# Case 13

Andrey Afonin, Roland Hildebrand

February 25, 2019

## Case 13

The minimal zero supports are given by (1, 2), (1, 3, 4), (1, 3, 5), (2, 4, 6), (3, 4, 6), (2, 5, 6). We have the symmetry $(123456) \mapsto (216453)$.

$$A = \begin{pmatrix} 1 & -1 & -\cos(\phi_2) & \cos(\phi_1+\phi_2) & \cos(\phi_2+\phi_3) & b_2 \\ -1 & 1 & b_1 & \cos(\phi_4+\phi_5) & \cos(\phi_5+\phi_6) & -\cos(\phi_5) \\ -\cos(\phi_2) & b_1 & 1 & -\cos(\phi_1) & -\cos(\phi_3) & \cos(\phi_1+\phi_4) \\ \cos(\phi_1+\phi_2) & \cos(\phi_4+\phi_5) & -\cos(\phi_1) & 1 & b_3 & -\cos(\phi_4) \\ \cos(\phi_2+\phi_3) & \cos(\phi_5+\phi_6) & -\cos(\phi_3) & b_3 & 1 & -\cos(\phi_6) \\ b_2 & -\cos(\phi_5) & \cos(\phi_1+\phi_4) & -\cos(\phi_4) & -\cos(\phi_6) & 1 \end{pmatrix}$$

The zeros are given by the columns:

$$(u_1,u_2,u_3,u_4,u_5,u_6) = \begin{pmatrix}1\\1\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}\sin(\phi_1)\\0\\\sin(\phi_1+\phi_2)\\\sin(\phi_2)\\0\\0\end{pmatrix}\begin{pmatrix}\sin(\phi_3)\\0\\\sin(\phi_3+\phi_2)\\0\\\sin(\phi_2)\\0\end{pmatrix}\begin{pmatrix}0\\\sin(\phi_4)\\0\\\sin(\phi_5)\\0\\\sin(\phi_4+\phi_5)\end{pmatrix}\begin{pmatrix}0\\0\\\sin(\phi_4)\\\sin(\phi_1+\phi_4)\\0\\\sin(\phi_1)\end{pmatrix}\begin{pmatrix}0\\\sin(\phi_6)\\0\\0\\\sin(\phi_5)\\\sin(\phi_5+\phi_6)\end{pmatrix}$$

1. $$Au_1 = \begin{pmatrix} 0 \\ 0 \\ b_1-\cos(\phi_2) \\ \cos(\phi_1+\phi_2)+\cos(\phi_4+\phi_5) \\ \cos(\phi_2+\phi_3)+\cos(\phi_5+\phi_6) \\ b_2-\cos(\phi_5) \end{pmatrix} \Rightarrow \begin{cases} b_1-\cos(\phi_2)\geq 0 \\ \cos(\phi_1+\phi_2)+\cos(\phi_4+\phi_5)\geq 0 \\ \cos(\phi_2+\phi_3)+\cos(\phi_5+\phi_6)\geq 0 \\ b_2-\cos(\phi_5)\geq 0 \end{cases}$$

2. $$Au_2 = \begin{pmatrix} 0 \\ -\sin(\phi_1)+b_1\sin(\phi_1+\phi_2)+\sin(\phi_2)\cos(\phi_4+\phi_5) \\ 0 \\ 0 \\ b_3\sin(\phi_2)-\sin(\phi_2)\cos(\phi_1-\phi_3) \\ b_2\sin(\phi_1)+\sin(\phi_1)\cos(\phi_1+\phi_2+\phi_4) \end{pmatrix} \Rightarrow \begin{cases} -\sin(\phi_1)+b_1\sin(\phi_1+\phi_2)+\sin(\phi_2)\cos(\phi_4+\phi_5) \geq \\ b_3-\cos(\phi_1-\phi_3)\geq 0 \\ b_2+\cos(\phi_1+\phi_2+\phi_4)\geq 0 \end{cases}$$

