On the geometry of 3-dimensional convex cones

Roland Hildebrand

Laboratoire Jean Kuntzmann / CNRS

Seminar Institut Fourier

November 17, 2022

Monge-Ampère equation

Definition

A regular convex cone $K \subset \mathbb{R}^n$ is a closed convex cone with nonempty interior and containing no lines.

Theorem

Let $K \subset \mathbb{R}^n$ be a regular convex cone. Then the PDE

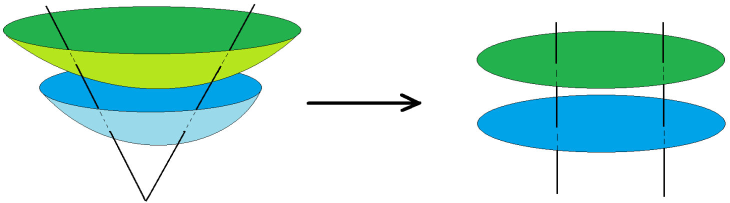
$$\det F'' = e^{2nF}, \quad F|_{\partial K} = +\infty$$

has a unique convex solution on the interior of K.
The level surfaces of F are affine spheres which are asymptotic to ∂K and form a homothetic family.

the solution F is invariant under unimodular automorphisms of K, and logarithmically homogeneous

$$F(tx) = F(x) - \log t, \quad t > 0, \; x \in K^o$$

Metric splitting



interior K^o is diffeomorphic to a direct product of a level surface and a radial ray

Theorem (Loftin 2002)

Under the above diffeomorphism the Riemannian metric defined on K^o by the Hessian F'' splits into a direct product $g = h \oplus s$, where h is the Blaschke metric of the level surface and s the trivial 1-dimensional metric on the ray.

Blaschke metric and cubic form

the Blaschke metric h, i.e. the restriction of F'' to a level surface, is a complete Riemannian metric

it is projectively invariant if we identify the surface with a proper convex domain in $\mathbb{R}P^{n-1}$

the restriction of F''' to the surface is the cubic form C

given h and C the level surfaces of F and the cone K can be recovered up to an unimodular linear isomorphism in $SL(n, \mathbb{R})$

not every pair (h, C) corresponds to an affine sphere
a necessary condition is that C is trace-less with respect to h,

$$h^{ij} C_{ijk} = 0$$

Riemann surfaces

for 3-dimensional cones K the level surfaces M of F are 2-dimensional
hence M is a non-compact simply connected Riemann surface

Uniformization theorem: Every simply connected Riemann surface is conformally equivalent to either the unit disc $\mathbb{D}$, or the complex plane $\mathbb{C}$, or the Riemann sphere S, equipped with either the hyperbolic metric, or the flat (parabolic) metric, or the spherical (elliptic) metric, respectively.

due to Klein, Riemann, Schwarz, Koebe, Poincaré, Hilbert, Weyl, Radó ... 1880–1920

only $\mathbb{D}$ and $\mathbb{C}$ are non-compact

Riemann surfaces

global chart with values in $\mathbb{D}$ or $\mathbb{C}$ exists and is unique up to automorphisms such that $h = e^u|dz|^2$
here $z = x + iy$, $|dz|^2 = dx^2 + dy^2$

u defines the conformal factor e^u

may use other simply connected domains which are conformally isomorphic (in case of $\mathbb{D}$)

if there is a symmetry group acting on the domain, we may use the (not simply connected) factor domain

Cubic differential

consider a conformal chart on M such that $h = e^u(dx^2 + dy^2)$

the trace-less cubic form

$$C = 2\left[\begin{pmatrix} U_1 & -U_2 \\ -U_2 & -U_1 \end{pmatrix}, \quad \begin{pmatrix} -U_2 & -U_1 \\ -U_1 & U_2 \end{pmatrix}\right]$$

has two independent components and can be represented by a cubic differential $U = U_1 + iU_2$: $C = 2Re(U(z)dz^3)$

under bi-holomorphic coordinate changes u, U transform like

$$U(w) = U(z)\left(\frac{dz}{dw}\right)^3, \; u(w) = u(z) + 2\log\left|\frac{dz}{dw}\right|$$

