Comparison of the PPT cone and the separable cone for $2 \otimes n$ systems

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Outline

- Definitions of the separable cone and the PPT cone
- Known results on the relation between PPT and separability
- New results

— Volume radii and homothetic images

— Block-Hankel matrices perturbed by rank 1 matrices

— Special case : approximation of convex functions with CGFs

Definitions

Consider the space of $nm \times nm$ complex hermitian matrices. A matrix A consists of $m \times m$ blocks A_{kl} of size $n \times n$ each.

Let Γ be the operator of *partial transposition* acting as

$$\Gamma \; : \; A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix} \mapsto A^{\Gamma} = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{m1} \\ A_{12} & A_{22} & \cdots & A_{m2} \\ \vdots & \vdots & & \vdots \\ A_{1m} & A_{2m} & \cdots & A_{mm} \end{pmatrix}$$

Define

PSD — the cone of positive semidefinite matrices

PPT — $\mathrm{PSD} \cap \Gamma(\mathrm{PSD})$

SEP — the convex hull of all matrices of the type $(x \otimes y)(x \otimes y)^*$, $x \in \mathbf{C}^m$, $y \in \mathbf{C}^n$

Inclusion relations and automorphisms

we have the inclusions $\mathrm{SEP} \subset \mathrm{PPT} \subset \mathrm{PSD} \subset \mathrm{PPT}^* \subset \mathrm{SEP}^*$

Proof : let $A = (x \otimes y)(x \otimes y)^* \in \mathrm{SEP}$, then $A \succeq 0$ and $A^\Gamma = (\bar{x} \otimes y)(\bar{x} \otimes y)^* \succeq 0$

hence $A \in \mathrm{PPT}$

Automorphism group : $A \in \mathrm{SEP} \Leftrightarrow (S \otimes I_n)A(S \otimes I_n)^*$,
$(I_m \otimes T)A(I_m \otimes T)^* \in \mathrm{SEP}$ for all $S \in GL(m, \mathbf{C})$, $T \in GL(n, \mathbf{C})$

Interpretations

Interpretation in terms of positive polynomials

$A \in \mathrm{SEP}^*$ if $p_A(x,y) = (x \otimes y)^* A (x \otimes y) \geq 0$ for all $x \in \mathbf{C}^n$, $y \in \mathbf{C}^m$

hence SEP^* is a cone of positive polynomials

one can show that PPT^* is the corresponding cone of sums of squares

PPT is used to approximate SEP

application : sets of mixed states of a composite quantum system consisting of a subsystem with m states and a subsystem with n states

Known results on the relation between PPT and SEP

Block-Hankel matrices

let $m = 2$

if we restrict x to be in $\mathbf{R}^2$ (instead of $x \in \mathbf{C}^2$), then all matrices are in the blockwise symmetric subspace $A_{12} = A_{21}$

SEP is the convex hull of rank 1 block-Hankel matrices

Γ amounts to the identity map, and PPT is the cone of positive semidefinite block-Hankel matrices

Theorem : PPT = SEP, *i.e. any positive semidefinite block-Hankel matrix is a sum of rank 1 positive semidefinite block-Hankel matrices.*

(spectral factorization theorem for quadratic matrix-valued polynomials in 1 variable)

the same hold for block-Töplitz matrices with $A_{11} = A_{22}$

Exactness in the complex case

Theorem (Woronowicz 1976) : *Let* $m = 2$*,* $n \leq 3$ *or* $m \leq 3$*,* $m = 2$*. Then PPT = SEP. If* $m = 2$*,* $n = 4$ *or* $m = 4$*,* $n = 2$*, then PPT* $\neq$ *SEP.*

Theorem (Terpstra 1938) : *If* $\min(n, m) \geq 3$*, then PPT* $\neq$ *SEP.*

trivially PSD = PPT = SEP if $\min(n, m) = 1$

hence PPT = SEP **if and only if** $\min(n, m) = 1$ or $m + n \leq 5$

$2 \otimes n$ case

we consider the case $m = 2$

Theorem (Gurvits 2003) : *Let*

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathrm{PPT}$$

Then

$$\begin{pmatrix} 2A_{11} & A_{12} \\ A_{21} & 2A_{22} \end{pmatrix} \in \mathrm{SEP}$$