3. $$Au_3 = \begin{pmatrix} 0 \\ -\sin(\phi_3)+b_1\sin(\phi_2+\phi_3)+\sin(\phi_2)\cos(\phi_5+\phi_6) \\ 0 \\ b_3\sin(\phi_2)-\sin(\phi_2)\cos(\phi_1-\phi_3) \\ 0 \\ b_2\sin(\phi_3)+\sin(\phi_1+\phi_3)\cos(\phi_1+\phi_4)-\sin(\phi_2)\cos(\phi_6) \end{pmatrix} \Rightarrow$$
$$\begin{cases} -\sin(\phi_3)+b_1\sin(\phi_2+\phi_3)+\sin(\phi_2)\cos(\phi_5+\phi_6)\geq 0 \\ b_3-\cos(\phi_1-\phi_3)\geq 0 \\ b_2\sin(\phi_3)+\sin(\phi_1+\phi_3)\cos(\phi_1+\phi_4)-\sin(\phi_2)\cos(\phi_6)\geq 0 \end{cases}$$

4. $$Au_4 = \begin{pmatrix} -\sin(\phi_4)+\sin(\phi_5)\cos(\phi_1+\phi_2)+b_2\sin(\phi_4+\phi_5) \\ 0 \\ b_1\sin(\phi_4)+\sin(\phi_4)\cos(\phi_1+\phi_4+\phi_5) \\ 0 \\ b_3\sin(\phi_5)-\sin(\phi_5)\cos(\phi_4-\phi_6) \\ 0 \end{pmatrix} \Rightarrow$$
$$\begin{cases} -\sin(\phi_4)+\sin(\phi_5)\cos(\phi_1+\phi_2)+b_2\sin(\phi_4+\phi_5)\geq 0 \\ b_1+\cos(\phi_1+\phi_4+\phi_5)\geq 0 \\ b_3-\cos(\phi_4-\phi_6)\geq 0 \end{cases}$$

5. $Au_5 = \begin{pmatrix} b_2\sin(\phi_1)+\sin(\phi_1)\cos(\phi_1+\phi_2+\phi_4) \\ b_1\sin(\phi_4)+\sin(\phi_4)\cos(\phi_1+\phi_4+\phi_5) \\ 0 \\ 0 \\ b_3\sin(\phi_1+\phi_4)-\sin(\phi_4)\cos(\phi_3)-\sin(\phi_1)\cos(\phi_6) \\ 0 \end{pmatrix} \Rightarrow$

$$\begin{cases} b_2+\cos(\phi_1+\phi_2+\phi_4)\geq 0 \\ b_1+\cos(\phi_1+\phi_4+\phi_5)\geq 0 \\ b_3\sin(\phi_1+\phi_4)-\sin(\phi_4)\cos(\phi_3)-\sin(\phi_1)\cos(\phi_6)\geq 0 \end{cases}$$

6. $Au_6 = \begin{pmatrix} \sin(\phi_5)\cos(\phi_2+\phi_3)-\sin(\phi_6)+b_2\sin(\phi_5+\phi_6) \\ 0 \\ -\sin(\phi_5)\cos(\phi_3)+b_1\sin(\phi_6)+\sin(\phi_5+\phi_6)\cos(\phi_1+\phi_4) \\ b_3\sin(\phi_5)-\sin(\phi_5)\cos(\phi_4-\phi_6) \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$$\begin{cases} \sin(\phi_5)\cos(\phi_2+\phi_3)-\sin(\phi_6)+b_2\sin(\phi_5+\phi_6)\geq 0 \\ -\sin(\phi_5)\cos(\phi_3)+b_1\sin(\phi_6)+\sin(\phi_5+\phi_6)\cos(\phi_1+\phi_4)\geq 0 \\ b_3-\cos(\phi_4-\phi_6)\geq 0 \end{cases}$$