Wang's equation

compatibility requirements on u, U [C.-P. Wang 1991]:

$$\frac{\partial U}{\partial \bar{z}} = 0,$$
$$|U|^2 = \frac{1}{2}e^{3u} - \frac{1}{4}e^{2u}\Delta u = \frac{1}{2}e^{3u}(1 + \mathrm{K})$$

here $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, $\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}$,
$\mathrm{K} \in [-1, 0]$ is the Gaussian curvature

the function U is hence holomorphic

it is called holomorphic cubic differential

- for given U, the equation is an elliptic PDE on u
- if (u, U) is a solution, then $(u, e^{i\varphi}U)$ is also a solution for all constant φ

Relation between cones and solutions (u, U)

Theorem: (follows from [Simon, Wang 1993])

- Let $K \subset \mathbb{R}^3$ be a regular convex cone. Then the solution F of the Monge-Ampère PDE on K^o defines a solution (u, U) on a simply-connected domain $M \subset \mathbb{C}$ with complete Riemannian metric $h = e^u|dz|^2$ up to bi-holomorphic isomorphisms of the domain.
- Every simply connected non-compact Riemann surface M with complete metric $h = e^u|dz|^2$ and holomorphic cubic differential U satisfying Wang's equation corresponds to a regular convex cone $K \subset \mathbb{R}^3$, up to unimodular linear isomorphisms.

complete solutions (u, U) up to bi-holomorphisms

$$\Leftrightarrow$$

regular convex cones up to unimodular isomorphisms

Recovery of the cone K

let (u, U) be a complete solution on $M \subset \mathbb{C}$

construct a surface immersion $f : M \to \mathbb{R}^3$ by integrating

$$f_{zz} = u_z f_z - U e^{-u} f_{\bar{z}}, \quad f_{z\bar{z}} = \frac{1}{2} e^u f, \quad f_{\bar{z}\bar{z}} = -\bar{U} e^{-u} f_z + u_{\bar{z}} f_{\bar{z}}$$

with arbitrary non-degenerate initial condition (f_x, f_y, f)

the surface $f[M]$ will be asymptotic to a cone $K \subset \mathbb{R}^3$

different initial conditions lead to isomorphic cones

Frame equations

equivalently, integrate

$$F_x = F \begin{pmatrix} -e^{-u} Re\, U & \frac{u_y}{2} + e^{-u} Im\, U & e^{u/2} \\ -\frac{u_y}{2} + e^{-u} Im\, U & e^{-u} Re\, U & 0 \\ e^{u/2} & 0 & 0 \end{pmatrix}$$

$$F_y = F \begin{pmatrix} e^{-u} Im\, U & -\frac{u_x}{2} + e^{-u} Re\, U & 0 \\ \frac{u_x}{2} + e^{-u} Re\, U & -e^{-u} Im\, U & e^{u/2} \\ 0 & e^{u/2} & 0 \end{pmatrix}$$

with unimodular initial $F = (e^{-u/2} f_x, e^{-u/2} f_y, f) \in SL(3, \mathbb{R})$

the unimodular matrix function $F(z)$ is called moving frame

Associated family and duality

for given u, the form U is determined up to a constant factor $e^{i\varphi}$ this yields an associated family of (isomorphism classes of) cones $K \subset \mathbb{R}^3$

Definition

Let $K \subset \mathbb{R}^n$ be a regular convex cone. The dual cone of K is defined as

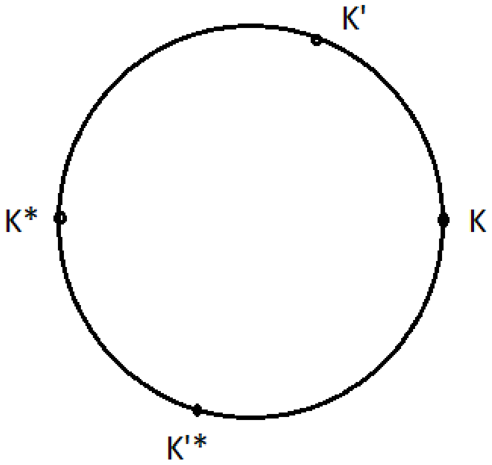
$$K^* = \{y \in (\mathbb{R}^n)^* \mid \langle x, y \rangle \geq 0 \ \forall \ x \in K\}.$$

if the moving frame $F(z)$ defines a surface asymptotic to ∂K, then F^{-T} defines a surface asymptotic to ∂K^*

the matrix function F^{-T} satisfies the same moving frame equations as F but with U replaced by $-U$

