# Hessian potentials with parallel derivatives

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# Outline

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# Optimization problems

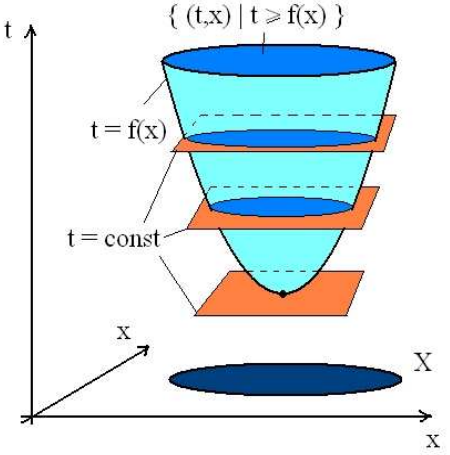
minimize objective function with respect to constraints

$$\min_{x \in X} f(x)$$

in convex optimization problems, $f$ and $X$ are assumed convex

$X \subset \mathbb{R}^n$ is called the feasible set

# Linear objective function




$f(x)$ can be assumed linear

otherwise minimize $t$ over the epigraph

# Definition of barriers

## Definition

Let $X \subset \mathbb{R}^n$ be a regular convex set. A barrier for $X$ is a smooth function $F : X^o \to \mathbb{R}$ such that

- $F''(x) \succ 0$ (convexity)
- $\lim_{x \to \partial X} F(x) = +\infty$ (boundary behaviour)

$F''$ defines a Hessian metric on $X^o$

# Interior-point methods using barriers

$$\min_{x \in X} \langle c, x \rangle$$

constrained convex program

let $F(x) = +\infty$ for all $x \notin X^o$

$$\min_x \tau \langle c, x \rangle + F(x)$$

unconstrained program, $\tau > 0$ a parameter
by convexity and boundary behaviour of $F$ this program is convex

the minimizer $x^*_\tau$ of the unconstrained program tends to the minimizer $x^*$ of the constrained program as $\tau \to +\infty$

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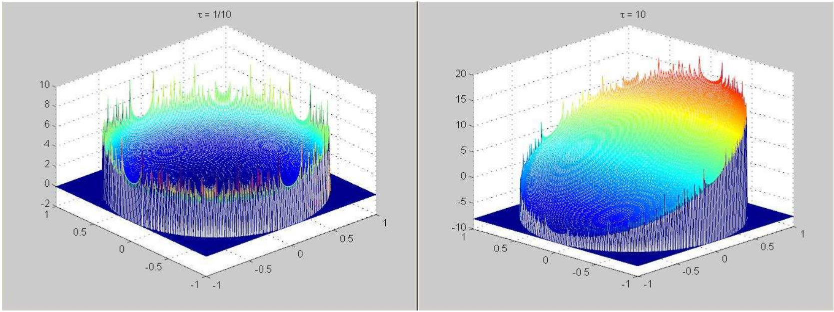



plots of $\tau\langle c, x\rangle + F(x)$ for

$X = \{x \in \mathbb{R}^2 \mid ||x||_2^2 \leq 1\}$, $\langle c, x\rangle = x_1$, $F(x) = -\log(1 - ||x||_2^2)$

# Outline

# Conic programs

## Definition

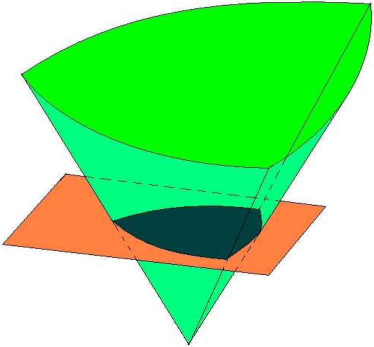
A regular convex cone $K \subset \mathbb{R}^n$ is a closed convex cone having nonempty interior and containing no lines.