From inequalities on $\phi$ we get $\begin{cases} \phi_1+\phi_2+\phi_4+\phi_5\leq\pi \\ \phi_2+\phi_3+\phi_5+\phi_6\leq\pi \end{cases}$

If $b_1\geq\cos(\phi_2)$ all conditions are fulfilled on $b_1$, except 6.2
If $b_2\geq\cos(\phi_5)$ all conditions are fulfilled on $b_1$, except 3.3

**Copositivity:** Sets $(2,3,4),(2,3,5),(1,4,5),(2,4,5),(3,4,5),(2,3,4,5),(1,3,6),(2,3,6),(1,4,6),(1,5,6)$, $(3,5,6),(4,5,6),(1,4,5,6)$ have to be considered.

1. $I=(2,3,4): u=e_3+e_4, b_1\geq\cos(\phi_2), \cos(\phi_2)+\cos(\phi_4+\phi_5)\geq 0 => \phi_2+\phi_4+\phi_5\leq\pi$ - True
2. $I=(2,3,5): u=e_3+e_5, b_1\geq\cos(\phi_2), \cos(\phi_2)+\cos(\phi_5+\phi_6)\geq 0 => \phi_2+\phi_6+\phi_5\leq\pi$ - True
3. $I=(1,4,5): u=e_1+e_5(\phi_3\geq\phi_1), \cos(\phi_1+\phi_2)+\cos(\phi_1-\phi_3)\geq 0 => \phi_2+\phi_3\leq\pi$
   $u=e_1+e_4(\phi_3<\phi_1)$
4. $I=(2,4,5): u=e_2+e_4(\phi_4\geq\phi_6), \cos(\phi_5+\phi_6)+\cos(\phi_4-\phi_6)\geq 0 => \phi_5+\phi_4\leq\pi$
   $u=e_2+e_5(\phi_4<\phi_6)$
5. $I=(2,3,4,5),(3,4,5): u=e_3+e_4(\phi_3\geq\phi_1), -\cos(\phi_3)+\cos(\phi_1-\phi_3)\geq 0 => -\phi_1\leq 0$
   $u=e_3+e_5(\phi_1>\phi_3), -\cos(\phi_1)+\cos(\phi_1-\phi_3)\geq 0 => -\phi_3\leq 0$
6. $I=(1,3,6): u=e_1+e_3, \cos(\phi_5)+\cos(\phi_1+\phi_4)\geq 0 => \phi_1+\phi_4+\phi_5\leq\pi$ - True
7. $I=(2,3,6): u=e_2+e_6, \cos(\phi_2)+\cos(\phi_1+\phi_4)\geq 0 => \phi_1+\phi_4+\phi_2\leq\pi$ - True
8. $I=(1,4,6): u=e_4+e_6, \cos(\phi_1+\phi_2)-\cos(\phi_1+\phi_2+\phi_4)\geq 0 => \phi_4\geq 0$ - True
9. $I=(1,5,6): u=e_5+e_6, b_2\geq\cos(\phi_5), \cos(\phi_2+\phi_3)+\cos(\phi_5)\geq 0 => \phi_2+\phi_3+\phi_5\leq\pi$ - True
10. $I=(3,5,6): \phi_3-\phi_1-\phi_4+\phi_6\geq 0$
11. $I=(1,4,5,6),(4,5,6): u=e_5+e_6(\phi_4\geq\phi_6), \cos(\phi_4-\phi_6)\geq\cos(\phi_4) => -\phi_6\leq 0$
    $u=e_4+e_6(\phi_6>\phi_4), \cos(\phi_4-\phi_6)\geq\cos(\phi_6) => -\phi_4\leq 0$- True

That proves copositivity with new condition on $\phi_i: \phi_3-\phi_1-\phi_4+\phi_6>0$. The inequality is strict in order to prevent a minimal zero with support $\{3,5,6\}$ from appearing. It also follows that $\phi_1+\phi_2+\phi_4+\phi_5<\pi$ is fulfilled automatically.
Consider this condition and get $b_1=\cos(\phi_2), b_2=\cos(\phi_5)$