if (u, U) corresponds to K, then $(u, -U)$ corresponds to K^*

Associated family and duality



the associated family permits to define "fractional" dual cones

Conditions on U

existence and uniqueness results for u given U

- Wang 1997; Loftin 2001; Labourie 2007: for a holomorphic function on a compact Riemann surface of genus $g \geq 2$ there exists a unique solution (extends to universal cover)
- Benoist, Hulin 2014: let U be holomorphic on $\mathbb{D}$ such that $|U|^{2/3}|dz|^2$ is bounded with respect to the uniformizing hyperbolic metric, then there exists a unique complete solution u such that $|u - \log \frac{4}{(1-|z|^2)^2}|$ is bounded
- Dumas, Wolf 2015: let U be a polynomial on $\mathbb{C}$, then there exists a unique complete solution u
- Wan, Au 1994; Q. Li 2019: let U be holomorphic on $\mathbb{D}$, then there exists a unique complete solution u
- Q. Li 2019: let $U \not\equiv 0$ be holomorphic on $\mathbb{C}$, then there exists a unique complete solution u

there is no solution for $U \equiv 0$ on $\mathbb{C}$

Structure of the solution

if $U \neq 0$, then a solution is given by

$$e^u = 2^{1/3}|U|^{2/3}$$

this corresponds to a flat metric
however, even if $U \neq 0$ everywhere, this solution may be incomplete

the Blaschke metric is flat if and only if $U \equiv const \neq 0$ and $M = \mathbb{C}$
this case yields the cone $\mathbb{R}^3_+$

if $U \equiv 0$, then $\mathrm{K} \equiv -1$ and $e^u|dz|^2$ is the metric of hyperbolic space, this yields the cone $K = L_3$

generally, $e^u \sim |U|^{2/3}$ where $|U|$ is large and the metric is close to hyperbolic where $|U|$ is small

Main problem

holomorphic functions U on domains $M \subset \mathbb{C}$ (except $U \equiv 0$ on $\mathbb{C}$) up to bi-holomorphisms

$$\Leftrightarrow$$

regular convex cones $K \subset \mathbb{R}^3$ up to unimodular isomorphisms

for non-simply-connected Riemann surfaces, pass to the universal cover

interior points of M correspond one-to-one to interior rays of K
boundary points of M (including the infinitely far point) correspond to the boundary rays of K, but not one-to-one

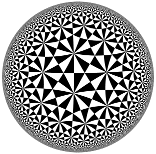
Problem: study this relationship in more detail

in particular: which cones correspond to $M = \mathbb{C}$

Known results

Dumas, Wolf 2015: polynomials U of degree k on $\mathbb{C}$ correspond to polyhedral cones K with $k+3$ extreme rays
$U = z^k$ corresponds to the cone over the regular $(k+3)$-gon

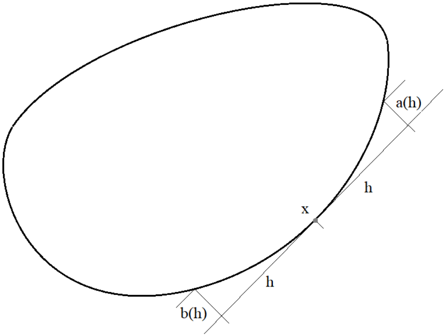
Wang 1997; Loftin 2001; Labourie 2007:
holomorphic functions on a compact Riemann surface of genus $g \geq 2$ correspond to cones K such that ∂K is C^1, but in general nowhere C^2

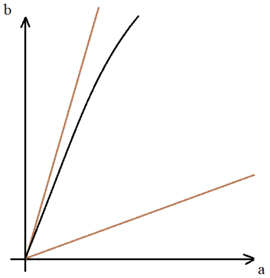


Benoist, Hulin 2014: the following are equivalent:

- $\sup_M \mathrm{K} < 0$
- $\mathbb{R}^3_+$ is not in the closure of the orbit of K under $SL(3,\mathbb{R})$
- M is conformally equivalent to $\mathbb{D}$ and U is bounded in the hyperbolic metric
- ∂K is C^1 and quasi-symmetric

