Case 41

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Zeros and linear conditions

The minimal zero supports are given by (1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 3, 6), (2, 4, 6), (3, 4, 6), (3, 5, 6)

$$A = \begin{pmatrix} 1 & -\cos(\phi_2) & -\cos(\phi_1) & \cos(\phi_2+\phi_3) & \cos(\phi_2+\phi_4) & \cos(\phi_1+\phi_5) \\ -\cos(\phi_2) & 1 & \cos(\phi_1+\phi_2) & -\cos(\phi_3) & -\cos(\phi_4) & \cos(\phi_3+\phi_6) \\ -\cos(\phi_1) & \cos(\phi_1+\phi_2) & 1 & \cos(\phi_5+\phi_6) & \cos(\phi_5+\phi_7) & -\cos(\phi_5) \\ \cos(\phi_2+\phi_3) & -\cos(\phi_3) & \cos(\phi_5+\phi_6) & 1 & b_1 & -\cos(\phi_6) \\ \cos(\phi_2+\phi_4) & -\cos(\phi_4) & \cos(\phi_5+\phi_7) & b_1 & 1 & -\cos(\phi_7) \\ \cos(\phi_1+\phi_5) & \cos(\phi_3+\phi_6) & -\cos(\phi_5) & -\cos(\phi_6) & -\cos(\phi_7) & 1 \end{pmatrix}$$

The zeros are given by the columns:

$$(u_1,u_2,u_3,u_4,u_5,u_6,u_7) = \begin{pmatrix} \sin(\phi_1+\phi_2) \\ \sin(\phi_1) \\ \sin(\phi_2) \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \sin(\phi_3) \\ \sin(\phi_2+\phi_3) \\ 0 \\ \sin(\phi_2) \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \sin(\phi_4) \\ \sin(\phi_2+\phi_4) \\ 0 \\ 0 \\ \sin(\phi_2) \\ 0 \end{pmatrix} \begin{pmatrix} \sin(\phi_5) \\ 0 \\ \sin(\phi_1+\phi_5) \\ 0 \\ 0 \\ \sin(\phi_1) \end{pmatrix} \begin{pmatrix} 0 \\ \sin(\phi_6) \\ 0 \\ \sin(\phi_3+\phi_6) \\ 0 \\ \sin(\phi_3) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \sin(\phi_6) \\ \sin(\phi_5) \\ 0 \\ \sin(\phi_5+\phi_6) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \sin(\phi_7) \\ 0 \\ \sin(\phi_5) \\ \sin(\phi_5+\phi_7) \end{pmatrix}$$

1. $$Au_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sin(\phi_2)\cos(\phi_1+\phi_2+\phi_3)+\sin(\phi_2)\cos(\phi_5+\phi_6) \\ \sin(\phi_2)\cos(\phi_1+\phi_2+\phi_4)+\sin(\phi_2)\cos(\phi_5+\phi_7) \\ \sin(\phi_1)\cos(\phi_1+\phi_2+\phi_5)+\sin(\phi_1)\cos(\phi_3+\phi_6) \end{pmatrix} \Rightarrow \begin{cases} \cos(\phi_1+\phi_2+\phi_3) \ge -\cos(\phi_5+\phi_6) \\ \cos(\phi_1+\phi_2+\phi_4) \ge -\cos(\phi_5+\phi_7) \\ \cos(\phi_1+\phi_2+\phi_5) \ge -\cos(\phi_3+\phi_6) \end{cases}$$

2. $$Au_2 = \begin{pmatrix} 0 \\ 0 \\ \sin(\phi_2)\cos(\phi_1+\phi_2+\phi_3)+\sin(\phi_2)\cos(\phi_5+\phi_6) \\ 0 \\ b_1\sin(\phi_2)-\sin(\phi_2)\cos(\phi_3-\phi_4) \\ \sin(\phi_3)\cos(\phi_1+\phi_5)+\sin(\phi_3)\cos(\phi_2+\phi_3+\phi_6) \end{pmatrix} \Rightarrow \begin{cases} \cos(\phi_1+\phi_2+\phi_3) \ge -\cos(\phi_5+\phi_6) \\ b_1 \ge \cos(\phi_3-\phi_4) \\ \cos(\phi_1+\phi_5) \ge -\cos(\phi_2+\phi_3+\phi_6) \end{cases}$$