The condition $\phi_3 - \phi_1 - \phi_4 + \phi_6 > 0$ implies that

$$\cos(\phi_1 - \phi_3) \le \frac{\sin\phi_4\cos\phi_3 + \sin\phi_1\cos\phi_6}{\sin(\phi_1+\phi_4)} \quad \Leftrightarrow \quad \phi_3 \ge \phi_1 + \phi_4 + \phi_6,$$

$$\cos(\phi_4 - \phi_6) \le \frac{\sin\phi_4\cos\phi_3 + \sin\phi_1\cos\phi_6}{\sin(\phi_1+\phi_4)} \quad \Leftrightarrow \quad \phi_6 \ge \phi_1 + \phi_4 + \phi_3,$$

$$\cos(\phi_1 - \phi_3) \ge \cos(\phi_4 - \phi_6) \quad \Leftrightarrow \quad \phi_1 + \phi_6 \ge \phi_3 + \phi_4.$$

It follows that $b_3 \ne \frac{\sin\phi_4\cos\phi_3+\sin\phi_1\cos\phi_6}{\sin(\phi_1+\phi_4)}$, since the first two conditions together contradict $\phi_i > 0$. Hence $b_3 = \cos(\phi_1 - \phi_3)$ if $\phi_1 + \phi_6 \ge \phi_3 + \phi_4$ and $b_3 = \cos(\phi_4 - \phi_6)$ if $\phi_1 + \phi_6 \le \phi_3 + \phi_4$.

## Parametrization

We obtain

$$A = \begin{pmatrix} 1 & -1 & -\cos(\phi_2) & \cos(\phi_1+\phi_2) & \cos(\phi_2+\phi_3) & \cos\phi_5 \\ -1 & 1 & \cos\phi_2 & \cos(\phi_4+\phi_5) & \cos(\phi_5+\phi_6) & -\cos(\phi_5) \\ -\cos(\phi_2) & \cos\phi_2 & 1 & -\cos(\phi_1) & -\cos(\phi_3) & \cos(\phi_1+\phi_4) \\ \cos(\phi_1+\phi_2) & \cos(\phi_4+\phi_5) & -\cos(\phi_1) & 1 & b_3 & -\cos(\phi_4) \\ \cos(\phi_2+\phi_3) & \cos(\phi_5+\phi_6) & -\cos(\phi_3) & b_3 & 1 & -\cos(\phi_6) \\ \cos\phi_5 & -\cos(\phi_5) & \cos(\phi_1+\phi_4) & -\cos(\phi_4) & -\cos(\phi_6) & 1 \end{pmatrix}, \quad (1)$$

$b_3 = \max\{\cos(\phi_1-\phi_3), \cos(\phi_4-\phi_6)\}$, $\phi_i > 0$, $\phi_3+\phi_6 > \phi_1+\phi_4$, $\phi_2+\phi_3+\phi_5+\phi_6 \le \pi$.

## Extremality

$X = FPF^T$

$$X = \begin{pmatrix} b_{11} & b_{13} & b_{14} & b_{15} & * \\ b_{31} & b_{33} & b_{34} & b_{35} & b_{36} \\ b_{41} & b_{43} & b_{44} & * & b_{46} \\ b_{51} & b_{53} & * & b_{55} & * \\ * & b_{63} & b_{64} & * & b_{66} \end{pmatrix}, F = \begin{pmatrix} f_1 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{pmatrix}, \begin{cases} u_{21}f_1 + u_{23}f_3 + u_{24}f_4 = 0 \\ u_{31}f_1 + u_{33}f_3 + u_{35}f_5 = 0 \\ u_{53}f_3 + u_{54}f_4 + u_{56}f_6 = 0 \end{cases}, F = \begin{pmatrix} \sin(\phi_1+\phi_2) & 0 \\ -\sin(\phi_1) & -\sin(\phi_2) \\ 0 & \sin(\phi_1+\phi_2) \\ \sin(\phi_1-\phi_3) & \sin(\phi_2+\phi_3) \\ \sin(\phi_4) & -\sin(\phi_1+\phi_2+\phi_4) \end{pmatrix}$$