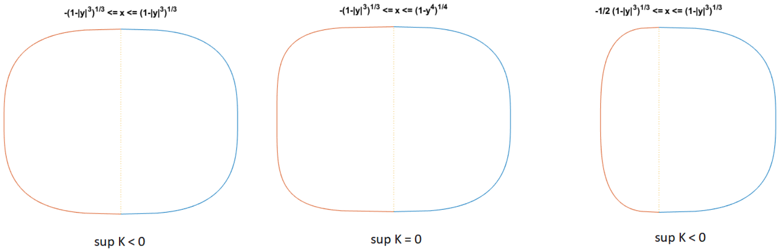
Quasi-symmetric convex sets





the curve $(a(h), b(h))$ has to be enclosed in a sector bounded away form the coordinate axes, for every point x of the boundary

Examples



the $||\cdot||_p$ unit ball is quasi-symmetric convex even if one half is linearly scaled

combining different p-norms leads to loss of quasi-symmetry

Local results

let M be a Riemann surface with a puncture z_0, and let the holomorphic function U on M have a pole of order k at z_0
let $\Pi : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}P^2$ be the natural projection

the boundary portion of the universal cover of M at z_0 corresponds in $\Pi[K]$ to

- a piece with finite Hilbert volume if $k \leq 2$ [Benoist, Hulin 2013]
- a piece of either a straight line segment or a corner if $k = 3$ [Loftin 2004, 2019]
- a polyhedral piece with $k - 3$ vertices if $k \geq 4$ [Nie 2018 preprint]

Representation of cones

$SL(3,\mathbb{R})$-orbits of sufficiently smooth regular convex cones can be represented by 3-rd order linear ODEs

$$\dddot{y} + 2\alpha\dot{y} + (\dot{\alpha} + \beta)y = 0$$

$\alpha(t), \beta(t)$ are 2π-periodic functions, $y : \mathbb{R} \to \mathbb{R}^3$

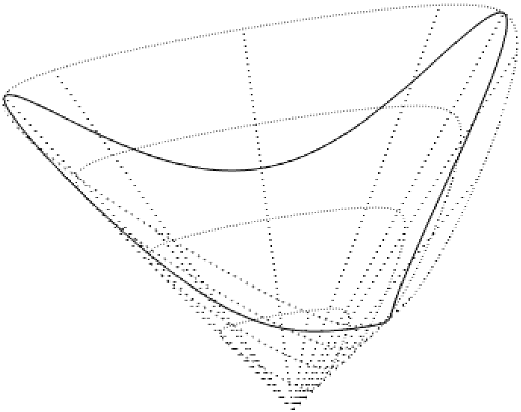
K is obtained as the convex conic hull of the solution curve

different initial values lead to isomorphic cones

reparametrizations of the time parameter:

- $\alpha(t) \equiv const \leq \frac{1}{2}$ can be achieved [H. 2020]
- $\beta(t)$ transforms as cubic differential [Halphen, Wilczynski, ...]
- splitting α/β corresponds to symmetric and skew-symmetric part of differential operator [Ovsienko, Tabachnikov]

ODE can in some cases be obtained from U



can be used also to represent smooth pieces of conic boundaries

Example: constant coefficients

vector-valued solution $y(t) = (e^{c_1 t}, e^{c_2 t}, e^{c_3 t})$, $c_1 > c_2 > c_3$

set $p = \frac{c_1 - c_3}{c_2 - c_3}$, $q = \frac{c_1 - c_3}{c_1 - c_2}$, $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \in (1, +\infty)$

the solution then satisfies $y_2 = y_1^{1/p} y_3^{1/q}$ and lies on the boundary of the power cone

$$K_p = \{(x, y, z) \mid |z| \leq x^{1/p} y^{1/q},\ x, y \geq 0\}$$

special case $p = 2$: c_i equidistant, $K_p \simeq L_3$

if $c_1 > c_2 = c_3$, then with $\tau = (c_1 - c_2)t$

$$y = (e^{c_1 t}, e^{c_2 t}, (c_1 - c_2) t e^{c_2 t}) = e^{c_2 t}(e^{\tau}, 1, \tau)$$

curve lies on the boundary of the exponential cone

$$K_{\exp} = \overline{\{(x, y, z) \mid y/z \geq e^{x/z},\ z > 0\}}$$

Semi-homogeneous cones

Definition

A regular convex cone $K \subset \mathbb{R}^3$ is called semi-homogeneous if it has a non-trivial continuous automorphism group.