3. $$Au_3 = \begin{pmatrix} 0 \\ 0 \\ \sin(\phi_2)\cos(\phi_1+\phi_2+\phi_4)+\sin(\phi_2)\cos(\phi_5+\phi_7) \\ b_1\sin(\phi_2)-\sin(\phi_2)\cos(\phi_3-\phi_4) \\ 0 \\ \sin(\phi_4)\cos(\phi_1+\phi_5)+\sin(\phi_2+\phi_4)\cos(\phi_3+\phi_6)-\sin(\phi_2)\cos(\phi_7) \end{pmatrix} \Rightarrow \begin{cases} \cos(\phi_1+\phi_2+\phi_4) \ge -\cos(\phi_5+\phi_7) \\ b_1 \ge \cos(\phi_3-\phi_4) \\ \sin(\phi_4)\cos(\phi_1+\phi_5)+\sin(\phi_2+\phi_4)\cos(\phi_3+ \end{cases}$$

4. $$Au_4 = \begin{pmatrix} 0 \\ \sin(\phi_1)\cos(\phi_1+\phi_2+\phi_5)+\sin(\phi_1)\cos(\phi_3+\phi_6) \\ 0 \\ \sin(\phi_5)\cos(\phi_2+\phi_3)+\sin(\phi_5)\cos(\phi_1+\phi_5+\phi_6) \\ \sin(\phi_5)\cos(\phi_2+\phi_4)+\sin(\phi_5)\cos(\phi_1+\phi_5+\phi_7) \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \cos(\phi_1+\phi_2+\phi_5) \ge -\cos(\phi_3+\phi_6) \\ \cos(\phi_2+\phi_3) \ge -\cos(\phi_1+\phi_5+\phi_6) \\ \cos(\phi_2+\phi_4) \ge -\cos(\phi_1+\phi_5+\phi_7) \end{cases}$$

5. $Au_5 = \begin{pmatrix} \sin(\phi_3)\cos(\phi_1+\phi_5)+\sin(\phi_3)\cos(\phi_2+\phi_3+\phi_6) \\ 0 \\ \sin(\phi_6)\cos(\phi_1+\phi_2)+\sin(\phi_6)\cos(\phi_3+\phi_5+\phi_6) \\ 0 \\ b_1\sin(\phi_3+\phi_6)-\sin(\phi_6)\cos(\phi_4)-\sin(\phi_3)\cos(\phi_7) \\ 0 \end{pmatrix} \Rightarrow$

$\begin{cases} \cos(\phi_1+\phi_5) \geq -\cos(\phi_2+\phi_3+\phi_6) \\ \cos(\phi_1+\phi_2) \geq -\cos(\phi_3+\phi_5+\phi_6)) \\ b_1\sin(\phi_3+\phi_6)-\sin(\phi_6)\cos(\phi_4)-\sin(\phi_3)\cos(\phi_7) \geq 0 \end{cases}$

6. $Au_6 = \begin{pmatrix} \sin(\phi_5)\cos(\phi_2+\phi_3)+\sin(\phi_5)\cos(\phi_1+\phi_5+\phi_6) \\ \sin(\phi_6)\cos(\phi_1+\phi_2)+\sin(\phi_6)\cos(\phi_3+\phi_5+\phi_6) \\ 0 \\ 0 \\ b_1\sin(\phi_5)-\sin(\phi_5)\cos(\phi_6-\phi_7) \\ 0 \end{pmatrix} \Rightarrow$

$\begin{cases} \cos(\phi_2+\phi_3) \geq -\cos(\phi_1+\phi_5+\phi_6) \\ \cos(\phi_1+\phi_2) \geq -\cos(\phi_3+\phi_5+\phi_6) \\ b_1 \geq \cos(\phi_6-\phi_7) \end{cases}$

7. $Au_7 = \begin{pmatrix} \sin(\phi_5)\cos(\phi_2+\phi_4)+\sin(\phi_5)\cos(\phi_1+\phi_5+\phi_7) \\ -\sin(\phi_5)\cos(\phi_4)+\sin(\phi_7)\cos(\phi_1+\phi_2)+\sin(\phi_5+\phi_7)\cos(\phi_3+\phi_6) \\ 0 \\ b_1\sin(\phi_5)-\sin(\phi_5)\cos(\phi_6-\phi_7) \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$\begin{cases} \cos(\phi_2+\phi_4) \geq -\cos(\phi_1+\phi_5+\phi_7) \\ -\sin(\phi_5)\cos(\phi_4)+\sin(\phi_7)\cos(\phi_1+\phi_2)+\sin(\phi_5+\phi_7)\cos(\phi_3+\phi_6) \geq 0 \\ b_1 \geq \cos(\phi_6-\phi_7) \end{cases}$

From inequalities on ϕ_i we get:

$\begin{cases} \phi_1+\phi_2+\phi_3+\phi_5+\phi_6 \leq \pi \\ \phi_1+\phi_2+\phi_4+\phi_5+\phi_7 \leq \pi \end{cases} \Rightarrow$

Copositivity:

Sets $(1,3,4), (2,3,4), (1,3,5), (2,3,5), (1,4,5), (2,4,5), (3,4,5), (1,3,4,5), (2,3,4,5), (1,2,6), (2,3,6), (1,4,6), (1,5,6), (2,5,6), (4,5,6), (1,4,5,6)$ have to be considered.