$Y = GQG^T$

$$Y = \begin{pmatrix} b_{11} & b_{12} & * & * & b_{16} \\ b_{21} & b_{22} & b_{24} & b_{25} & b_{26} \\ * & b_{42} & b_{44} & * & b_{46} \\ * & b_{52} & * & b_{55} & b_{56} \\ b_{61} & b_{62} & b_{64} & b_{65} & b_{66} \end{pmatrix}, G = \begin{pmatrix} g_1 \\ g_2 \\ g_4 \\ g_5 \\ g_6 \end{pmatrix}, \begin{cases} u_{42}g_2 + u_{44}g_4 + u_{46}g_6 = 0 \\ u_{62}g_2 + u_{65}g_5 + u_{66}g_6 = 0 \\ g_2 = -g_1 \\ (Bu_1)_i = 0 : b_{13} = -b_{23}, \\ b_{51} = -b_{52}(\phi_2+\phi_3+\phi_5+\phi_6 = \pi) \end{cases}, G = \begin{pmatrix} -\sin(\phi_4+\phi_5) & 0 \\ \sin(\phi_4+\phi_5) & 0 \\ 0 & \sin(\phi_4+\phi_5 \\ \sin(\phi_4-\phi_6) & \sin(\phi_5+\phi_6 \\ -\sin(\phi_4) & -\sin(\phi_5) \end{pmatrix}$$

Necessary equalities, that are constrains on elements of matrices P and Q:

$$\begin{cases} x_{11} = y_{11} \\ x_{44} = y_{44} \\ x_{55} = y_{55} \\ x_{66} = y_{66} \\ x_{64} = y_{64} \end{cases} \Rightarrow$$

$$\begin{cases} \sin^2(\phi_1+\phi_2)p_{11} = \sin^2(\phi_4+\phi_5)q_{11} \\ \sin^2(\phi_1+\phi_2)p_{22} = \sin^2(\phi_4+\phi_5)q_{22} \\ \sin^2(\phi_1-\phi_3)p_{11} + 2\sin(\phi_1-\phi_3)\sin(\phi_2+\phi_3)p_{12} + \sin^2(\phi_2+\phi_3)p_{22} = \sin^2(\phi_4-\phi_6)q_{11} + 2\sin(\phi_5+\phi_6)\sin(\phi_4-\phi_6)q_{12} + \\ +\sin^2(\phi_5+\phi_6)q_{22} \\ \sin^2(\phi_4)p_{11} - 2\sin(\phi_4)\sin(\phi_1+\phi_2+\phi_4)p_{12} + \sin^2(\phi_1+\phi_2+\phi_4)p_{22} = \sin^2(\phi_4)q_{11} + 2\sin(\phi_4)\sin(\phi_5)q_{12} + \sin^2(\phi_5)q_{22} \\ \sin(\phi_1+\phi_2)(\sin(\phi_4)p_{21} - \sin(\phi_1+\phi_2+\phi_4)p_{22}) = \sin(\phi_4+\phi_5)(-\sin(\phi_4)q_{12} - \sin(\phi_5)q_{22}) \end{cases}$$

Let there $B$ be a solution such that $b_{44} = 0$. Then it follows, that $p_{22} = 0$, $q_{22} = 0$ and we get matrix of above system of equations on $p_{11}, p_{12}, q_{11}, q_{12}$:

$$\begin{pmatrix} \sin^2(\phi_1+\phi_2) & 0 & -\sin^2(\phi_4+\phi_5) & 0 \\ \sin^2(\phi_1-\phi_3) & 2\sin(\phi_1-\phi_3)\sin(\phi_2+\phi_3) & -\sin^2(\phi_4-\phi_6) & -2\sin(\phi_5+\phi_6)\sin(\phi_4-\phi_6) \\ \sin(\phi_4) & -2\sin(\phi_1+\phi_2+\phi_4) & -\sin(\phi_4) & -2\sin(\phi_5) \\ 0 & \sin(\phi_1+\phi_2) & 0 & \sin(\phi_4+\phi_5) \end{pmatrix}$$

