Case 5

Andrey Afonin, Roland Hildebrand

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Basic structure

The index set of the supports of the minimal zeros is given by $\{\{1,2\},\{1,3\},\{2,4\},\{3,4,5\},\{4,5,6\},\{5,6,1\}\}$. It follows that A has the form

$$A = \begin{pmatrix} 1 & -1 & -1 & 1 & \cos(\phi_3+\phi_4) & -\cos\phi_4 \\ -1 & 1 & 1 & -1 & a_{25} & a_{26} \\ -1 & 1 & 1 & -\cos\phi_1 & \cos(\phi_1+\phi_2) & a_{36} \\ 1 & -1 & -\cos\phi_1 & 1 & -\cos\phi_2 & \cos(\phi_2+\phi_3) \\ \cos(\phi_3+\phi_4) & a_{25} & \cos(\phi_1+\phi_2) & -\cos\phi_2 & 1 & -\cos\phi_3 \\ -\cos\phi_4 & a_{26} & a_{36} & \cos(\phi_2+\phi_3) & -\cos\phi_3 & 1 \end{pmatrix}.$$

Here $\phi_1,\phi_2,\phi_3,\phi_4 \in (0,\pi)$, and $\phi_1+\phi_2, \phi_2+\phi_3, \phi_3+\phi_4 < \pi$.

The minimal zeros $u_1,\dots,u_6$ are given by the columns of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \sin\phi_3 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \sin\phi_2 & 0 & 0 \\ 0 & 0 & 1 & \sin(\phi_1+\phi_2) & \sin\phi_3 & 0 \\ 0 & 0 & 0 & \sin\phi_1 & \sin(\phi_2+\phi_3) & \sin\phi_4 \\ 0 & 0 & 0 & 0 & \sin\phi_2 & \sin(\phi_3+\phi_4) \end{pmatrix}.$$

First order conditions

We have the inequality

$$(Au_2)_5 = \cos(\phi_3+\phi_4) + \cos(\phi_1+\phi_2) \geq 0,$$

which yields $\phi_1+\phi_2+\phi_3+\phi_4 \leq \pi$. This also implies

$$(Au_4)_1 = -\sin\phi_2+\sin(\phi_1+\phi_2)+\sin\phi_1\cos(\phi_3+\phi_4) \geq -\sin\phi_2+\sin(\phi_1+\phi_2)-\sin\phi_1\cos(\phi_1+\phi_2) = (1-\cos\phi_1)\sin(\phi_1+\phi_2) > 0$$

$$(Au_5)_1 = \sin\phi_3 + \sin(\phi_2+\phi_3)\cos(\phi_3+\phi_4) - \sin\phi_2\cos\phi_4 = \sin\phi_3(1+\cos(\phi_2+\phi_3+\phi_4)) > 0$$

$$(Au_6)_4 = \sin\phi_3 - \sin\phi_4\cos\phi_2 + \cos(\phi_2+\phi_3)\sin(\phi_3+\phi_4) = \sin\phi_3(1+\cos(\phi_2+\phi_3+\phi_4)) > 0$$

Let us determine the by now still unknown elements a_{25}, a_{26}, a_{36}.

The conditions $(Au_1)_5, (Au_3)_5, (Au_4)_2 \geq 0$ involve the element a_{25} only and imply

$$a_{25} \geq \max\left(-\cos(\phi_3+\phi_4), \cos\phi_2, \frac{-\sin\phi_2+\sin(\phi_1+\phi_2)}{\sin\phi_1}\right) = \cos\phi_2.$$

The other two terms in the maximum are strictly smaller, hence $(Au_1)_5, (Au_4)_2 > 0$.

The conditions $(Au_1)_6, (Au_3)_6 \geq 0$ involve the element a_{26} only and imply

$$a_{26} \geq \max(\cos\phi_4, -\cos(\phi_2+\phi_3)) = \cos\phi_4$$

The other term in the maximum is strictly smaller, hence $(Au_3)_6 > 0$.