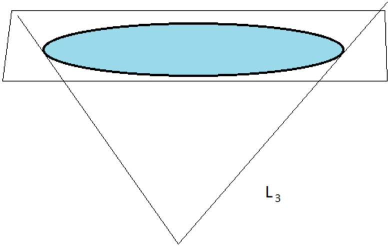
classification in [H. 2014]
U has to be constant on orbits, hence $U \equiv const$

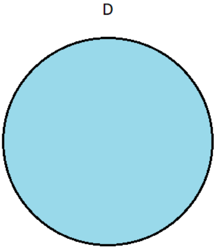
M	$U \equiv const$	K
$\mathbb{D}$	0	L_3
$\mathbb{C}$	1	$\mathbb{R}^3_+$
$\lvert Re\, z\rvert < \frac{l}{2}$	$e^{i\varphi}$	asymmetric power cone
$Re\, z > 0$	$e^{i\varphi}$, $\lvert\varphi\rvert < \frac{\pi}{2}$	half power cone
$Re\, z > 0$	$\pm i$	exponential cone
$Re\, z > 0$	$e^{i\varphi}$, $\lvert\varphi\rvert > \frac{\pi}{2}$	dual of half power cone

solution u given by Weierstrass $\wp$ functions [Z. Lin, E. Wang 2016]

representing ODE $\dddot{y} + 2\alpha\dot{y} + \beta y = 0$ has constant coefficients

Lorentz cone

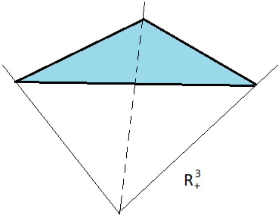


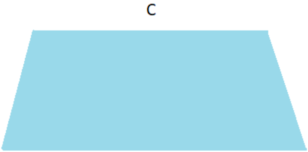


$M = \mathbb{D}$, $U \equiv 0$

$\mathbb{D}$ with the Klein model is isometric to the circular section

Orthant

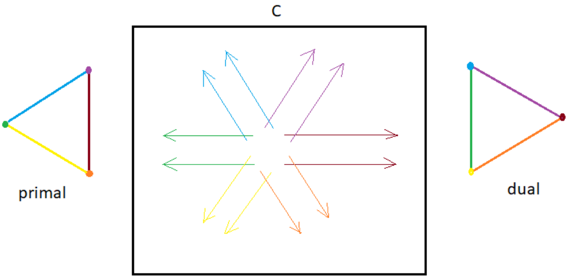




$M = \mathbb{C},\ U \equiv 1$

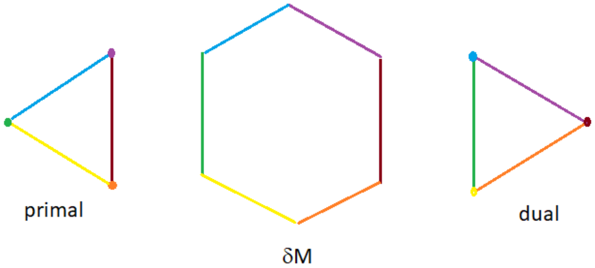
the surface $xyz = 1$ over the triangle is mapped to $\mathbb{C}$ by

$$(x, y, z) \mapsto (\log x, \log y, \log z)$$



only geodesics with angles $\frac{k\pi}{3}$ tend to interior points of the primal and dual edges

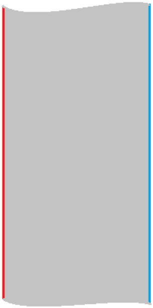
these critical directions divide the plane into sectors with similar convergence behaviour at ∞ (Stokes' phenomenon)

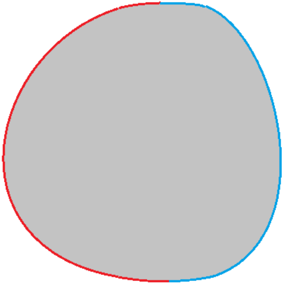


in $\mathbb{R}P^2 \times (\mathbb{R}P^2)^*$ the boundary ∂M is a hexagon

the differential $U\,dz^3$ increases its argument by π per vertex of the hexagon

Power cone

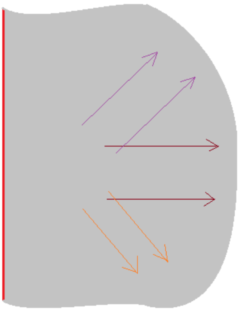


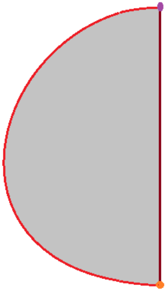


$$M = \{z \mid |Re\, z| < \tfrac{l}{2}\},\ U \equiv e^{i\varphi}$$
$$K = \{(x, y, z) \mid -c_1 x^{1/p} y^{1/q} \leq z \leq c_2 x^{1/p} y^{1/q},\ \ x, y \geq 0\}$$