Set $p_{11} = \sin^2(\phi_4+\phi_5)$, $p_{12} = a \cdot \sin(\phi_4+\phi_5)$, $q_{11} = \sin^2(\phi_1+\phi_2)$, $q_{12} = -a \cdot \sin(\phi_1+\phi_2)$. Then the 1st and 4th rows in the above system equal 0. Note that $p_{11} = 0$ would imply $\sin(\phi_1+\phi_2+\phi_4+\phi_5) = 0$, and hence $P, Q$ have the above form for $a = \frac{\sin^2(\phi_4+\phi_5)-\sin^2(\phi_1+\phi_2)}{2\sin(\phi_1+\phi_2+\phi_4+\phi_5)}$.

$$\begin{cases} \sin^2(\phi_1-\phi_3)\sin^2(\phi_4+\phi_5) + a * \sin(\phi_4+\phi_5) * 2\sin(\phi_1-\phi_3)\sin(\phi_2+\phi_3) - \sin^2(\phi_4-\phi_6)\sin^2(\phi_1+\phi_2) + 2\sin(\phi_5+ \\ \sin(\phi_4)(-\sin^2(\phi_1+\phi_2)+\sin^2(\phi_4+\phi_5)) + 2a * (\sin(\phi_5)\sin(\phi_1+\phi_2) - \sin(\phi_4+\phi_5)\sin(\phi_1+\phi_2+\phi_4)) = 0 \end{cases}$$

$$\frac{\sin(\phi_5)\sin(\phi_1+\phi_2) - \sin(\phi_4+\phi_5)\sin(\phi_1+\phi_2+\phi_4)}{\sin(\phi_5+\phi_6)\sin(\phi_4-\phi_6)\sin(\phi_1+\phi_2) + \sin(\phi_4+\phi_5)\sin(\phi_1-\phi_3)\sin(\phi_2+\phi_3)}$$

$=$

$$\frac{\sin(\phi_4)(\sin^2(\phi_1+\phi_2) - \sin^2(\phi_4+\phi_5))}{\sin^2(\phi_4-\phi_6)\sin^2(\phi_1+\phi_2) - \sin^2(\phi_1-\phi_3)\sin^2(\phi_4+\phi_5)}$$

$\Rightarrow$

$$\frac{\sin(\phi_4)\sin(\phi_1+\phi_2+\phi_4+\phi_5)}{-\sin(\phi_1+\phi_2+\phi_4+\phi_5) + \sin(\phi_1-\phi_3+\phi_4-\phi_6)\cos(\phi_2+\phi_3-\phi_5-\phi_6) + \sin(\phi_2+\phi_3+\phi_5+\phi_6) * \cos(\phi_1-\phi_3-\phi_4+\phi}$$

$=$

$$\frac{\sin(\phi_4)\sin(\phi_1+\phi_2+\phi_4+\phi_5)\sin(-}{-\sin(\phi_1+\phi_2+\phi_4+\phi_5)\sin(-\phi_1-\phi_2+\phi_4+\phi_5) - \sin(\phi_1-\phi_3+\phi_4-\phi_6)\sin(\phi_1-\phi_3-\phi_4+\phi_6) - 0.5(-\sin(2\phi_1+\phi_2-}$$