Proof :

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \succeq 0, \begin{pmatrix} A_{11} & \pm A_{21} \\ \pm A_{12} & A_{22} \end{pmatrix} \succeq 0$$

$$\Rightarrow \begin{pmatrix} 2A_{11} & A_{12}+A_{21} \\ A_{12}+A_{21} & 2A_{22} \end{pmatrix} \in \mathrm{SEP}, \begin{pmatrix} 2A_{11} & i(A_{12}-A_{21}) \\ i(A_{12}-A_{21}) & 2A_{22} \end{pmatrix} \in \mathrm{SEP}$$

$$\Rightarrow \begin{pmatrix} 2A_{11} & A_{12}\pm A_{21} \\ \pm A_{12}+A_{21} & 2A_{22} \end{pmatrix} \in \mathrm{SEP} \ \Box$$

$HT_{Diag}(2) \otimes Id_{n\times n}[\mathrm{PPT}] \subset \mathrm{SEP}$

here $HT_{Diag}(2)$ is the mapping that contracts the space of 2×2 hermitian matrices by a factor of 2 in the directions orthogonal to the subspace of diagonal matrices

the homothetic image of PPT with respect to some subspace is contained in SEP

New results : homothetic images in the general case

Let m, n be arbitrary and let $A \in \mathrm{SEP}^*$. Then for any $x \in \mathbf{C}^m$ we have $A_x = (x \otimes I_n)^* A (x \otimes I_n) \succeq 0$ (because $y^*(x \otimes I_n)^* A (x \otimes I_n) y = (x \otimes y)^* A (x \otimes y) \geq 0$ for all $y \in \mathbf{C}^n$). Therefore

$$(xx^*) \otimes A_x \in \mathrm{SEP}$$

Let x be normally distributed. Then

$$\mathbf{E}[(xx^*) \otimes A_x] = const \cdot HT_{I_m}(m+1) \otimes Id_{n \times n}(A) \in \mathrm{SEP}$$

Theorem : $HT_{I_m}(m+1) \otimes Id_{n \times n}[\mathrm{SEP}^*] \subset \mathrm{SEP}$.

Let $R_V(K) = (Vol(K)/Vol(B_1))^{1/dim(K)}$ denote the volume radius of a convex body K, B_1 is the unit ball.

Theorem :
$R_V(\mathrm{SEP} \cap \{trace = 1\}) \geq R_V(\mathrm{SEP}^* \cap \{trace = 1\}) \cdot (m+1)^{-(m^2-1)n^2/(m^2n^2-1)}$

Homothetic images for $m = 2$

Let $m = 2$ and consider

$$A = \begin{pmatrix} A_0 + A_1 & A_2 + iA_3 \\ A_2 - iA_3 & A_0 - A_1 \end{pmatrix} \in \text{PPT}$$

Then

$$\begin{pmatrix} A_0 + A_1 & A_2 \\ A_2 & A_0 - A_1 \end{pmatrix}, \begin{pmatrix} A_0 + A_1 & iA_3 \\ -iA_3 & A_0 - A_1 \end{pmatrix}, \begin{pmatrix} A_0 & A_2 + iA_3 \\ A_2 - iA_3 & A_0 \end{pmatrix}$$

are in SEP. Hence

$$\begin{pmatrix} \frac{3}{2}A_0 + A_1 & A_2 + iA_3 \\ A_2 - iA_3 & \frac{3}{2}A_0 - A_1 \end{pmatrix} \in \text{SEP}$$

Theorem : $HT_{I_2}(3/2) \otimes Id_{n\times n}[\text{PPT}] \subset$SEP.

Theorem : $R_V(\text{SEP}\cap\{trace = 1\}) \geq R_V(\text{PPT}\cap\{trace = 1\}) \cdot (2/3)^{3/(4-n^{-2})}$ ($\rightarrow\approx 0.738$).

Perturbed block Hankel matrices

Consider again the case $m = 2$.

