter of logarithmic homogeneity = self-concordance parameter

level surfaces are centro-affine and homothetic

a level surface determines F up to an additive constant if we take the minimal ν

Equivalence with centro-affine objects

by [Loftin, 2001]

- $g|_M = -\nu h$
- $T|_M = -\nu C$

h — centro-affine metric, ∇ — centro-affine connection,
$C = \nabla h$ — cubic form

Corollary Under homothety, $g|_M$ and $T|_M$ are identical for different level surfaces.

We obtain a natural projectively flat Codazzi structure (∇, h) on the level surfaces of F.

Self-concordance

Theorem [H., 2011] Let $M \subset \mathbb{R}^n$ be a concave centro-affine hypersurface which is asymptotic to a regular convex cone $K \subset \mathbb{R}^n$. Then M defines a logarithmically homogeneous self-concordant barrier with parameter ν if and only if $|C(u,u,u)| \leq 2\gamma ||u||_h^{3/2}$ for all $u \in TM$, where $\gamma = \frac{\nu-2}{\sqrt{\nu-1}}$.

by [Pick,Berwald, 1923]
$\nu = 2 \Leftrightarrow \gamma = 0 \Leftrightarrow C = 0 \Leftrightarrow K = L_n$

Corollary Let $K \subset \mathbb{R}^n$, $n \geq 2$, be a regular convex cone. For every self-concordant log-homogeneous barrier on K, $\nu \geq 2$.

The Lorentz cone with its barrier is the simplest cone.
interior-point algorithms should be able to solve CQP with a single conic constraint in one step!

Duality

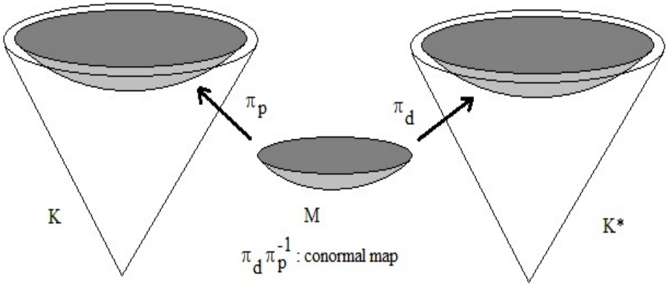
Legendre transform $F \mapsto F^*$ corresponds to

duality of centro-affine hypersurface immersions $M \to \mathbb{R}^n$ and $M \to \mathbb{R}_n$

defined by the *conormal map*.

In the absence of a volume form on $\mathbb{R}^n$ the cornormal map is defined up to homothety.

Primal and dual centro-affine immersion



M endowed with a dual pair of projectively flat Codazzi structures

consider M as submanifold in a product of projective spaces

Projective space

$\mathbb{P}^{n-1}$ — projective space, $\mathbb{P}_{n-1}$ — dual projective space

$\pi : \mathbb{R}^n \setminus \{0\} \to \mathbb{P}^{n-1}$, $\pi_* : \mathbb{R}_n \setminus \{0\} \to \mathbb{P}_{n-1}$ — projections

$C = \pi[K]$
$C^* = \pi_*[K^*]$

$C \subset \mathbb{P}^{n-1}$, $C^* \subset \mathbb{P}_{n-1}$ compact convex sets containing no projective lines

Barriers as submanifolds

Definition We call the $2(n-1)$-dimensional manifold $\mathcal{M} = \{(x,p) \,|\, x \not\perp p\} \subset \mathbb{P}^{n-1} \times \mathbb{P}_{n-1}$ the *Cross-ratio manifold*.

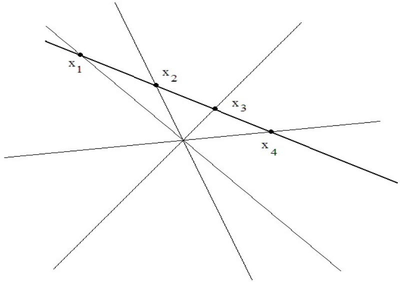
with $p = F'(x)$: $M = \{(\pi(x), \pi_*(p)) \,|\, x \in K^o\} \subset \mathcal{M}$ s.t.