1. $I = (4,5,6), b_1 \geq \cos(\phi_6-\phi_7) => \phi_6+\phi_7 \geq |\phi_6-\phi_7|$ performs strictly
2. $I = (1,4,5,6), (1,5,6) : u = e_5+e_6 : \cos(\phi_6-\phi_7) \geq \cos(\phi_6), \cos(\phi_2+\phi_4) \geq -\cos(\phi_1+\phi_5) => \phi_2+\phi_4+\phi_1+\phi_5 \leq \pi$ performs
3. $I = (2,5,6) : \phi_4+\pi-\phi_3-\phi_6+\phi_7 \geq \pi => \phi_4+\phi_7 \geq \phi_3+\phi_6$
4. $I = (1,4,6) : \phi_6+\pi-\phi_2-\phi_3+\pi-\phi_1-\phi_5 \geq \pi => \pi \geq \phi_1+\phi_5-\phi_6+\phi_2+\phi_3$ performs strictly
5. $I = (1,3,4,5), (1,3,4), (1,3,5) : u = e_1+e_3 : 3)\pi \geq \phi_1+\phi_2+\phi_6+\phi_5$
   $5)\pi \geq \phi_2+\phi_4+\phi_5+\phi_7$ performs strictly
6. $I = (2,3,4,5), (2,3,4), (2,4,5) : u = e_2+e_4 : 3)\pi \geq \phi_1+\phi_2+\phi_6+\phi_5$
   $5)\cos(\phi_3-\phi_4) \geq \cos(\phi_4) => \phi_4 \geq |\phi_3-\phi_4|$ performs strictly
7. $I = (2,3,5) : \phi_4+\pi-\phi_1-\phi_2+\pi-\phi_5-\phi_7 \geq \pi => \pi \geq \phi_1+\phi_5-\phi_4+\phi_2+\phi_7$ performs strictly
8. $I = (1,4,5) : \pi-|\phi_3-\phi_4|+\pi-\phi_2-\phi_3+\pi-\phi_2-\phi_4 \geq \pi => 2\pi \geq 2\phi_2+\phi_3+\phi_4+|\phi_3-\phi_4|$ performs strictly

9. $I = (3,4,5) : \pi - |\phi_6 - \phi_7| + \pi - \phi_5 - \phi_7 + \pi - \phi_5 - \phi_6 \geq \pi => 2\pi \geq 2\phi_5 + \phi_6 + \phi_7 + |\phi_6 - \phi_7|$ performs strictly

10. $I = (1,2,6) : \phi_2 + \pi - \phi_1 - \phi_5 + \pi - \phi_3 - \phi_6 \geq \pi => \pi \geq \phi_1 + \phi_5 - \phi_2 + \phi_3 + \phi_6$ performs strictly

11. $I = (2,3,6) : \phi_5 + \pi - \phi_1 - \phi_2 + \pi - \phi_3 - \phi_6 \geq \pi => \pi \geq \phi_1 - \phi_5 + \phi_2 + \phi_3 + \phi_6$ performs strictly

That proves copositivity with additional condition on $\phi_i : \phi_4 + \phi_7 \geq \phi_3 + \phi_6$, using it we get conditions on $\phi_i : \begin{cases} \phi_1 + \phi_2 + \phi_4 + \phi_5 + \phi_7 \leq \pi \\ \phi_4 + \phi_7 \geq \phi_3 + \phi_6 \end{cases}$ All remain inequalities fulfilled. If $\phi_4 + \phi_7 = \phi_3 + \phi_6$, then there is additional minimal zero (2, 5, 6) $\Rightarrow \phi_4 + \phi_7 > \phi_3 + \phi_6$
There is a symmetry $123456 \leftrightarrow 361452$, that means for $\phi_i : \phi_2 \leftrightarrow \phi_5, \phi_3 \leftrightarrow \phi_6, \phi_4 \leftrightarrow \phi_7$ and it follows we can let $b_1 = \cos(\phi_6 - \phi_7)$

