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The minimal zero supports are given by (1, 2), (1, 3, 4), (1, 3, 5), (1, 4, 6), (2, 5, 6), (3, 5, 6), (4 , 5, 6)

$$A = \begin{pmatrix} 1 & -1 & -\cos(\phi_2) & -\cos(\phi_1) & \cos(\phi_2+\phi_3) & \cos(\phi_1+\phi_4) \\ -1 & 1 & b_1 & b_2 & \cos(\phi_5+\phi_6) & -\cos(\phi_6) \\ -\cos(\phi_2) & b_1 & 1 & \cos(\phi_1+\phi_2) & -\cos(\phi_3) & \cos(\phi_3+\phi_5) \\ -\cos(\phi_1) & b_2 & \cos(\phi_1+\phi_2) & 1 & \cos(\phi_4+\phi_5) & -\cos(\phi_4) \\ \cos(\phi_2+\phi_3) & \cos(\phi_5+\phi_6) & -\cos(\phi_3) & \cos(\phi_4+\phi_5) & 1 & -\cos(\phi_5) \\ \cos(\phi_1+\phi_4) & -\cos(\phi_6) & \cos(\phi_3+\phi_5) & -\cos(\phi_4) & -\cos(\phi_5) & 1 \end{pmatrix}$$

The zeros are given by the columns:

$$(u_1, u_2, u_3, u_4, u_5, u_6, u_7) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \sin(\phi_1+\phi_2) \\ 0 \\ \sin(\phi_1) \\ \sin(\phi_2) \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \sin(\phi_3) \\ 0 \\ \sin(\phi_2+\phi_3) \\ 0 \\ \sin(\phi_2) \\ 0 \end{pmatrix} \begin{pmatrix} \sin(\phi_4) \\ 0 \\ 0 \\ \sin(\phi_1+\phi_4) \\ 0 \\ \sin(\phi_1) \end{pmatrix} \begin{pmatrix} 0 \\ \sin(\phi_5) \\ 0 \\ 0 \\ \sin(\phi_6) \\ \sin(\phi_5+\phi_6) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \sin(\phi_5) \\ 0 \\ \sin(\phi_3+\phi_5) \\ \sin(\phi_3) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sin(\phi_5) \\ \sin(\phi_4) \\ \sin(\phi_4+\phi \end{pmatrix}$$

1. $$Au_1 = \begin{pmatrix} 0 \\ 0 \\ b_1 - \cos(\phi_2) \\ b_2 - \cos(\phi_1) \\ \cos(\phi_2+\phi_3) + \cos(\phi_5+\phi_6) \\ \cos(\phi_1+\phi_4) - \cos(\phi_6) \end{pmatrix} => \begin{cases} b_1 \geq \cos(\phi_2) \\ b_2 \geq \cos(\phi_1) \\ \cos(\phi_2+\phi_3) + \cos(\phi_5+\phi_6) \geq 0 \\ \cos(\phi_1+\phi_4) - \cos(\phi_6) \geq 0 \end{cases}$$

2. $$Au_2 = \begin{pmatrix} 0 \\ b_1 * \sin(\phi_1) + b_2 * \sin(\phi_2) - \sin(\phi_1+\phi_2) \\ 0 \\ 0 \\ \sin(\phi_2)(\cos(\phi_1+\phi_2+\phi_3) + \cos(\phi_4+\phi_5)) \\ \sin(\phi_1)(\cos(\phi_1+\phi_2+\phi_4) + \cos(\phi_3+\phi_5)) \end{pmatrix} => \begin{cases} b_1 * \sin(\phi_1) + b_2 * \sin(\phi_2) - \sin(\phi_1+\phi_2) \geq 0 \\ \cos(\phi_1+\phi_2+\phi_3) + \cos(\phi_4+\phi_5) \geq 0 \\ \cos(\phi_1+\phi_2+\phi_4) + \cos(\phi_3+\phi_5) \geq 0 \end{cases}$$

3. $$Au_3 = \begin{pmatrix} 0 \\ -\sin(\phi_3) + b_1 * \sin(\phi_2+\phi_3) + \sin(\phi_2)\cos(\phi_5+\phi_6) \\ 0 \\ \sin(\phi_2)(\cos(\phi_1+\phi_2+\phi_3) + \cos(\phi_4+\phi_5)) \\ 0 \\ \sin(\phi_3)(\cos(\phi_2+\phi_3+\phi_5) + \cos(\phi_1+\phi_4) \end{pmatrix} =>$$
$$\begin{cases} -\sin(\phi_3) + b_1 * \sin(\phi_2+\phi_3) + \sin(\phi_2)\cos(\phi_5+\phi_6) \geq 0 \\ \cos(\phi_1+\phi_2+\phi_3) + \cos(\phi_4+\phi_5) \geq 0 \\ \cos(\phi_2+\phi_3+\phi_5) + \cos(\phi_1+\phi_4) \geq 0 \end{cases}$$