## Definition

A conic program over a regular convex cone $K \subset \mathbb{R}^n$ is an optimization problem of the form

$$\min_{x \in K} \langle c, x \rangle : \quad Ax = b.$$

# Geometric interpretation



the feasible set is the intersection of $K$ with an affine subspace

# Symmetric cones

example: conic programs over $K = \mathbb{R}^n_+$

feasible set is a convex polyhedron $\to$ **linear program** (LP)

$\mathbb{R}^n_+$ is self-dual: $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$

and homogeneous: $\mathrm{Aut}(\mathbb{R}^n_+)$ acts transitively on $\mathbb{R}^n_{++}$

**Definition**

A self-dual, homogeneous convex cone is called symmetric.

theory of IP methods most advanced over symmetric cones

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A self-dual, homogeneous convex cone is called symmetric.

theory of IP methods most advanced over symmetric cones

# Classification of symmetric cones

**Theorem (Vinberg, 1960; Koecher, 1962)**

*Every symmetric cone can be represented as a direct product of a finite number of the following irreducible symmetric cones:*

- *Lorentz (or second order) cone* $L_n = \left\{(x_0, \dots, x_{n-1}) \,|\, x_0 \geq \sqrt{x_1^2 + \dots + x_{n-1}^2}\right\}$
- *matrix cones* $S_+(n)$, $H_+(n)$, $Q_+(n)$ *of real, complex, or quaternionic hermitian positive semi-definite matrices*
- *Albert cone* $O_+(3)$ *of octonionic hermitian positive semi-definite* $3 \times 3$ *matrices*

# Canonical barriers

barriers on irreducible symmetric cones

- Lorentz cone $L_n$: $F(x) = -\log(x_0^2 - x_1^2 - \cdots - x_{n-1}^2)$
- matrix cones: $F(X) = -\log\det X$

barriers on reducible symmetric cones
weighted sums of the barriers on the irreducible components

example: $K = \mathbb{R}_+^n$, $F(x) = -\sum_{k=1}^n \log x_k$

# Programs over symmetric cones

conic programs over symmetric cones are efficiently solvable by interior-point methods [Nesterov, Nemirovski, 1994]

- linear programs (LP) over $\mathbb{R}_+^n$ $\sim 10^6$ variables
- conic quadratic programs (CQP) over $L_n \sim 10^4$ variables
- semi-definite programs (SDP) over $S_+(n) \sim 10^2$ variables

structure can greatly increase tractable sizes

free (CLP, LiPS, SDPT3, SeDuMi, ...) and commercial (CPLEX, MOSEK, ...) solvers available

increasingly used in engineering sciences and industry

**What is so special about symmetric cones?**

**How to characterize the canonical barriers on symmetric cones?**

**Is there a local characterization of these barriers?**

# Outline

# Jordan algebras

an algebra $A$ is a vector space $V$ ($\dim V < \infty$) equipped with a bilinear operation $\bullet : V \times V \to V$

## Definition

An algebra $J$ is a Jordan algebra if

- $x \bullet y = y \bullet x$ for all $x, y \in J$ (commutativity)
- $x^2 \bullet (x \bullet y) = x \bullet (x^2 \bullet y)$ for all $x, y \in J$ (Jordan identity)

where $x^2 = x \bullet x$.

## Definition

A Jordan algebra is formally real or Euclidean if $\sum_{k=1}^m x_k^2 = 0$ implies $x_k = 0$ for all $k, m$.

# Unital and simple Jordan algebras

**Definition**

A Jordan algebra is called unital if it possesses a unit element $e$, satisfying $u \bullet e = u$ for all $u \in J$.

**Definition**

A Jordan algebra is called simple if it is not nil and has no non-trivial ideal.

**Definition**

A Jordan algebra is called semi-simple if it is a direct product of simple Jordan algebras.

