Case 30

The minimal zero supports are given by (1, 2, 3), (1, 2, 4), (1, 3, 5), (2, 4, 5), (3, 4, 6), (3, 5, 6)

$$A = \begin{pmatrix} 1 & -cos(\phi_2) & -cos(\phi_1) & cos(\phi_2+\phi_3) & cos(\phi_1+\phi_4) & b_1 \\ -cos(\phi_2) & 1 & cos(\phi_1+\phi_2) & -cos(\phi_3) & cos(\phi_3+\phi_5) & b_2 \\ -cos(\phi_1) & cos(\phi_1+\phi_2) & 1 & -cos(\phi_7) & -cos(\phi_4) & -cos(\phi_6) \\ cos(\phi_2+\phi_3) & -cos(\phi_3) & -cos(\phi_7) & 1 & -cos(\phi_5) & cos(\phi_6+\phi_7) \\ cos(\phi_1+\phi_4) & cos(\phi_3+\phi_5) & -cos(\phi_4) & -cos(\phi_5) & 1 & cos(\phi_4+\phi_6) \\ b_1 & b_2 & -cos(\phi_6) & cos(\phi_6+\phi_7) & cos(\phi_4+\phi_6) & 1 \end{pmatrix}$$

The zeros are given by the columns:

$$(u_1,u_2,u_3,u_4,u_5,u_6,u_7) = \begin{pmatrix} sin(\phi_1+\phi_2) \\ sin(\phi_1) \\ sin(\phi_2) \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} sin(\phi_3) \\ sin(\phi_2+\phi_3) \\ 0 \\ sin(\phi_2) \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} sin(\phi_4) \\ 0 \\ sin(\phi_1+\phi_4) \\ 0 \\ sin(\phi_1) \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ sin(\phi_5) \\ 0 \\ sin(\phi_3+\phi_5) \\ sin(\phi_3) \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ sin(\phi_6+\phi_7) \\ sin(\phi_6) \\ 0 \\ sin(\phi_7) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ sin(\phi_4+\phi_6) \\ 0 \\ sin(\phi_6) \\ sin(\phi_4) \end{pmatrix}$$

1. $$Au_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ sin(\phi_2)cos(\phi_1+\phi_2+\phi_3) - sin(\phi_2)cos(\phi_7) \\ sin(\phi_1)cos(\phi_1+\phi_2+\phi_4) + sin(\phi_1)cos(\phi_3+\phi_5) \\ b_2 sin(\phi_1) + b_1 sin(\phi_1+\phi_2) - sin(\phi_2)cos(\phi_6) \end{pmatrix}$$
$$=> \begin{cases} cos(\phi_1+\phi_2+\phi_3) \geq cos(\phi_7) \\ cos(\phi_1+\phi_2+\phi_4) \geq -cos(\phi_3+\phi_5) \\ b_2 sin(\phi_1) + b_1 sin(\phi_1+\phi_2) - sin(\phi_2)cos(\phi_6) \geq 0 \end{cases}$$

2. $$Au_2 = \begin{pmatrix} 0 \\ 0 \\ sin(\phi_2)cos(\phi_1+\phi_2+\phi_3) - sin(\phi_2)cos(\phi_7) \\ 0 \\ sin(\phi_3)cos(\phi_1+\phi_4) + sin(\phi_3)cos(\phi_2+\phi_3+\phi_5) \\ b_1 sin(\phi_3) + b_2 sin(\phi_2+\phi_3) + sin(\phi_2)cos(\phi_6+\phi_7) \end{pmatrix}$$
$$=> \begin{cases} cos(\phi_1+\phi_2+\phi_3) \geq cos(\phi_7) \\ cos(\phi_1+\phi_4) \geq -cos(\phi_2+\phi_3+\phi_5) \\ b_1 sin(\phi_3) + b_2 sin(\phi_2+\phi_3) + sin(\phi_2)cos(\phi_6+\phi_7) \geq 0 \end{cases}$$

3. $$Au_3 = \begin{pmatrix} 0 \\ sin(\phi_1)cos(\phi_1+\phi_2+\phi_4) + sin(\phi_1)cos(\phi_3+\phi_5) \\ 0 \\ sin(\phi_4)cos(\phi_2+\phi_3) - sin(\phi_1)cos(\phi_5) - sin(\phi_1+\phi_4)cos(\phi_7) \\ 0 \\ b_1 sin(\phi_4) - sin(\phi_4)cos(\phi_1-\phi_6) \end{pmatrix}$$
$$=> \begin{cases} cos(\phi_1+\phi_2+\phi_4) \geq -cos(\phi_3+\phi_5) \\ sin(\phi_4)cos(\phi_2+\phi_3) - sin(\phi_1)cos(\phi_5) - sin(\phi_1+\phi_4)cos(\phi_7) \geq 0 \\ b_1 \geq cos(\phi_1-\phi_6) \end{cases}$$