The conditions $(Au_2)_6, (Au_4)_6, (Au_5)_3, (Au_6)_3 \geq 0$ involve the element a_{36} only and imply

$$a_{36} \geq \max\left(\cos\phi_4, \frac{-\sin(\phi_1+\phi_2)\cos(\phi_2+\phi_3)+\sin\phi_1\cos\phi_3}{\sin\phi_2}, \frac{\sin\phi_3\cos\phi_1-\cos(\phi_1+\phi_2)\sin(\phi_2+\phi_3)}{\sin\phi_2}, \frac{\sin\phi_3-\sin\phi_4\cos(\phi_1+\phi_2)}{\sin(\phi_3+\phi_4)}\right) = \cos\phi_4$$

The other three terms in the maximum are strictly smaller, hence $(Au_4)_6, (Au_5)_3, (Au_6)_3 > 0$.

The conditions which involve more than one unknown element then become

$$(Au_5)_2 = -\sin\phi_3 + a_{25}\sin(\phi_2+\phi_3) + a_{26}\sin\phi_2 \geq -\sin\phi_3 + \cos\phi_2\sin(\phi_2+\phi_3) + \cos\phi_4\sin\phi_2 = \sin\phi_2(\cos(\phi_2+\phi_3)+\cos\phi_4) > 0$$

$$(Au_6)_2 = -\sin\phi_3 + a_{25}\sin\phi_4 + a_{26}\sin(\phi_3+\phi_4) \geq -\sin\phi_3 + \cos\phi_2\sin\phi_4 + \cos\phi_4\sin(\phi_3+\phi_4) = \sin\phi_4(\cos(\phi_3+\phi_4)+\cos\phi_2) > 0$$

and are trivially satisfied. The irreducibility conditions of A with respect to E_{25}, E_{26}, E_{36} then imply

$$a_{25} = \cos\phi_2, \quad a_{26} = a_{36} = \cos\phi_4.$$

Copositivity

Copositivity of

$$A = \begin{pmatrix} 1 & -1 & -1 & 1 & \cos(\phi_3+\phi_4) & -\cos\phi_4 \\ -1 & 1 & 1 & -1 & \cos\phi_2 & \cos\phi_4 \\ -1 & 1 & 1 & -\cos\phi_1 & \cos(\phi_1+\phi_2) & \cos\phi_4 \\ 1 & -1 & -\cos\phi_1 & 1 & -\cos\phi_2 & \cos(\phi_2+\phi_3) \\ \cos(\phi_3+\phi_4) & \cos\phi_2 & \cos(\phi_1+\phi_2) & -\cos\phi_2 & 1 & -\cos\phi_3 \\ -\cos\phi_4 & \cos\phi_4 & \cos\phi_4 & \cos(\phi_2+\phi_3) & -\cos\phi_3 & 1 \end{pmatrix} \quad (1)$$

will be checked by the criterion in Theorem 4.6 of [1]. For each non-empty index set $I \subset \{1,\dots,6\}$ we have to find a vector $v \in \mathbb{R}^6$ such that I contains the support of v and is contained in the nonnegative support of Av.

For I of size 1 or 2 we may take $v = \sum_{i\in I} e_i$. For I containing the support of a minimal zero u we may take $v = u$. The other index sets I are

$$\{2,3,5\}, \{1,4,5\}, \{2,3,6\}, \{1,4,6\}, \{3,4,6\}, \{2,5,6\}, \{3,5,6\}, \{2,3,5,6\}. \quad (2)$$

For these index sets we may choose the vectors

$$e_2 - e_3,\ e_1 - e_4,\ e_3 - e_2,\ e_4 - e_1,\ e_3 + e_4,\ e_5 + e_6,\ e_5 + e_6,\ e_5 + e_6,$$

respectively.

This proves copositivity.

Absence of other minimal zeros

By construction there are no further minimal zeros with supports of size 1 or 2. Index sets which contain a minimal zero support are excluded by definition. Hence the index sets which remain to be checked are those in (2).