# Power associativity

let $L_u$ be the operator of multiplication with $u$

then the Jordan identity is equivalent to $[L_u, L_{u^2}] = 0$

define $u^{m+1} = u \bullet u^m$

**Theorem (Jordan, von Neumann, Wigner 1934)**

*Let $J$ be a Jordan algebra. Then for every $u \in J$, $u^r \bullet u^s = u^{r+s}$ for all $r, s \geq 1$.*

the subspace spanned by the powers $u, u^2, \dots$ is an associative subalgebra

# Examples of Jordan algebras

let $Q$ be a real symmetric matrix and $e \in \mathbb{R}^n$ such that $e^T Qe = 1$

the quadratic factor $\mathcal{J}_n(Q)$ is the space $\mathbb{R}^n$ equipped with the multiplication

$$x \bullet y = e^T Qx \cdot y + e^T Qy \cdot x - x^T Qy \cdot e$$

let $\mathcal{H}$ be an algebra of Hermitian matrices over a real coordinate algebra ($\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$; for $\mathbb{O}$ of size $\leq 3$)
then the corresponding Hermitian Jordan algebra is the vector space underlying $\mathcal{H}$ equipped with the multiplication

$$A \bullet B = \frac{AB + BA}{2}$$

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# Euclidean Jordan algebras

**Theorem (Jordan, von Neumann, Wigner 1934)**

*Every Euclidean Jordan algebra is a direct product of simple Jordan algebras of the following types:*

- *quadratic factor with matrix $Q$ of signature $+ - \cdots -$*
- *real symmetric matrices*
- *complex Hermitian matrices*
- *quaternionic Hermitian matrices*
- *octonionic Hermitian $3 \times 3$ matrices*

# Trace forms

**Definition**

Let $J$ be a Jordan algebra. A symmetric bilinear form $\gamma$ on $J$ is called trace form if $\gamma(u, v \bullet w) = \gamma(u \bullet v, w)$ for all $u, v, w \in J$.

**Theorem (Köcher)**

*Let $J$ be a unital Jordan algebra. The symmetric bilinear form*

$$\tau(u, v) = tr\, L_{u \bullet v}$$

*is a trace form, called the generic bilinear trace form.*

**Theorem (Köcher)**

*A Jordan algebra $J$ is semi-simple if and only if its generic bilinear trace form is non-degenerate.*

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# Generic minimum polynomial

for every $u$ in a unital Jordan algebra there exists $m$ such that

- $u^0, u^1, \dots, u^{m-1}$ are linearly independent ($u^0 := e$)
- $u^m = \sigma_1 u^{m-1} - \sigma_2 u^{m-2} + \dots - (-1)^m \sigma_m u^0$

$p_u(\lambda) = \lambda^m - \sigma_1 \lambda^{m-1} + \dots + (-1)^m \sigma_m$ is the minimum polynomial of $u$

**Theorem (Jacobson, 1963)**

*There exists a unique minimal polynomial $p(\lambda) = \lambda^m - \sigma_1(u)\lambda^{m-1} + \dots + (-1)^m \sigma_m(u)$, the generic minimum polynomial, such that $p_u | p$ for all $u$. The coefficient $\sigma_k(u)$ is homogeneous of degree $k$ in $u$. The coefficient $t(u) = \sigma_1(u)$ is called generic trace and the coefficient $n(u) = \sigma_m(u)$ the generic norm.*

# Outline

# Symmetric cones and Euclidean Jordan algebras

**Theorem (Vinberg, 1960; Koecher, 1962)**

*The symmetric cones are exactly the cones of squares of Euclidean Jordan algebras,* $K = \{x^2 \mid x \in J\}$.

by $\frac{\partial x^2}{\partial x} = 2L_x$ the boundary of $K$ is composed of elements satisfying $\det L_x = 0$

# Barriers on symmetric cones

on irreducible symmetric cones the canonical barrier is proportional to

$$F(x) = -\log n(x)$$

on reducible symmetric cones $K = K_1 \times \cdots \times K_r$ the canonical barriers are given by

$$F(x) = -\sum_{k=1}^{r} \alpha_k \log n_k(x_k)$$

with $x_k$ the components of $x$ and $n_k$ the generic norm of the algebra corresponding to $K_k$