- $U = \pm 1$: $p = 2$
- $U = \pm i$: symmetric power cone ($c_1 = c_2$)

Half-power cone

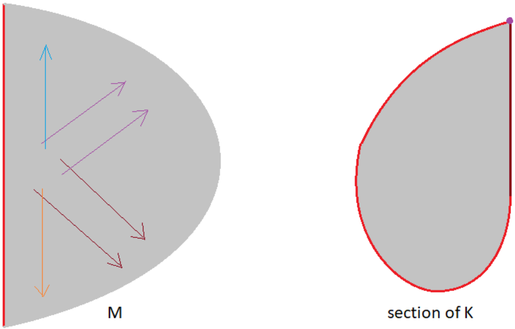




$$M = \{z \mid Re\, z > 0\},\ U \equiv e^{i\varphi},\ Re\, U > 0$$
$$K = \{(x, y, z) \mid -cx^{1/p}y^{1/q} \leq z \leq 0,\ x, y \geq 0\}$$

- $U = 1$: $p = 2$
- $U \to \pm i$: $p \to +\infty$

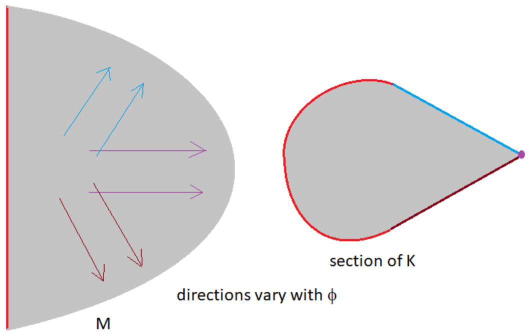
Exponential cone



$M = \{z \mid Re\, z > 0\}$, $U \equiv i$
$\varphi = \frac{\pi}{2}$: directions rotate by $-\frac{\pi}{6}$ to keep argument of $U\, dz^3$ constant

the second corner disappeared because the critical direction points along ∂M

Dual of half-power cone



$$M = \{z \mid Re\, z > 0\},\ U \equiv e^{i\varphi},\ Re\, U < 0$$
$$K = \{(x, y, z) \mid -cx^{1/p}y^{1/q} \leq z,\ \ x, y \geq 0\}$$

- $U = -1$: $p = 2$
- $U \to \pm i$: $p \to +\infty$

Self-associated cones

Definition

A regular convex cone $K \subset \mathbb{R}^3$ is called self-associated if it is linearly isomorphic to all its associated cones.

classification in [H. 2022]

$|U|$ has to be constant on orbits, phase changes

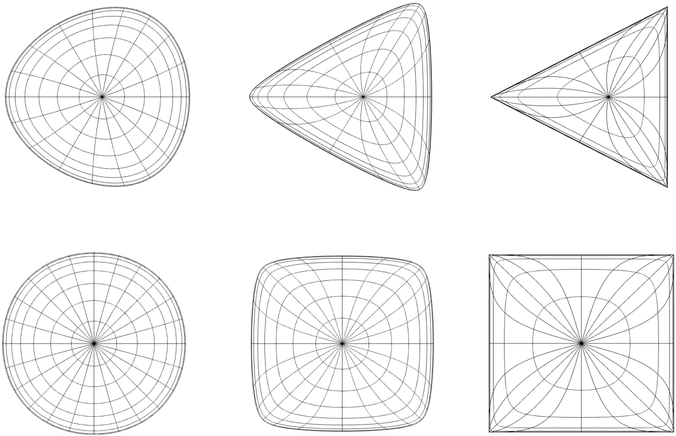
type	M	parameter	U
elliptic	$\|z\| < R$	$R \in (0, +\infty]$	z^k
parabolic	$Re\, z < b$	$b \in (-\infty, +\infty]$	e^z
hyperbolic	$a < Re\, z < b$	$-\infty < a < b \leq +\infty$	e^z

type defined by spectrum of generator of automorphism group of K

solution u given by degenerate Painlevé III (D_7) transcendents

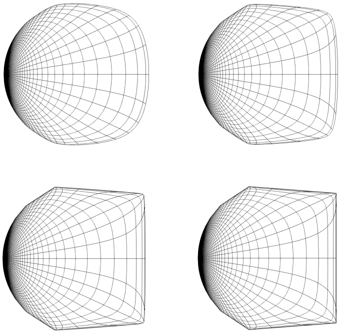
representing ODE $\dddot{y} + 2\alpha\dot{y} + \beta \cdot \sin t \cdot y = 0$, $\alpha, \beta = const$