4. $$Au_4 = \begin{pmatrix} 0 \\ b_2 * \sin(\phi_1+\phi_4) - \sin(\phi_4) - \sin(\phi_1)\cos(\phi_6) \\ \sin(\phi_1)(\cos(\phi_1+\phi_2+\phi_4) + \cos(\phi_3+\phi_5)) \\ 0 \\ \sin(\phi_4)(\cos(\phi_2+\phi_3) + \cos(\phi_1+\phi_4+\phi_5)) \\ 0 \end{pmatrix} =>$$
$$\begin{cases} b_2 * \sin(\phi_1+\phi_4) - \sin(\phi_4) - \sin(\phi_1)\cos(\phi_6) \geq 0 \\ \cos(\phi_1+\phi_2+\phi_4) + \cos(\phi_3+\phi_5) \geq 0 \\ \cos(\phi_2+\phi_3) + \cos(\phi_1+\phi_4+\phi_5) \geq 0 \end{cases}$$

5. $$Au_5 = \begin{pmatrix} -\sin(\phi_5) + \sin(\phi_6)\cos(\phi_2+\phi_3) - \sin(\phi_5+\phi_6)\cos(\phi_1+\phi_4) \\ 0 \\ b_1 * \sin(\phi_5) + \cos(\phi_3+\phi_5+\phi_6)\sin(\phi_5) \\ b_2 * \sin(\phi_5) - \cos(\phi_4-\phi_6)\sin(\phi_5) \\ 0 \\ 0 \end{pmatrix} =>$$

$$\begin{cases} -sin(\phi_5)+sin(\phi_6)cos(\phi_2+\phi_3)-sin(\phi_5+\phi_6)cos(\phi_1+\phi_4)\geq 0\\ b_1\geq -cos(\phi_3+\phi_5+\phi_6)\\ b_2\geq cos(\phi_4-\phi_6)\end{cases}$$

6. $Au_6=\begin{pmatrix} sin(\phi_3)cos(\phi_1+\phi_4)+sin(\phi_3)cos(\phi_2+\phi_3+\phi_5)\\ b_1 sin(\phi_5)+sin(\phi_5)cos(\phi_3+\phi_5+\phi_6)\\ 0\\ sin(\phi_5)cos(\phi_3+\phi_4+\phi_5)+sin(\phi_5)cos(\phi_1+\phi_2)\\ 0\\ 0\end{pmatrix} =>$

$$\begin{cases} cos(\phi_1+\phi_4)+cos(\phi_2+\phi_3+\phi_5)\geq 0\\ b_1\geq -cos(\phi_3+\phi_5+\phi_6)\\ cos(\phi_3+\phi_4+\phi_5)+cos(\phi_1+\phi_2)\geq 0\end{cases}$$

7. $Au_7=\begin{pmatrix} sin(\phi_4)cos(\phi_2+\phi_3)+sin(\phi_4)cos(\phi_1+\phi_4+\phi_5)\\ b_2 sin(\phi_5)-sin(\phi_5)cos(\phi_4-\phi_6)\\ 0\\ sin(\phi_5)cos(\phi_3+\phi_4+\phi_5)+sin(\phi_5)cos(\phi_1+\phi_2)\\ 0\\ 0\end{pmatrix} =>$

$$\begin{cases} cos(\phi_2+\phi_3)+cos(\phi_1+\phi_4+\phi_5)\geq 0\\ b_2\geq cos(\phi_4-\phi_6)\\ cos(\phi_3+\phi_4+\phi_5)+cos(\phi_1+\phi_2)\geq 0\end{cases}$$

From inequalities on ϕ_i we get:

$$\begin{cases} \phi_1+\phi_2+\phi_3+\phi_4+\phi_5\leq\pi\\ \phi_2+\phi_3+\phi_5+\phi_6\leq\pi\\ \phi_6\geq\phi_1+\phi_4\end{cases} => \begin{cases} b_1=cos(\phi_2)\\ b_2=cos(\phi_1)\end{cases}$$

All other inequalities are fulfilled.

Parametrization

We arrive at the parametrization

$$A=\begin{pmatrix} 1 & -1 & -cos(\phi_2) & -cos(\phi_1) & cos(\phi_2+\phi_3) & cos(\phi_1+\phi_4)\\ -1 & 1 & cos(\phi_2) & cos(\phi_1) & cos(\phi_5+\phi_6) & -cos(\phi_6)\\ -cos(\phi_2) & cos(\phi_2) & 1 & cos(\phi_1+\phi_2) & -cos(\phi_3) & cos(\phi_3+\phi_5)\\ -cos(\phi_1) & cos(\phi_1) & cos(\phi_1+\phi_2) & 1 & cos(\phi_4+\phi_5) & -cos(\phi_4)\\ cos(\phi_2+\phi_3) & cos(\phi_5+\phi_6) & -cos(\phi_3) & cos(\phi_4+\phi_5) & 1 & -cos(\phi_5)\\ cos(\phi_1+\phi_4) & -cos(\phi_6) & cos(\phi_3+\phi_5) & -cos(\phi_4) & -cos(\phi_5) & 1\end{pmatrix}$$

with conditions on ϕ_i : $\phi_i\in(0,\pi),\phi_1+\phi_2+\phi_3+\phi_4+\phi_5\leq\pi,\phi_2+\phi_3+\phi_5+\phi_6\leq\pi,\phi_6\geq\phi_1+\phi_4$