# Exponential map

define the exponential map

$$\exp(u) = \sum_{k=0}^{\infty} \frac{u^k}{k!}$$

**Theorem (Köcher)**

*Let $J$ be a Euclidean Jordan algebra and $K$ its cone of squares. Then the exponential map is injective and its image is the interior of $K$,*

$$\exp[J] = K^o.$$

# Logarithm

let $J$ be a Euclidean Jordan algebra with cone of squares $K$

then we can define the logarithm

$$\log : K^o \to J$$

as the inverse of the exponential map

for Euclidean Jordan algebras with cone of squares $K$ we have

$$\log n(x) = t(\log x) = \tau(e, \log x)$$

for all $x \in K^o$

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# Barriers on reducible cones

let $K = K_1 \times \cdots \times K_r$ be a symmetric cone corresponding to an algebra $J$

the canonical barriers on $K$ have the form

$$F(x) = -\sum_{k=1}^{r} \alpha_k \log n_k(x_k) = -\sum_{k=1}^{r} \alpha_k \tau_k(e_k, \log x_k) = \tau(z, \log x)$$

with $z = -\sum_{k=1}^{r} \alpha_k e_k$ a central element of $J$

# Central elements and trace forms

**Theorem (Köcher)**

*Let $J$ be a semi-simple Euclidean Jordan algebra. Then every trace form $\gamma$ on $J$ has the form*

$$\gamma(u,v) = \tau(z \bullet u, v)$$

*with $z$ some central element of $J$.*
*The trace form $\gamma$ is non-degenerate if and only if $z$ is invertible.*

for a Euclidean Jordan algebra $J$ every central element is of the form $z = \sum_{k=1}^{r} \alpha_k e_k$

- $z$ invertible if and only if all $\alpha_k \neq 0$
- $\gamma$ positive definite if and only if all $\alpha_k < 0$

# Barriers and trace forms

## Corollary

*Let $K$ be a symmetric cone and $J$ the corresponding Euclidean Jordan algebra. Then every canonical barrier on $K$ can be expressed as*

$$F(x) = \gamma(e, \log x)$$

*with $\gamma$ a positive definite trace form.*
*On the other hand, for every positive definite trace form $\gamma$ the function $F(x)$ is a canonical barrier on $K$.*

# Notation for derivatives

let $F : U \to \mathbb{R}$ be a smooth function on $U \subset \mathbb{A}^n$, where $\mathbb{A}^n$ is the $n$-dimensional affine real space

we note $\frac{\partial F}{\partial x^\alpha} = F_{,\alpha}$, $\frac{\partial^2 F}{\partial x^\alpha \partial x^\beta} = F_{,\alpha\beta}$ etc.

note $F^{,\alpha\beta}$ for the inverse of the Hessian

we adopt the Einstein summation convention over repeating indices, e.g.,

$$F^{,\alpha\beta}F_{,\beta\gamma} := \sum_{\beta=1}^{n} F^{,\alpha\beta}F_{,\beta\gamma} = \delta_\gamma^\alpha$$

# Outline

# Hessian metrics

## Definition

Let $U \subset \mathbb{A}^n$ be a domain equipped with a pseudo-metric $h$. Then $h$ is called Hessian if there locally exists a smooth function $F$ such that $h = F''$. The function $F$ is called Hessian potential.

for every $x \in U$, $h$ defines a symmetric bilinear form

$$h_x : T_xU \times T_xU \to \mathbb{R}, \qquad h_x : (u, v) \mapsto h_x(u, v) = \partial_u \partial_v F = F_{,\alpha\beta} u^\alpha v^\beta$$