4. $$Au_4 = \begin{pmatrix} sin(\phi_3)cos(\phi_1+\phi_4) + sin(\phi_3)cos(\phi_2+\phi_3+\phi_5) \\ 0 \\ -sin(\phi_3)cos(\phi_4) + sin(\phi_5)cos(\phi_1+\phi_2) - sin(\phi_7)cos(\phi_3+\phi_5) \\ 0 \\ 0 \\ b_2 sin(\phi_5) + sin(\phi_3)cos(\phi_4+\phi_6) + sin(\phi_3+\phi_5)cos(\phi_6+\phi_7) \end{pmatrix}$$
$$=> \begin{cases} cos(\phi_1+\phi_4) \geq -cos(\phi_2+\phi_3+\phi_5) \\ -sin(\phi_3)cos(\phi_4) + sin(\phi_5)cos(\phi_1+\phi_2) - sin(\phi_7)cos(\phi_3+\phi_5) \geq 0 \\ b_2 sin(\phi_5) + sin(\phi_3)cos(\phi_4+\phi_6) + sin(\phi_3+\phi_5)cos(\phi_6+\phi_7) \geq 0 \end{cases}$$

5. $Au_5 = \begin{pmatrix} sin(\phi_6)cos(\phi_2+\phi_3) + b_1 sin(\phi_7) - sin(\phi_6+\phi_7)cos(\phi_1) \\ -sin(\phi_6)cos(\phi_3) + b_2 sin(\phi_7) + sin(\phi_6+\phi_7)cos(\phi_1+\phi_2) \\ 0 \\ 0 \\ -sin(\phi_6)cos(\phi_5) - sin(\phi_6)cos(\phi_4-\phi_7) \\ 0 \end{pmatrix} =>$

$$\begin{cases} sin(\phi_6)cos(\phi_2+\phi_3) + b_1 sin(\phi_7) - sin(\phi_6+\phi_7)cos(\phi_1) \geq 0 \\ -sin(\phi_6)cos(\phi_3) + b_2 sin(\phi_7) + sin(\phi_6+\phi_7)cos(\phi_1+\phi_2) \geq 0 \\ -cos(\phi_5) \geq cos(\phi_4-\phi_7) \end{cases}$$

6. $Au_6 = \begin{pmatrix} b_1 sin(\phi_4) - sin(\phi_4)cos(\phi_1-\phi_6) \\ b_2 sin(\phi_4) + sin(\phi_4+\phi_6)cos(\phi_1+\phi_2) + sin(\phi_6)cos(\phi_3+\phi_5) \\ 0 \\ -sin(\phi_6)cos(\phi_5) - sin(\phi_6)cos(\phi_4-\phi_7) \\ 0 \\ 0 \end{pmatrix} =>$

$$\begin{cases} b_1 \geq cos(\phi_1-\phi_6) \\ b_2 sin(\phi_4) + sin(\phi_4+\phi_6)cos(\phi_1+\phi_2) + sin(\phi_6)cos(\phi_3+\phi_5) \geq 0 \\ -cos(\phi_5) \geq cos(\phi_4-\phi_7) \end{cases}$$

Copositivity:

Sets $(1,3,4), (2,3,4), (1,2,5), (2,3,5), (1,4,5), (3,4,5), (1,2,6), (1,3,6), (2,3,6), (1,4,6), (2,4,6), (1,5,6), (2,5,6), (1,2,5,6), (4,5,6), (1,4,5,6)$ have to be considered.

1. $I = (1,3,4) : \phi_1 + \phi_7 \geq \phi_2 + \phi_3$
2. $I = (1,3,4) : \phi_3 + \phi_7 \geq \phi_1 + \phi_2$
3. $I = (1,2,5) : \pi \geq \phi_1 + \phi_4 + \phi_3 + \phi_5 - \phi_2$
4. $I = (2,3,5) : \pi \geq \phi_1 - \phi_4 + \phi_3 + \phi_5 + \phi_2$
5. $I = (1,4,5) : \pi \geq \phi_1 + \phi_4 + \phi_3 - \phi_5 + \phi_2$
6. $I = (3,4,5) : \pi \leq \phi_7 + \phi_4 + \phi_5$
7. $I = (4,5,6) : \pi \geq 2\phi_6 + \phi_7 + \phi_4 - \phi_5$

Consider above inequalities:

1. from sets (1,4,5) and (2,3,5) $=> \pi \geq \phi_1 + \phi_3 + \phi_2$ and it follows from inequality 1.1 $\phi_1 + \phi_3 + \phi_2 \leq \phi_7$
2. from set (4,5,6) and $\phi_1 + \phi_3 + \phi_2 \leq \phi_7 => \pi \geq \phi_1 + \phi_3 + \phi_2 + 2\phi_6 + \phi_4 - \phi_5$ and it follows from inequality 1.1 $\phi_1 + \phi_3 + \phi_2 + \phi_4 + \phi_5 \leq \pi$
3. from inequality 6.1 follows $\pi \leq |\phi_4 - \phi_7| + \phi_5$: if $\phi_4 \geq \phi_7 => \phi_4 + \phi_5 - \phi_7 \geq \pi$, but $\phi_1 + \phi_3 + \phi_2 + \phi_4 + \phi_5 \leq \pi =>$ this case doesn't hold and $\phi_4 \leq \phi_7 => -\phi_4 + \phi_5 + \phi_7 \geq \pi$