The presence of a minimal zero with support I implies that the submatrix A_I is positive semi-definite with co-rank 1 and positive kernel vector. The submatrices $A_{\{2,3,5\}}, A_{\{1,4,5\}}, A_{\{1,4,6\}}, A_{\{2,3,5,6\}}$ are not PSD. A PSD matrix $\begin{pmatrix} 1 & -\cos\xi_1 & -\cos\xi_2 \\ -\cos\xi_1 & 1 & -\cos\xi_3 \\ -\cos\xi_2 & -\cos\xi_3 & 1 \end{pmatrix}$ with positive kernel vector must satisfy $\xi_1 + \xi_2 + \xi_3 = \pi$. For the remaining index sets this yields the conditions

$$3\pi - 2\phi_4 = \pi,\ 2\pi + \phi_1 - \phi_2 - \phi_3 - \phi_4 = \pi,\ 2\pi - \phi_2 + \phi_3 - \phi_4 = \pi,\ 2\pi - \phi_1 - \phi_2 + \phi_3 - \phi_4 = \pi,$$

respectively, which all do not hold.

Thus there do not exist additional minimal zeros.

Extremality

We use the extremality criterion Theorem 17 point 5 in [2]. The matrix A is extremal whenever every matrix B satisfying $(Bu_i)_j = 0$ whenever $(Au_i)_j = 0$ is proportional to A. Let us consider the elements $(Au_i)_j$.

The following elements are always zero:

$$(Au_1)_{1,2,3,4,6},\ (Au_2)_{1,2,3,6},\ (Au_3)_{1,2,4,5},\ (Au_4)_{3,4,5},\ (Au_5)_{4,5,6},\ (Au_6)_{1,5,6}. \tag{3}$$

The following elements may become zero:

$$(Au_2)_5 = 0 \quad \text{if} \quad \phi_1 + \phi_2 + \phi_3 + \phi_4 = \pi.$$

The following elements are always positive:

$$(Au_1)_5,\ (Au_2)_4,\ (Au_3)_{3,6},\ (Au_4)_{1,2,6},\ (Au_5)_{1,2,3},\ (Au_6)_{2,3,4}.$$

Since $a_{11} = 1$ and $B = A$ is always a solution, we may assume without loss of generality that $b_{11} = 0$, by possibly adding a multiple of A to a solution B. The relations on B from (3) then yield

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & b_{15} & b_{16} \\ 0 & 0 & 0 & 0 & b_{25} & -b_{16} \\ 0 & 0 & 0 & b_{34} & b_{35} & -b_{16} \\ 0 & 0 & b_{34} & 0 & -b_{25} & b_{46} \\ b_{15} & b_{25} & b_{35} & -b_{25} & b_{55} & b_{56} \\ b_{16} & -b_{16} & -b_{16} & b_{46} & b_{56} & b_{66} \end{pmatrix}$$

with the additional conditions

$$\begin{pmatrix} & & & \sin(\phi_1+\phi_2) & \sin\phi_1 & & & & \\ & & -\sin\phi_1 & \sin\phi_2 & & & & & \\ & & -\sin(\phi_1+\phi_2) & & \sin\phi_2 & & \sin\phi_1 & & \\ & & -\sin(\phi_2+\phi_3) & & & \sin\phi_2 & & & \\ & & -\sin\phi_3 & & & & \sin(\phi_2+\phi_3) & \sin\phi_2 & \\ & & & & & \sin\phi_3 & & \sin(\phi_2+\phi_3) & \sin\phi_2 \\ \sin\phi_4 & \sin(\phi_3+\phi_4) & & & & & & & \\ \sin\phi_3 & & & & & & \sin\phi_4 & \sin(\phi_3+\phi_4) & \\ & \sin\phi_3 & & & & & & \sin\phi_4 & \sin(\phi_3+\phi_4) \end{pmatrix} \begin{pmatrix} b_{15} \\ b_{16} \\ b_{25} \\ b_{34} \\ b_{35} \\ b_{46} \\ b_{55} \\ b_{56} \\ b_{66} \end{pmatrix} = 0$$