Theorem : *Let*

$$A = \begin{pmatrix} A_{11} & A_{12} + izz^* \\ A_{12} - izz^* & A_{22} \end{pmatrix} \in \text{PPT}$$

with A_{12} *hermitian,* $z \in \mathbf{C}^n$*. Then* A *is separable.*

Idea of proof : Decompose A in two separable matrices

$$A = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{12} & \tilde{A}_{22} \end{pmatrix} + B_{2\times 2} \otimes zz^*$$

such that $B \succeq 0$ and $\det B$ is maximized.

Hence equality between SEP and PPT is maintained if a block-Hankel matrix is perturbed by a separable rank 1 matrix — extension of the spectral factorization theorem (works also for perturbed block-Töplitz matrices).

Approximation of PPT by SEP for $m = 2$

Let $m = 2$.

Theorem : *Let P be PPT. Then there exists $c \geq 1$ and S in SEP such that*

$$S \preceq P \preceq cS$$

Proof : Let $P_{11} = I_n$ without restriction of generality. Let P be of rank r. Factor P as

$$P = \begin{pmatrix} I_n & 0 \\ W & Z \end{pmatrix} \begin{pmatrix} I_n & 0 \\ W & Z \end{pmatrix}^*$$

Then $S \preceq P \preceq cS$ if and only if

$$S = \begin{pmatrix} I_n & 0 \\ W & Z \end{pmatrix} M \begin{pmatrix} I_n & 0 \\ W & Z \end{pmatrix}^*$$

with $M \preceq I \preceq cM$.

But

$$\begin{pmatrix} I_n & 0 \\ W & Z \end{pmatrix} zz^* \begin{pmatrix} I_n & 0 \\ W & Z \end{pmatrix}^* \in \mathrm{SEP}$$

if and only if z is an eigenvector of the matrix pencil $(W\ Z) + \lambda(I_n\ 0)$ (including $\lambda = \infty$). Complete $(W\ Z)$ to a square diagonalizable matrix and let $z_1, \ldots, z_r$ be its eigenvectors.

Look for $M = \sum_{k=1}^{r} c_k z_k z_k^*$, $c_k > 0$ such that $M \preceq I$. Since M is regular, we get $M \preceq I \preceq cM$ for some $c \geq 1$. $\square$

Bounded quality of approximation in a special case

Consider the following $(n+1)$-dimensional subspace of the space of $2n \times 2n$ matrices :

$$A_D = \begin{pmatrix} I_n & 0_{n\times 1} \\ 0_{n\times 1} & I_n \end{pmatrix} D_{(n+1)\times(n+1)} \begin{pmatrix} I_n & 0_{n\times 1} \\ 0_{n\times 1} & I_n \end{pmatrix}^T$$

with D diagonal. A_D is PPT if the sequence of its diagonal elements is logarithmically convex. A_D is SEP if its diagonal can be extended to a positive semidefinite Hankel matrix.

Theorem : *Let $A_D \in PPT$. Then there exists $S_D \in SEP$ such that*

$$S_D \preceq A_D \preceq 4S_D$$

Idea of proof :

For $n \to \infty$ the sets of logarithms of the corresponding sequences tend to the set of convex functions and the set of cumulant generating functions (CGFs), that is *log-sum-exp*s of linear functions.

Approximation of convex functions by CGFs

Theorem : *Let f be a continuous convex function on a real interval. Then there exists a CGF g defined on the same interval and a constant c,* **not dependent on** f*, such that $||f - g||_{C^0} \leq c$. The best of such constants c^* is bounded by $\ln 2/2 \leq c^* \leq \ln 2$.*

If we consider convex functions in higher dimensions, the theorem is no more valid.

Conclusions

several new results on the relations between PPT cones and separable cones were obtained

- inclusions of homothetic images and relations between the volume radii of SEP^* and SEP in the general case and between PPT and SEP for $m = 2$
- extension of the spectral factorization theorem to the case of $2n \times 2n$ block-Hankel (block-Töplitz) matrices perturbed by rank 1 matrices
- approximation of convex functions by CGFs

Applications : quantum information, mathematical programming (semidefinite relaxations)