- $\dim M = n-1$
- $\pi : M \to C^o$ bijective
- $\pi_* : M \to (C^*)^o$ bijective
- $\partial M = \Delta$

$\Delta = \{(\pi(x), \pi_*(p)) \,|\, x \in \partial K \setminus \{0\},\ p \in \partial K^* \setminus \{0\},\ x \perp p\} \subset (\mathbb{P}^{n-1} \times \mathbb{P}_{n-1}) \setminus \mathcal{M} = \partial\mathcal{M}$ depends only on K

Which submanifolds M define self-concordant barriers?

Cross-ratio

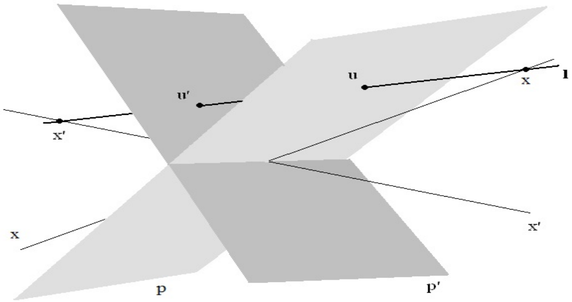


x_1, x_2, x_3, x_4 points on the projective line

$$(x_1, x_2; x_3, x_4) = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_2 - x_3)(x_1 - x_4)}$$

Two-point function

[Ariyawansa,Davidon,McKennon '99]: instead of 4 collinear points use 2 points and 2 dual points — *quadra-bracket*



$z = (x, p), z' = (x', p') \in \mathcal{M} \subset \mathbb{P}^{n-1} \times P_{n-1}$

$$(z; z') = (z'; z) = (u, x'; u', x)$$

Cross-ratio manifold

for $z \approx z'$

$$(z; z') = g(z' - z, z' - z) + O(||z' - z||^3)$$

defines a pseudo-Riemannian metric of neutral signature on $\mathcal{M}$

involution of tangent space $J : T\mathcal{M} \to T\mathcal{M}$
$J : u = (u_x, u_p) \mapsto (u_x, -u_p)$

define $\omega(X, Y) = g(JX, Y)$

Theorem [H., 2011] ω is a symplectic form (closed, skew-symmetric, non-degenerated) which is compatible with g (parallel with respect to the Levi-Civita connection D of g: $D\omega = 0$).

$\mathcal{M}$ becomes a homogeneous *para-Kähler manifold*

Lagrangian submanifolds

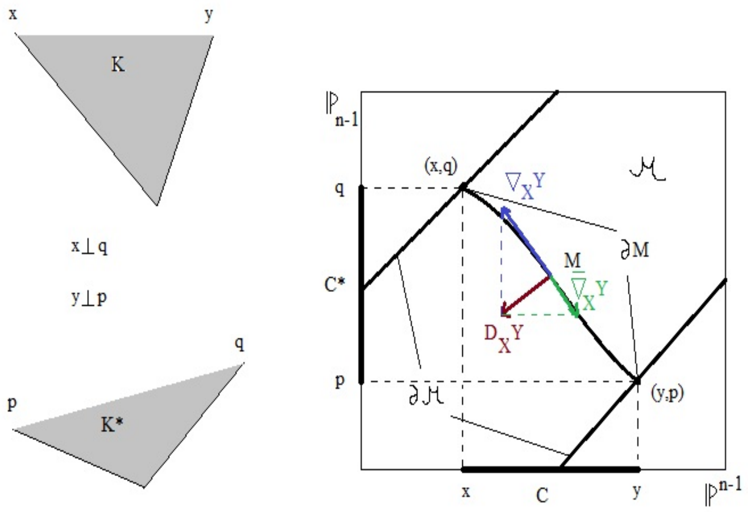
Definition A $(n-1)$-dimensional submanifold $M \subset \mathcal{M}$ is called *Lagrangian* if $\omega|_M = 0$.

Theorem [H., 2011] Up to homothety, there is a 1-to-1 correspondence between the Lagrangian submanifolds of $\mathcal{M}$ and the centro-affine immersions in $\mathbb{R}^n$.