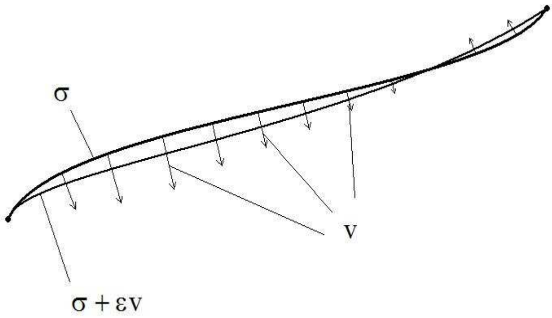
# Geodesics

for every curve $\sigma : [0, T] \to U$, the length is given by

$$\mathcal{L}(\sigma(\cdot)) = \int_0^T \sqrt{h_{\sigma(t)}(\dot{\sigma}(t), \dot{\sigma}(t))}\, dt$$

**Definition**

A stationary point of the length functional $\mathcal{L}$ with respect to variations vanishing at the endpoints is called geodesic.




stationary point means

$$\left.\frac{d\mathcal{L}(\sigma(\cdot)+\varepsilon v(\cdot))}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

for all vector fields $v(t)$ along the curve $\sigma$ satisfying $v(0) = v(T) = 0$

# Christoffel symbols

the Euler-Lagrange equation for the length functional is

$$\frac{d^2\sigma^\alpha}{dt^2} + \frac{1}{2} F^{,\alpha\delta} F_{,\beta\gamma\delta} \frac{d\sigma^\beta}{dt} \frac{d\sigma^\gamma}{dt} = 0$$

the coefficients at the first derivatives $\dot{\sigma}$ are the Christoffel symbols

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} F^{,\alpha\delta} F_{,\beta\gamma\delta}$$

the geodesic equation becomes

$$\frac{d^2\sigma^\alpha}{dt^2} + \Gamma^\alpha_{\beta\gamma} \frac{d\sigma^\beta}{dt} \frac{d\sigma^\gamma}{dt} = 0$$

# Parallel vector transport

let $\sigma : [0, T]$ be a curve and $v \in T_{\sigma(0)}U$ a tangent vector at the starting point

the parallel transport of the vector $v$ along the curve $\sigma$ is defined by the ODE

$$\frac{dv^\alpha}{dt} + \Gamma^\alpha_{\beta\gamma} v^\beta \frac{d\sigma^\gamma}{dt} = 0$$

with $w^\alpha = \frac{d\sigma^\alpha}{dt}$ the geodesic equation becomes

$$\frac{dw^\alpha}{dt} + \Gamma^\alpha_{\beta\gamma} w^\beta \frac{d\sigma^\gamma}{dt} = 0$$

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# Parallel transport of forms

let $C_t : T_{\sigma(t)}U \times \cdots \times T_{\sigma(t)}U \to \mathbb{R}$ be a multilinear form along a curve $\sigma$

the form $C$ is parallel along $\sigma$ if for all parallel vector fields $u_t, \dots, v_t$ along $\sigma$ the value $C_t(u_t, \dots, v_t)$ is constant

this leads to the ODE

$$\frac{dC_{\alpha_1 \dots \alpha_r}}{dt} - \sum_{k=1}^{r} \Gamma^{\beta}_{\alpha_k \gamma} C_{\alpha_1 \dots \beta \dots \alpha_r} \frac{d\sigma^\gamma}{dt} = 0$$

where $\beta$ takes the place of the index $\alpha_k$

# Parallel vector fields and forms

a vector field $u^\alpha$ is parallel if it is parallel along every curve

this is equivalent to the PDE

$$v^\alpha_{,\beta} + \Gamma^\alpha_{\beta\gamma} v^\gamma = 0$$

a form $C_{\alpha_1\dots\alpha_r}$ is parallel if it is parallel along every curve

$$C_{\alpha_1\dots\alpha_r,\beta} - \sum_{k=1}^r \Gamma^\gamma_{\alpha_k\beta} C_{\alpha_1\dots\gamma\dots\alpha_r} = 0$$

parallel vector fields may not exist on a given Riemannian manifold

# Metric tensor

the metric of a pseudo-Riemannian manifold is always parallel

hence the second derivative $F''$ of a Hessian potential is always parallel

**What does parallelism of other derivatives imply?**

# Outline

1. Conic optimization
   - Convex programs
   - Conic programs
2. Jordan algebras and symmetric cones
   - Jordan algebras
   - Symmetric cones
3. Hessian metrics
   - Parallel transport
   - Parallel first derivative
   - Parallel third derivative