0.1 Parametrization:

$$A = \begin{pmatrix} 1 & -\cos(\phi_2) & -\cos(\phi_1) & \cos(\phi_2+\phi_3) & \cos(\phi_2+\phi_4) & \cos(\phi_1+\phi_5) \\ -\cos(\phi_2) & 1 & \cos(\phi_1+\phi_2) & -\cos(\phi_3) & -\cos(\phi_4) & \cos(\phi_3+\phi_6) \\ -\cos(\phi_1) & \cos(\phi_1+\phi_2) & 1 & \cos(\phi_5+\phi_6) & \cos(\phi_5+\phi_7) & -\cos(\phi_5) \\ \cos(\phi_2+\phi_3) & -\cos(\phi_3) & \cos(\phi_5+\phi_6) & 1 & \cos(\phi_6-\phi_7) & -\cos(\phi_6) \\ \cos(\phi_2+\phi_4) & -\cos(\phi_4) & \cos(\phi_5+\phi_7) & \cos(\phi_6-\phi_7) & 1 & -\cos(\phi_7) \\ \cos(\phi_1+\phi_5) & \cos(\phi_3+\phi_6) & -\cos(\phi_5) & -\cos(\phi_6) & -\cos(\phi_7) & 1 \end{pmatrix}$$

with conditions : $\phi_i \in (0,\pi), \phi_1 + \phi_2 + \phi_4 + \phi_5 + \phi_7 \leq \pi, \phi_4 + \phi_7 > \phi_3 + \phi_6, |\phi_4 - \phi_3| \geq |\phi_7 - \phi_6|$, or

$$A = \begin{pmatrix} 1 & -\cos(\phi_2) & -\cos(\phi_1) & \cos(\phi_2+\phi_3) & \cos(\phi_2+\phi_4) & \cos(\phi_1+\phi_5) \\ -\cos(\phi_2) & 1 & \cos(\phi_1+\phi_2) & -\cos(\phi_3) & -\cos(\phi_4) & \cos(\phi_3+\phi_6) \\ -\cos(\phi_1) & \cos(\phi_1+\phi_2) & 1 & \cos(\phi_5+\phi_6) & \cos(\phi_5+\phi_7) & -\cos(\phi_5) \\ \cos(\phi_2+\phi_3) & -\cos(\phi_3) & \cos(\phi_5+\phi_6) & 1 & \cos(\phi_3-\phi_4) & -\cos(\phi_6) \\ \cos(\phi_2+\phi_4) & -\cos(\phi_4) & \cos(\phi_5+\phi_7) & \cos(\phi_3-\phi_4) & 1 & -\cos(\phi_7) \\ \cos(\phi_1+\phi_5) & \cos(\phi_3+\phi_6) & -\cos(\phi_5) & -\cos(\phi_6) & -\cos(\phi_7) & 1 \end{pmatrix}$$

with conditions : $\phi_i \in (0,\pi), \phi_1 + \phi_2 + \phi_4 + \phi_5 + \phi_7 \leq \pi, \phi_4 + \phi_7 > \phi_3 + \phi_6, |\phi_4 - \phi_3| \leq |\phi_7 - \phi_6|$

Extremality:

The submatrix A_{12346} is isomorphic to a T-matrix. Any B in the face of A is hence such that B_{12346} is proportional to A_{12346}. In particular, only the diagonal element B_{55} may differ from the other diagonal elements of B. But either of the zeros u_3, u_7 ensures that B_{55} equals the other diagonal elements. Hence B is proportional to A and A is extremal.

Result

The extremal copositive matrices with the given minimal zero support set are given by $A = \begin{pmatrix} 1 & -\cos(\phi_2) & -\cos(\phi_1 \\ -\cos(\phi_2) & 1 & \cos(\phi_1 + \\ -\cos(\phi_1) & \cos(\phi_1+\phi_2) & 1 \\ \cos(\phi_2+\phi_3) & -\cos(\phi_3) & \cos(\phi_5 + \\ \cos(\phi_2+\phi_4) & -\cos(\phi_4) & \cos(\phi_5 + \\ \cos(\phi_1+\phi_5) & \cos(\phi_3+\phi_6) & -\cos(\phi_5 \end{pmatrix}$

under the conditions : $\phi_i \in (0,\pi), \phi_1 + \phi_2 + \phi_4 + \phi_5 + \phi_7 \leq \pi, \phi_4 + \phi_7 > \phi_3 + \phi_6$.

References

[1] Peter J.C. Dickinson. *A new certificate for copositivity. Optimization Online.*

[2] Peter J.C. Dickinson and Roland Hildebrand. *Considering copositivity locally.* J.Math. Anal.Appl., 437(2):11841195, 2016

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