$\Rightarrow \sin(-\phi_1-\phi_2+\phi_4+\phi_5)(\sin(\phi_1-\phi_3+\phi_4-\phi_6)\cos(\phi_2+\phi_3-\phi_5-\phi_6) + \sin(\phi_2+\phi_3+\phi_5+\phi_6) * \cos(\phi_1-\phi_3-\phi_4+\phi_6)) =$
$-\sin(\phi_1-\phi_3+\phi_4-\phi_6)\sin(\phi_1-\phi_3-\phi_4+\phi_6) - 0.5(-\sin(2\phi_1+\phi_2-\phi_3-2\phi_4-\phi_5+\phi_6)\sin(\phi_2+\phi_3+\phi_5+\phi_6) - \sin(2\phi_1+\phi_2-\phi_3+2\phi_4+\phi_5-\phi_6)\sin(\phi_2+\phi_3-\phi_5-\phi_6))$

$0.5\sin(\phi_1-\phi_3+\phi_4-\phi_6)(-\sin(\phi_1-\phi_3-\phi_4+\phi_6) - \sin(\phi_1+2\phi_2+\phi_3-\phi_4-2\phi_5-\phi_6)) + 0.5\sin(\phi_2+\phi_3+\phi_5+\phi_6)(-\sin(2\phi_1+\phi_2-\phi_3-2\phi_4-\phi_5+\phi_6) - \sin(\phi_2+\phi_3-\phi_5-\phi_6)) = -\sin(\phi_1-\phi_3+\phi_4-\phi_6)\sin(\phi_1-\phi_3-\phi_4+\phi_6) - 0.5(-\sin(2\phi_1+\phi_2-\phi_3-2\phi_4-\phi_5+\phi_6)\sin(\phi_2+\phi_3+\phi_5+\phi_6) - \sin(2\phi_1+\phi_2-\phi_3+2\phi_4+\phi_5-\phi_6)\sin(\phi_2+\phi_3-\phi_5-\phi_6))$

$\sin(\phi_1-\phi_3+\phi_4-\phi_6)(\sin(\phi_1-\phi_3-\phi_4+\phi_6) - \sin(\phi_1+2\phi_2+\phi_3-\phi_4-2\phi_5-\phi_6)) + \sin(\phi_2+\phi_3-\phi_5-\phi_6)(\sin(2\phi_1+\phi_2-\phi_3+2\phi_4+\phi_5-\phi_6) - \sin(\phi_2+\phi_3+\phi_5+\phi_6))$

$\sin(\phi_2+\phi_3-\phi_5-\phi_6)\sin(-\phi_1+\phi_3-\phi_4+\phi_6)(\cos(\phi_1+\phi_2-\phi_4-\phi_5) - \cos(\phi_1+\phi_2+\phi_4+\phi_5)) = 0$

The second and third factor are always positive. Hence the coefficient matrix is degenerate if and only if $\phi_2+\phi_3 = \phi_5+\phi_6$. If this condition is not fulfilled, then $A$ is extremal.

We shall henceforth assume $\phi_2+\phi_3 = \phi_5+\phi_6$. There are further conditions which involve $b_3$.

If only $(Au_2)_5 = (Au_3)_4 = 0$, then we get the conditions

$$\sin\phi_1 b_{15} + \sin(\phi_1+\phi_2)b_{35} + \sin\phi_2 b_{45} = \sin\phi_3 b_{14} + \sin(\phi_2+\phi_3)b_{34} + \sin\phi_2 b_{45} = 0,$$

which are equivalent to $X_{45} = b_{45}$, and no further conditions on $P, Q$ arise.

If only $(Au_4)_5 = (Au_6)_4 = 0$, then the ensuing conditions on $B$ are equivalent to $Y_{45} = b_{45}$ and no further conditions on $P, Q$ arise.

If $(Au_2)_5 = (Au_3)_4 = 0$ and $(Au_4)_5 = (Au_6)_4 = 0$, or equivalently $\phi_1+\phi_6 = \phi_3+\phi_4$, then we get the condition $X_{45} = Y_{45}$ on $P, Q$, which under the condition $b_{44} = 0$ is expressed by

$$\sin(\phi_1+\phi_2)\sin(\phi_1-\phi_3)p_{12} = \sin(\phi_4+\phi_5)\sin(\phi_4-\phi_6)q_{12}.$$

We have $\sin(\phi_1-\phi_3) = \sin(\phi_4-\phi_6) \neq 0$, and the condition becomes $\sin(\phi_1+\phi_2)p_{12} - \sin(\phi_4+\phi_5)q_{12} = 0$. This is incompatible with the condition $x_{46} = y_{46}$ which gives a similar equation with reverted sign. Hence in this case there is only the trivial solution $B = 0$ and $A$ is extremal.