Copositivity/Absence of other minimal zeros:

Sets $(2,3,4),(2,3,5),(1,4,5),(2,4,5),(3,4,5),(2,3,4,5),(1,3,6),(2,3,6),(2,4,6),(3,4,6)$, $(2,3,4,6),(1,5,6)$ have to be considered.

1. $I=(2,3,4,5):u=e_3+e_5:cos(\phi_2)\geq -cos(\phi_5+\phi_6) => \phi_2\leq\pi-\phi_5-\phi_6$ -true
   $cos(\phi_1+\phi_2)+cos(\phi_4+\phi_5)\geq 0,\phi_1+\phi_2+\phi_4+\phi_5\leq\pi$
2. $I=(2,3,4,6):u=e_4+e_6:b_2\geq cos(\phi_4-\phi_6),cos(\phi_4-\phi_6)\geq cos(\phi_6) => |\phi_4-\phi_6|\leq\phi_6$ -true
   $cos(\phi_1+\phi_2)\geq -cos(\phi_3+\phi_5) => \phi_1+\phi_2\leq\phi_6+\phi_2\leq\pi-\phi_3-\phi_5$-true
3. $I=(1,3,6):\phi_2+\pi-\phi_1-\phi_4+\pi-\phi_3-\phi_5\geq\pi,\pi\geq\phi_1-\phi_2+\phi_3+\phi_4+\phi_5$ performs strictly
4. $I=(1,5,6):\phi_5+\pi-\phi_1-\phi_4+\pi-\phi_3-\phi_2\geq\pi,\pi\geq\phi_1+\phi_2+\phi_3+\phi_4-\phi_5$ performs strictly
5. $I=(1,4,5):\pi-\phi_4-\phi_5+\phi_1+\pi-\phi_2-\phi_3\geq\pi => \pi\geq -\phi_1+\phi_2+\phi_3+\phi_4+\phi_5$-performs strictly

6. $I = (2,3,4) : b_2 \geq cos(\phi_1) => \pi - \phi_2 + \pi - \phi_1 + \pi - \phi_1 - \phi_2 \geq \pi => \pi \geq \phi_1 + \phi_2$-performs strictly
7. $I = (2,4,5) : b_2 \geq cos(\phi_4 - \phi_6) => \pi - |\phi_4 - \phi_6| + \pi - \phi_4 - \phi_5 + \pi - \phi_5 - \phi_6 \geq \pi$ Considering cases $\phi_4 \geq \phi_6$ and $\phi_6 \geq \phi_4$ we get $\pi \geq \phi_4 + \phi_5$ and $\pi \geq \phi_6 + \phi_5$ correspondingly, that perform strictly.
8. $I = (2,3,6) : \pi - \phi_2 + \phi_6 + \pi - \phi_3 - \phi_5 \geq \pi => \pi \geq \phi_2 + \phi_3 + \phi_5 - \phi_6$ performs strictly

That proves copositivity.
All angle inequalities are satisfied strictly and the vectors for first 2 cases is not positive, hence there are no additional minimal zeros.

Extremality

If $\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 = \pi => Au_2 = 0$ and matrix A is positive semi-definite and as result non-exceptional.

The presence of the zeros $u_2, \ldots, u_6$ implies that the submatrix $B_{\{1,3,4,5,6\}}$ is proportional to the T-matrix $A_{\{1,3,4,5,6\}}$. The presence of the zero u_1 implies that the elements b_{11} and b_{22} are equal, and hence all diagonal elements b_{ii} are equal. We may then scale B such that it has unit diagonal. But by construction A was an extremal element of the polytope defined as intersection of the polyhedral cone defined by the first-order conditions with the unit diagonal affine subspace. Since B satisfies the same equality conditions as A, it lies in the same face of the polytope as A and must then coincide with A. Hence A is extremal.

Result

There are extreme copositive matrices with such set of zeros on the conditions: $\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 < \pi, \phi_2 + \phi_3 + \phi_5 + \phi_6 \leq \pi, \phi_6 \geq \phi_1 + \phi_4$

References

[1] Peter J.C. Dickinson. *A new certificate for copositivity. Optimization Online.*

[2] Peter J.C. Dickinson and Roland Hildebrand. *Considering copositivity locally.* J.Math. Anal.Appl., 437(2):1184–1195, 2016

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