# PDE

the first derivative of a Hessian potential is parallel if

$$F_{,\alpha\beta} - \Gamma^{\gamma}_{\alpha\beta} F_{,\gamma} = F_{,\alpha\beta} - \frac{1}{2} F^{,\gamma\delta} F_{,\alpha\beta\delta} F_{,\gamma} = 0$$

equivalently

$$2F''(\cdot,\cdot) = F'''(\cdot,\cdot,(F'')^{-1}F')$$

# Solution

with $e^\gamma = -F_{,\delta}F^{,\gamma\delta}$ the equation becomes

$$2F_{,\alpha\beta} = -F_{,\alpha\beta\delta}e^\delta$$

then

$$e^\gamma_{,\alpha} = -F_{,\alpha\delta}F^{,\gamma\delta} - F^{,\gamma\rho}F_{,\rho\sigma\alpha}e^\sigma = -\delta^\gamma_\alpha + 2F^{,\gamma\rho}F_{,\rho\alpha} = \delta^\gamma_\alpha$$

this integrates to $e = x + const$ with $x$ the position vector field

shift the coordinate system in $\mathbb{A}^n$ such that $x = e$

$$\begin{aligned} & F_{,\delta} + F_{,\gamma\delta}x^\gamma = (F_{,\gamma}x^\gamma)_{,\delta} = 0 \\ \Rightarrow \quad & F_{,\gamma}x^\gamma = const = \nu \\ \Rightarrow \quad & F(\alpha x) = \nu \log\alpha + F(x), \quad \alpha > 0 \end{aligned}$$

$F$ is logarithmically homogeneous

reverse implication holds too if $\det F'' \neq 0$

**Theorem (H., 2012)**

*Let $F : U \to \mathbb{R}$ be a $C^3$ function defined on some domain $U \subset \mathbb{A}^n$. Suppose that $F$ has a non-degenerate Hessian. Then the first derivative $F'$ is parallel with respect to the Hessian metric $F''$ if and only if $F$ is logarithmically homogeneous with respect to some central point.*

# Outline

# PDE

the third derivative of a Hessian potential is parallel if

$$F_{,\alpha\beta\gamma\delta} - \Gamma^{\rho}_{\alpha\delta} F_{,\rho\beta\gamma} - \Gamma^{\rho}_{\beta\delta} F_{,\alpha\rho\gamma} - \Gamma^{\rho}_{\gamma\delta} F_{,\alpha\beta\rho} = 0$$

equivalently we obtain the 4-th order quasi-linear PDE

$$F_{,\alpha\beta\gamma\delta} = \frac{1}{2} F^{,\rho\sigma} \left( F_{,\alpha\beta\rho} F_{,\gamma\delta\sigma} + F_{,\alpha\gamma\rho} F_{,\beta\delta\sigma} + F_{,\alpha\delta\rho} F_{,\beta\gamma\sigma} \right)$$

# Integrability condition

differentiating with respect to $x^\eta$ and substituting the fourth order derivatives by the right-hand side, we get