Theorem [H., 2011] The projection of the Levi-Civita connection D of g on a Lagrangian submanifold $M \subset \mathcal{M}$ along $ker(d\pi)$ and $ker(d\pi_*)$ defines two projectively flat affine connections $\nabla, \bar{\nabla}$ on M. $(\nabla, g|_M)$ and $(\bar{\nabla}, g|_M)$ are dual Codazzi structures. The cubic form $C = \nabla(g|_M)$ can be expressed by the second fundamental form II of M by

$$C(X, Y, Z) = -2\omega(II(X, Y), Z)$$

Two-dimensional case



Geometric characterisation

a self-concordant barrier for K (and K^*) is determined by a submanifold M satisfying

- Lagrangian: $\omega|_M = 0$
- $\partial M = \Delta$
- concavity: $g|_M \prec 0$
- self-concordance: $C = \nabla g = -2\omega II$ uniformly bounded

Local approximation

The second fundamental form II of a submanifold M of a Riemannian manifold at a point $\hat{x} \in M$ measures the deviation of M from the tangent geodesic submanifold at M.

Theorem [H., 2011] Lagrangian geodesic submanifolds of M are totally geodesic.

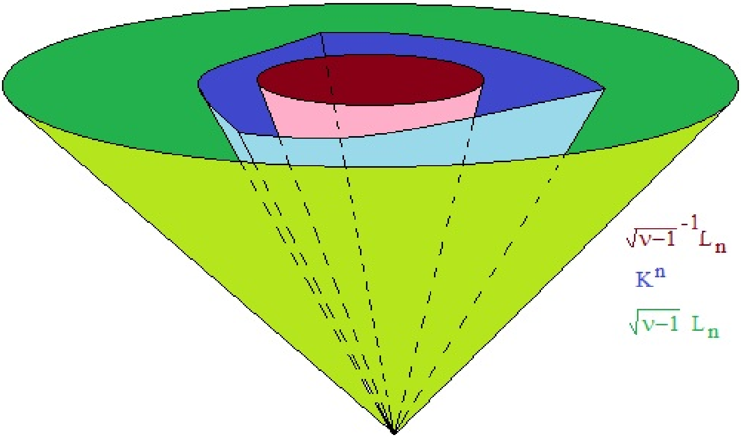
At a given point $\hat{x} \in M$ the tangent totally geodesic submanifold defines the barrier of an approximating Lorentz cone to K (and K^*).

The projective self-concordance parameter $\gamma = \frac{\nu-2}{\sqrt{\nu-1}}$ measures the 2nd order deviation of M from this barrier.

Dikin ellipsoids

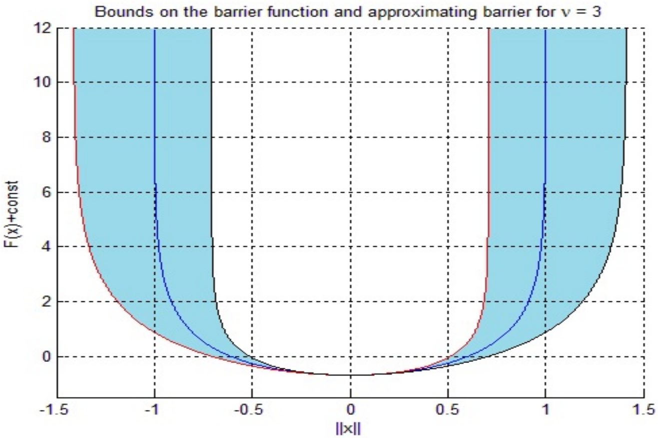
pass to coordinate system where approximating Lorentz cone is centered

inner and outer approximations of a cone K^n are equal to the approximating Lorentz cone scaled by $\sqrt{\nu - 1}$



Bounds on the barrier

on the section $x = (1, \tilde{x}^T)^T$
$F(x) \in [-(\nu - 1)\log(\sqrt{\nu - 1} \pm ||\tilde{x}||) - \log(1 \mp \sqrt{\nu - 1}||\tilde{x}||)]$



Distance function

Theorem [H., 2011] $D(z,z') = \sqrt{-(z;z')}$ real, symmetric, nonnegative, compatible with $-g|_M$, $D(z,z') = 0 \Leftrightarrow z = z'$, $\lim_{z' \to \partial M} D(z,z') = +\infty$.

can be used to

- measure the distance to the projective central path of a primal-dual feasible pair
- measure the progress of one iteration from a primal-dual feasible pair to the next one

not a real distance — violates triangle inequality

$$D(z_1,z_2) \geq D(z_1,z_0)\sqrt{1 + D^2(z_2,z_0)} + D(z_2,z_0)\sqrt{1 + D^2(z_1,z_0)}$$

for z_1, z_0, z_2 collinear in primal or dual projection

Universal barrier

[Nesterov and Nemirovski, 1994]

$$F(x) = const \cdot Vol(K^*(x))$$

$K^*(x) = \{p \in (\mathbb{R}^n)^* \mid \langle p, y - x \rangle \leq 1 \; \forall \; y \in K\}$

$F(x)$ self-concordant with parameter $\nu = O(n)$

does not behave well with respect to product operator and duality

Affine spheres

Theorem [Calabi], [An-Li] $\approx$ 1980 Let $K \subset \mathbb{R}^n$ be a regular convex cone. Then there exists, up to homothety, a unique concave centro-affine hypersurface immersion which is asymptotic to K s.t. $T_k = g^{ij} C_{ijk} = 0$.