Without loss of generality we may assume $b_{25} = \sin\phi_1 \sin\phi_2$, $b_{34} = \sin^2\phi_1$ according to the second equation. Then we obtain

$$b_{35} = -\sin\phi_1 \sin(\phi_1+\phi_2),\ b_{55} = 2\sin(\phi_1+\phi_2)\sin\phi_2,\ b_{46} = \sin\phi_1\sin(\phi_2+\phi_3),\ b_{56} = \sin\phi_1\sin\phi_3 - 2\sin(\phi_1+\phi_2)\sin(\phi_2+\phi_3),$$

$$b_{66} = \frac{-\sin\phi_3\sin\phi_1\sin(\phi_2+\phi_3) - \sin(\phi_2+\phi_3)(\sin\phi_1\sin\phi_3 - 2\sin(\phi_1+\phi_2)\sin(\phi_2+\phi_3))}{\sin\phi_2} = 2\sin(\phi_2+\phi_3)\sin(\phi_1+\phi_2+\phi_3),$$

$$\begin{aligned} b_{15} &= \frac{-2\sin(\phi_1+\phi_2)\sin\phi_2\sin\phi_4 - \sin(\phi_3+\phi_4)(\sin\phi_1\sin\phi_3 - 2\sin(\phi_1+\phi_2)\sin(\phi_2+\phi_3))}{\sin\phi_3} \\ &= 2\sin(\phi_1+\phi_2)\sin(\phi_2+\phi_3+\phi_4) - \sin(\phi_3+\phi_4)\sin\phi_1, \end{aligned}$$

$$\begin{aligned} b_{16} &= \frac{-\sin\phi_4(\sin\phi_1\sin\phi_3 - 2\sin(\phi_1+\phi_2)\sin(\phi_2+\phi_3)) - 2\sin(\phi_2+\phi_3)\sin(\phi_1+\phi_2+\phi_3)\sin(\phi_3+\phi_4)}{\sin\phi_3} \\ &= -2\sin(\phi_2+\phi_3)\sin(\phi_1+\phi_2+\phi_3+\phi_4) - \sin\phi_4\sin\phi_1. \end{aligned}$$

The remaining 7th equation then gives

$$\sin\phi_4(2\sin(\phi_1+\phi_2)\sin(\phi_2+\phi_3+\phi_4) - \sin(\phi_3+\phi_4)\sin\phi_1) + \sin(\phi_3+\phi_4)(-2\sin(\phi_2+\phi_3)\sin(\phi_1+\phi_2+\phi_3+\phi_4) - \sin\phi_4\sin\phi_1) = 0,$$

and after simplification

$$\sin\phi_3 \sin(\phi_2+\phi_3+\phi_4)\sin(\phi_1+\phi_2+\phi_3+\phi_4) = 0.$$

The first two factors are always positive, and there is a non-trivial solution if and only if $\phi_1+\phi_2+\phi_3+\phi_4 = \pi$.

In this case there is the additional equation $(Bu_2)_5 = 0$, however, which amounts to

$$b_{15}+b_{35} = 2\sin(\phi_1+\phi_2)\sin(\phi_2+\phi_3+\phi_4) - \sin(\phi_3+\phi_4)\sin\phi_1 - \sin\phi_1\sin(\phi_1+\phi_2) = 0$$

and is trivially satisfied.

Thus A is extremal if and only if $\phi_1+\phi_2+\phi_3+\phi_4 < \pi$.

Result

In Case 5 the extremal matrices with unit diagonal are given by (1) with $\phi_1,\phi_2,\phi_3,\phi_4 > 0$, $\phi_1+\phi_2+\phi_3+\phi_4 < \pi$.

References

[1] Peter J.C. Dickinson. A new certificate for copositivity. Optimization Online.

[2] Peter J.C. Dickinson and Roland Hildebrand. Considering copositivity locally. *J. Math. Anal. Appl.*, 437(2):1184–1195, 2016.