Finally, get system of inequalities on ϕ_i: $\begin{cases} \phi_1 + \phi_3 + \phi_2 + \phi_4 + \phi_5 \leq \pi \\ -\phi_4 + \phi_5 + \phi_7 \geq \pi \end{cases}$

Consider sets, that relates to b_2 :

1. $I = (2,4,6) : b_2 = cos(\phi) => b_2 \geq -cos(\phi_6 + \phi_7 - \phi_3)$

If $b_2 = \frac{-sin(\phi_3+\phi_5)cos(\phi_6+\phi_7) - sin(\phi_3)cos(\phi_4+\phi_6)}{sin(\phi_5)} \geq -cos(\phi_6 + \phi_7 - \phi_3)$ from inequality 4.3
$-cos(\phi_4 + \phi_6) \geq cos(\phi_5 + \phi_6 + \phi_7), \phi_5 + \phi_6 + \phi_7 \geq \pi => -\phi_4 + \phi_5 + \phi_7 \leq \pi$
- doesn't hold

If $b_2 = \frac{-sin(\phi_6+\phi_7)cos(\phi_1+\phi_2)+sin(\phi_6)cos(\phi_3)}{sin(\phi_7)} \geq -cos(\phi_6+\phi_7-\phi_3)$ from inequality 5.2
$cos(\phi_3-\phi_7) \geq cos(\phi_1+\phi_2) => -\phi_3+\phi_7 \leq \phi_1+\phi_2$
- doesn't hold

If $b_2 = \frac{-sin(\phi_6)cos(\phi_3+\phi_5)-sin(\phi_4+\phi_6)cos(\phi_1+\phi_2)}{sin(\phi_4)} \geq -cos(\phi_6+\phi_7-\phi_3)$ from inequality 6.2
$cos(\phi_1+\phi_2+\phi_4) \geq -cos(\phi_3+\phi_5) =>$
$sin(\phi_6)cos(\phi_1+\phi_2+\phi_4) - sin(\phi_4+\phi_6)cos(\phi_1+\phi_2) + cos(\phi_6+\phi_7-\phi_3)sin(\phi_4) \geq 0$
$cos(\phi_1+\phi_2-\phi_6) \leq cos(\phi_6+\phi_7-\phi_3) =>$
$\phi_1+\phi_2+\phi_3 \geq 2\phi_6+\phi_7, \phi_1+\phi_2-\phi_6 \geq 0$- doesn't hold
$-\phi_1-\phi_2+\phi_3 \geq \phi_7, \phi_1+\phi_2-\phi_6 \leq 0$- doesn't hold

If $b_2 = \frac{-sin(\phi_1+\phi_2)b_1+sin(\phi_2)cos(\phi_6)}{sin(\phi_1)} \geq -cos(\phi_6+\phi_7-\phi_3)$ from inequality 1.3 and 6.1 $=> b_1 \leq \frac{sin(\phi_1)cos(\phi_6+\phi_7-\phi_3)+sin(\phi_2)cos(\phi_6)}{sin(\phi_1+\phi_2)}$,
$b_1 = cos(\phi_1-\phi_6) => cos(\phi_6+\phi_7-\phi_3) \geq cos(\phi_1+\phi_2-\phi_6) => \phi_6+\phi_7-\phi_3 \leq |\phi_1+\phi_2-\phi_6|$
, but $\phi_1+\phi_3+\phi_2 \leq \phi_7 =>$ doesn't hold

If $b_2 = \frac{-sin(\phi_3)b_1-sin(\phi_2)cos(\phi_6+\phi_7)}{sin(\phi_2+\phi_3)} \geq -cos(\phi_6+\phi_7-\phi_3)$ from inequality 2.3 and 6.1
$=> b_1 \leq \frac{sin(\phi_2+\phi_3)cos(\phi_6+\phi_7-\phi_3)-sin(\phi_2)cos(\phi_6+\phi_7)}{sin(\phi_3)}, b_1 = cos(\phi_1-\phi_6) => cos(\phi_2+\phi_3-\phi_6-\phi_7) \geq cos(\phi_1-\phi_6) =>$
$\phi_6+\phi_7-\phi_3-\phi_2 \leq |\phi_1-\phi_6|$
, but $\phi_1+\phi_3+\phi_2 \leq \phi_7 =>$ doesn't hold

As result, we get, that there is no copositivity.

Result

There aren't any copositive matrices with such minimal zeros set.

References

[1] Peter J.C. Dickinson. *A new certificate for copositivity. Optimization Online.*

[2] Peter J.C. Dickinson and Roland Hildebrand. *Considering copositivity locally.* J.Math. Anal.Appl., 437(2):1184–1195, 2016

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