Elliptic type: compact sections



$M = \{z \mid |z| < R\}$, $U = z^k$, polar grid in M
$k = 0, 1$; $R = 1, 2, 4$ ($R = +\infty$: polyhedral cones)

Parabolic type: compact sections



$M = \{z \mid Re\, z < b\}$, $U = e^z$, uniform grid in M
$b = -2, -1, 0, 1$ ($b = +\infty$ or $M = \mathbb{C}$: ∞-gonal cone)
the whole boundary ∂K corresponds to $Re\, z = b$

∞-gonal cone

$M = \mathbb{C}$, $U = e^z$

corresponding cone K is the convex conic hull of the set

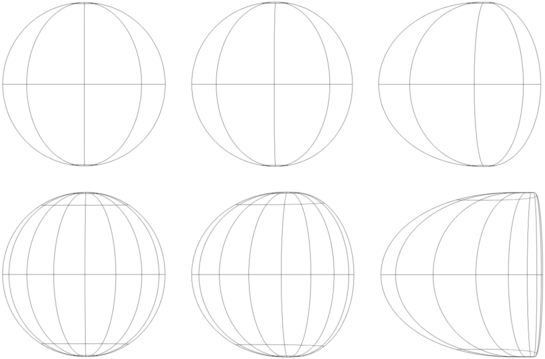
$$\{(1, n, n^2) \mid n \in \mathbb{Z}\}$$

compact section has infinitely many edges and vertices with a single accumulation point

- non-trivial automorphisms of K are isomorphic to the automorphisms of $\mathbb{Z}$ and form the infinite dihedral group D_∞
- K is self-dual

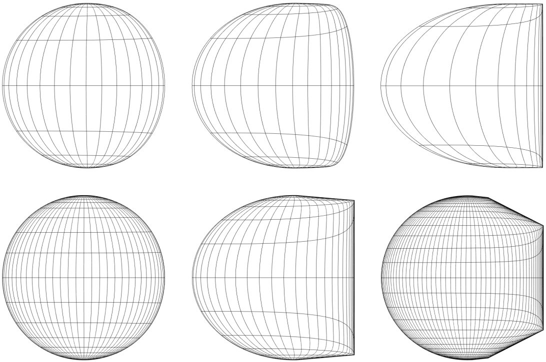
- lines $2k\pi i + \mathbb{R}$ tend to interior points of edges in K
- lines $(2k+1)\pi i + \mathbb{R}$ tend to interior points of edges in K^*

Hyperbolic type: compact sections



$M = \{z \mid a < Re\, z < b\}$, $U = e^z$, uniform grid in M
$(a, b) = (-3, 2); (-1, 0); (1, 2); (-4, -2); (-2, 0); (0, 2)$
∂K consists of two analytic pieces corresponding to $Re\, z = a, b$

Hyperbolic type: compact sections



$M = \{z \mid a < Re\, z < b\}$, $U = e^z$, uniform grid in M
$(a, b) = (-6, 2); (-4, 0); (-2, 2); (-12, -4); (-6, 2); (-14, 2)$
$b = +\infty$: polyhedral boundary piece

Cantor cone

let $M = \mathbb{C} \setminus \{-1, +1\}$, $U = \frac{cz(z^2-9)}{(z^2-1)^3}$, $c \in \mathbb{C}$

$C = 2Re\,(U\,dz^3)$ is invariant with respect to the symmetry group D_3 of the domain, generated by

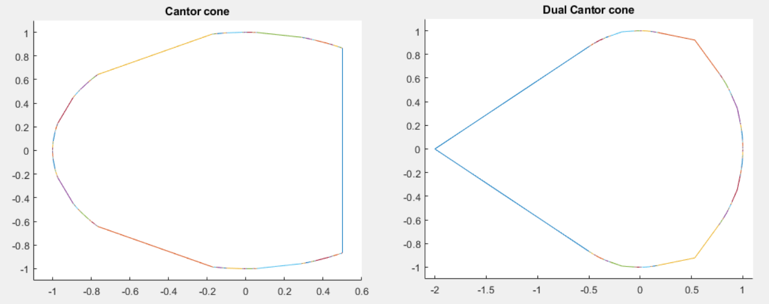
$$z \mapsto -z, \quad z \mapsto \frac{z-3}{z-1}$$