Affine hypersphere, can be computed by solving the Monge-Ampère equation $\det u'' = (-u)^{-(n+1)}$ on a compact section Ω of K with boundary condition $u|_{\partial\Omega} = 0$.

corresponds to the *minimal* Lagrangian submanifold $M \subset \mathcal{M}$ with $\partial M = \Delta$

Affine sphere barrier

properties of the barrier function corresponding to the affine sphere

- self-concordant with $\nu = O(n^2)$ (conservative — from results on PDEs)
- $\nu \log\det F'' = 2nF + const$ — characterizing equation
- dual barrier also affine sphere barrier
- $F_{K^n \times K^m} = \left(\frac{n}{\nu_n} F_{K^n} + \frac{m}{\nu_m} F_{K^m}\right) \cdot \max\left\{\frac{\nu_n}{n}, \frac{\nu_m}{m}\right\}$

classical barriers for $L_n, \mathbb{R}^n_+, S_+(n), H_+(n)$ are affine sphere barriers

Example: power cone

$p \in [2, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$

$$P_p = \{(x, y, z)^T \mid x^{1/p} y^{1/q} \geq |z|\} \subset \mathbb{R}^3$$

self-dual convex cone

[Nesterov, 2006]

$$F(x, y, z) = -\log(x^{2/p} y^{2/q} - z^2) - \log x - \log y$$

self-concordant with parameter $\nu = 4$

[Chares and Glineur, 2009]

$$F(x, y, z) = -\log(x^{2/p} y^{2/q} - z^2) - \frac{1}{q} \log x - \frac{1}{p} \log y$$

self-concordant with parameter $\nu = 3$

Power cone cont'd

[Chares and Glineur, 2009]

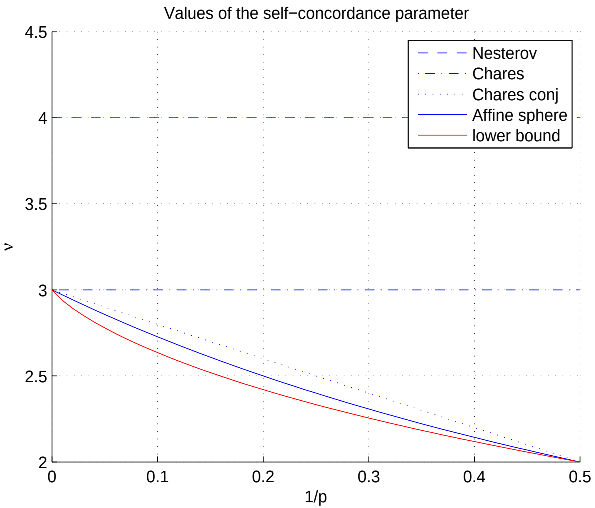
$$F(x,y,z) = -\log(x^{2/p}y^{2/q} - z^2) - \left(1 - \frac{2}{p}\right)\log x$$

conjectured to be self-concordant with parameter $\nu = 3 - \frac{2}{p}$

Affine hypersphere
given by the orbit of the curve
$\{(x,y,z)^T \mid px^2 - \frac{p+1}{3} = qy^2 - \frac{q+1}{3} = z^2\}$ under the action of the group generated by the Lie algebra element
$\mathrm{diag}(2p+q, -p-2q, q-p)$

self-concordant with parameter $\nu = \frac{3p}{p+1}$

Power cone cont'd



Outlook

What is done

- projective formulation of conic programming
- reduction to Lagrangian submanifolds of the cross-ratio manifold
- bounds on the divergence of the Lagrangian submanifold from totally geodesic approximation

What is to do

- fully projective interior-point methods
- additional structure when cone is symmetric

R. Hildebrand. Barriers on projective convex sets. To appear in AIMS Proceedings, Sept. 2011.