$$\begin{aligned} F_{,\alpha\beta\gamma\delta\eta} = \frac{1}{4}F^{,\rho\sigma}F^{,\mu\nu}\,(&F_{,\beta\eta\nu}F_{,\alpha\rho\mu}F_{,\gamma\delta\sigma} + F_{,\alpha\eta\mu}F_{,\rho\beta\nu}F_{,\gamma\delta\sigma} \\ + \quad & F_{,\gamma\eta\nu}F_{,\alpha\rho\mu}F_{,\beta\delta\sigma} + F_{,\alpha\eta\mu}F_{,\rho\gamma\nu}F_{,\beta\delta\sigma} + F_{,\beta\eta\nu}F_{,\gamma\rho\mu}F_{,\alpha\delta\sigma} \\ + \quad & F_{,\gamma\eta\mu}F_{,\rho\beta\nu}F_{,\alpha\delta\sigma} + F_{,\beta\eta\nu}F_{,\delta\rho\mu}F_{,\alpha\gamma\sigma} + F_{,\delta\eta\mu}F_{,\rho\beta\nu}F_{,\alpha\gamma\sigma} \\ + \quad & F_{,\delta\eta\nu}F_{,\alpha\rho\mu}F_{,\beta\gamma\sigma} + F_{,\alpha\eta\mu}F_{,\rho\delta\nu}F_{,\beta\gamma\sigma} + F_{,\delta\eta\nu}F_{,\gamma\rho\mu}F_{,\alpha\beta\sigma} \\ + \quad & F_{,\gamma\eta\mu}F_{,\rho\delta\nu}F_{,\alpha\beta\sigma}) \end{aligned}$$

anti-commuting $\delta, \eta$ gives the integrability condition

$$\begin{aligned} F^{,\rho\sigma}F^{,\mu\nu}\,(&F_{,\beta\eta\nu}F_{,\delta\rho\mu}F_{,\alpha\gamma\sigma} + F_{,\alpha\eta\mu}F_{,\rho\delta\nu}F_{,\beta\gamma\sigma} + F_{,\gamma\eta\mu}F_{,\rho\delta\nu}F_{,\alpha\beta\sigma} \\ &-F_{,\beta\delta\nu}F_{,\eta\rho\mu}F_{,\alpha\gamma\sigma} - F_{,\alpha\delta\mu}F_{,\rho\eta\nu}F_{,\beta\gamma\sigma} - F_{,\gamma\delta\mu}F_{,\rho\eta\nu}F_{,\alpha\beta\sigma}) = 0. \end{aligned}$$

# Simplification with Christoffel symbols

multiplying the integrability condition with $(F'')^{-1}$ we get

$$\begin{aligned} &\Gamma^{\eta}_{\alpha\mu}\Gamma^{\mu}_{\delta\rho}\Gamma^{\rho}_{\beta\gamma} + \Gamma^{\eta}_{\beta\mu}\Gamma^{\mu}_{\delta\rho}\Gamma^{\rho}_{\alpha\gamma} + \Gamma^{\eta}_{\gamma\mu}\Gamma^{\mu}_{\delta\rho}\Gamma^{\rho}_{\alpha\beta} \\ - \quad &\Gamma^{\mu}_{\alpha\delta}\Gamma^{\eta}_{\rho\mu}\Gamma^{\rho}_{\beta\gamma} - \Gamma^{\mu}_{\beta\delta}\Gamma^{\eta}_{\rho\mu}\Gamma^{\rho}_{\alpha\gamma} - \Gamma^{\mu}_{\gamma\delta}\Gamma^{\eta}_{\rho\mu}\Gamma^{\rho}_{\alpha\beta} = 0 \end{aligned}$$

this is satisfied if and only if

$$\Gamma^{\eta}_{\alpha\mu}\Gamma^{\mu}_{\delta\rho}\Gamma^{\rho}_{\beta\gamma}u^{\alpha}u^{\beta}u^{\gamma}v^{\delta} = \Gamma^{\mu}_{\alpha\delta}\Gamma^{\eta}_{\rho\mu}\Gamma^{\rho}_{\beta\gamma}u^{\alpha}u^{\beta}u^{\gamma}v^{\delta}$$