$|U|$ is invariant with respect to complex conjugation

at the punctures U has poles of order 3
each puncture corresponds to an edge or a vertex in ∂K
(dependent on the phase of c)

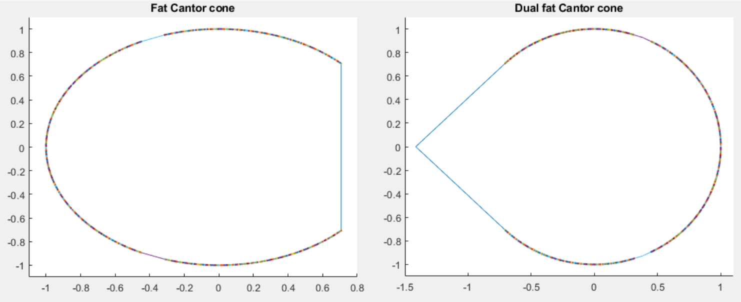
the union of edges and vertices is dense in ∂K
the symmetries determine the cone up to $SL(3,\mathbb{R})$ action and two parameters (corresponding to the choice of c)

universal cover of M is $\mathbb{D}$



compact affine section of Cantor cone: set of extreme rays has measure zero

- the cone can be computed by drawing an arbitrary edge and then acting by the symmetry group on it
- since the union of edges is dense, all other boundary rays appear in the limit
- extreme boundary rays are determined by the homotopy type of the path leading to the boundary point



compact affine section of fat Cantor cone: set of extreme rays has positive measure

Qualitative behaviour for "small" U

let $M = \mathbb{D}$, and let U be sufficiently regular at $\mathbb{T} = \partial\mathbb{D}$

Wang's equation reads

$$e^{2u}\Delta u = 2e^{3u} - 4|U|^2$$

let $u_0 = \log \frac{4}{(1-|z|^2)^2}$ correspond to the hyperbolic metric
set $\upsilon = u - u_0$, υ bounded [Benoist, Hulin 2014]

we propose the following approach:
Wang's equation can be written

$$-e^{-u_0}\Delta\upsilon + 2\upsilon = 4e^{-2\upsilon-3u_0}|U|^2 - 2(e^{\upsilon} - \upsilon - 1) =: f$$

operator on left-hand side has an explicit Green's function

$$f = 4e^{-2v-3u_0}|U|^2 - 2(e^v - v - 1)$$

$$v(z) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^1 f(z_0)\left(-\frac{((1+r^2)(1+r_0^2) - 4rr_0\cos(\varphi-\varphi_0))\log\frac{r^2+r_0^2-2rr_0\cos(\varphi-\varphi_0)}{1+r^2r_0^2-2rr_0\cos(\varphi-\varphi_0)}}{(1-r^2)(1-r_0^2)} - 2\right)\frac{4r_0}{(1-r_0^2)^2}\,dr_0 d\varphi_0$$

$$v(z) = \int_{\mathbb{D}} f(z_0)k(d(z-z_0))\,|dz_0|$$

for $r \to 1$ we get as the main term

$$v(z) = \frac{1}{6\pi}\int_0^{2\pi}\int_0^1 f(z_0)\frac{4r_0(1-r)^2}{(r_0^2 - 2r_0\cos(\varphi-\varphi_0)+1)^2}\,dr_0 d\varphi_0$$
$$= c(\varphi)(1-r)^2$$

v bounded $\Rightarrow$ f bounded $\Rightarrow$ $v \lesssim (1-r)^2 \Rightarrow$
$f \lesssim (1-r)^6|U|^2 + (1-r)^4$

by considering the asymptotics of the frame equations we get that along $\mathbb{T} \simeq \partial M$

$$\beta dt^3 = Re(U\, dz^3)$$

in particular, the smoothness of the cone boundary depends locally on the smoothness of U on $\mathbb{T}$

the coefficient α depends non-locally on U

Open problems

- characterize those cones which correspond to $M = \mathbb{C}$ (this would yield also a new description of entire functions)
- detail the connection between smoothness of U and ∂K
- connection to loop group methods

generalization to $n > 3$?

Thank you!