If $\phi_2 + \phi_3 + \phi_5 + \phi_6 = \pi$, then additional equalities are imposed on $B$. Namely, we also have $(Bu_1)_5 = (Bu_3)_2 = (Bu_6)_1 = 0$. Let $\phi_2 + \phi_3 = \phi_5 + \phi_6 = \frac{\pi}{2}$, then

$$A = \begin{pmatrix} 1 & -1 & -\cos(\phi_2) & \cos(\phi_1+\phi_2) & 0 & \cos\phi_5 \\ -1 & 1 & \cos\phi_2 & \cos(\phi_4+\phi_5) & 0 & -\cos(\phi_5) \\ -\cos(\phi_2) & \cos\phi_2 & 1 & -\cos(\phi_1) & -\sin\phi_2 & \cos(\phi_1+\phi_4) \\ \cos(\phi_1+\phi_2) & \cos(\phi_4+\phi_5) & -\cos(\phi_1) & 1 & b_3 & -\cos(\phi_4) \\ 0 & 0 & -\sin\phi_2 & b_3 & 1 & -\sin\phi_5 \\ \cos\phi_5 & -\cos(\phi_5) & \cos(\phi_1+\phi_4) & -\cos(\phi_4) & -\sin\phi_5 & 1 \end{pmatrix}, \qquad (2)$$

$b_3 = \max\{\sin(\phi_1 + \phi_2), \sin(\phi_4 + \phi_5)\}$.

The additional condition $X_{51} + Y_{52} = 0$ gives $\sin(\phi_1 - \phi_3)\sin(\phi_1 + \phi_2)p_{11} + \sin(\phi_1 + \phi_2)\sin(\phi_2 + \phi_3)p_{12} + \sin(\phi_4 + \phi_5)\sin(\phi_4 - \phi_6)q_{11} + \sin(\phi_4 + \phi_5)\sin(\phi_5 + \phi_6)q_{12} = 0$. With $p_{11} = \sin^2(\phi_4 + \phi_5)$, $p_{12} = a\sin(\phi_4 + \phi_5)$, $q_{11} = \sin^2(\phi_1 + \phi_2)$, $q_{12} = -a\sin(\phi_1 + \phi_2)$ gives

$$\sin(\phi_1 - \phi_3)\sin(\phi_4 + \phi_5) + \sin(\phi_4 - \phi_6)\sin(\phi_1 + \phi_2) + a(\sin(\phi_2 + \phi_3) - \sin(\phi_5 + \phi_6)) = 0.$$

This gives $-\cos(\phi_1+\phi_2)\sin(\phi_4+\phi_5) - \cos(\phi_4+\phi_5)\sin(\phi_1+\phi_2) = -\sin(\phi_1+\phi_2+\phi_4+\phi_5) = 0$, which is not possible. Hence in this case no solution $B$ with $b_{44} = 0$ exists, and $A$ is extremal.

## Result

The extremal matrices have the form (1) with $\phi_i > 0$, $\phi_2 + \phi_3 + \phi_5 + \phi_6 \le \pi$, $\phi_3 + \phi_6 > \phi_1 + \phi_4$, with either $\phi_2 + \phi_3 \neq \phi_5 + \phi_6$, or with $\phi_2 + \phi_3 = \phi_5 + \phi_6 = \frac{\pi}{2}$, or with $\phi_2 + \phi_3 = \phi_5 + \phi_6$, $\phi_1 + \phi_6 = \phi_3 + \phi_4$.