for all tangent vectors $u, v$

# Algebra defined by $F$

define a multiplication on the tangent space by $u \bullet v = \Gamma(u, v)$,

$$(u \bullet v)^\alpha = \Gamma^\alpha_{\beta\gamma} u^\beta v^\gamma$$

this defines a commutative algebra $J$

the integrability condition becomes

$$\Gamma(\Gamma(\Gamma(u,u),v),u) = \Gamma(\Gamma(u,v),\Gamma(u,u))$$

or

$$(u^2 \bullet v) \bullet u = (u \bullet v) \bullet u^2$$

it is equivalent to the Jordan identity

# Hessian metric as trace form

the Hessian metric $F''$ satisfies

$$\begin{aligned} F''(u \bullet v, w) &= F_{,\beta\gamma}\Gamma^{\beta}_{\delta\rho}u^{\delta}v^{\rho}w^{\gamma} = \frac{1}{2}F_{,\beta\gamma}F_{,\delta\rho\sigma}F^{,\sigma\beta}u^{\delta}v^{\rho}w^{\gamma} \\ &= \frac{1}{2}F_{,\delta\rho\gamma}u^{\delta}v^{\rho}w^{\gamma} = \frac{1}{2}F_{,\beta\delta}u^{\delta}F_{,\rho\gamma\sigma}F^{,\sigma\beta}v^{\rho}w^{\gamma} \\ &= F_{,\delta\beta}u^{\delta}\Gamma^{\beta}_{\rho\gamma}v^{\rho}w^{\gamma} = F''(u, v \bullet w). \end{aligned}$$

hence $F''$ is a trace form

**Theorem (H., 2012)**

*Let $F : U \to \mathbb{R}$ be a $C^5$ function defined on some domain $U \subset \mathbb{A}^n$. Suppose that $F$ has a non-degenerate Hessian. If the third derivative of $F$ is parallel with respect to the Hessian metric, then the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ of the Hessian metric define the structure tensor of a Jordan algebra, and the metric $F''$ is a trace form of this algebra.*

# Characterization of solutions

every pair $(J, \gamma)$ of a Jordan algebra $J$ and a non-degenerate trace form $\gamma$ on $J$ define

- a domain (of quasi-invertibility) $U \subset J$
- a closed 1-form $\zeta$ on $U \times \mathbb{R}$ up to a constant additive term

- the local potentials $\Phi$ of $\zeta$ are graphs of Hessian potentials $F$ with parallel 3rd derivative
- every such potential $F$ can be obtained in this way
- the transformation $F \leftrightarrow (J, \gamma)$ is invertible
- the Hessian metric $F''$ turns $U$ into a symmetric space

# Parallel first and third derivative

**Theorem (H., 2012)**

*Let $F : U \to \mathbb{R}$ be a Hessian potential with parallel 3rd derivative.*
*Then the Jordan algebra $J$ is unital if and only if $F$ is log-homogeneous, i.e., if the first derivative of $F$ is parallel.*

in this case

- $U$ is a domain of invertibility
- the value of $\gamma = F''$ on the unit element $e$ is the log-homogeneity parameter
- $F$ is locally a potential of the closed 1-form $\xi_x(\cdot) = \gamma(\cdot, x^{-1})$
- near $e$ we have $F(x) = \gamma(e, \log x) + const$

# Convexity

if in addition $F'' \succ 0$ then

- $J$ is a Euclidean Jordan algebra
- $U$ is a symmetric cone
- $F = \gamma(e, \log x) + const$ is globally defined on $U$

# Characterization of barriers

**Theorem (H., 2012)**

*Let $U \subset \mathbb{A}^n$ be a domain and $F : U \to \mathbb{R}^n$ a $C^5$ function. Then $U$ is a symmetric cone and $F$ a canonical barrier on it if and only if the following conditions hold simultaneously:*

- *$F''$ is a positive-definite Hessian metric on $U$*
- *the corresponding Riemannian space is complete*
- *the 1st derivative $F'$ is parallel with respect to the metric*
- *the 3rd derivative $F'''$ is parallel with respect to the metric*

self-concordance is a trivial consequence of these conditions

# Thank you