Cases: ???

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1 Toolbox

In this we use the notation:

$$\begin{aligned}\mathcal{COP}^n &:= \{A \in \mathcal{S}^n \mid \mathbf{v}^\mathsf{T} A \mathbf{v} \geq 0 \quad \text{for all } \mathbf{v} \in \mathbb{R}^n_+\},\\ \mathcal{S}^n_+ &:= \{A \in \mathcal{S}^n \mid \mathbf{v}^\mathsf{T} A \mathbf{v} \geq 0 \quad \text{for all } \mathbf{v} \in \mathbb{R}^n\}\\ &= \operatorname{conv}\{\mathbf{v}\mathbf{v}^\mathsf{T} \mid \mathbf{v} \in \mathbb{R}^n\},\\ \mathcal{S}^n_1 &:= \{A \in \mathcal{S}^n \mid a_{ii} = 1 \text{ and } |a_{ij}| \leq 1 \quad \text{for all } i,j = 1,\ldots,n\}.\end{aligned}$$

1.1 Limiting the entries of A

We begin by considering the entries of a copositive matrix.

Lemma 1.1. *Consider $A \in \mathcal{COP}^n$ such that $a_{ii} = 1$ for all i. Then $a_{ij} \geq -1$ for all i,j.*

Proof. For all i,j we have $0 \leq (\mathbf{e}_i + \mathbf{e}_j)^\mathsf{T} A(\mathbf{e}_i + \mathbf{e}_j) = 2 + 2a_{ij}$. ☐

Lemma 1.2 ([1, Lemma 3.1])**.** *Consider $A \in \mathcal{S}^n$ such that $a_{ii} = 1$ for all i. Then $A \notin \mathcal{COP}^n$ if and only if there exists $\mathcal{I} \subseteq \{1,\ldots,n\}$ such that $A_\mathcal{I} \notin \mathcal{COP}^{|\mathcal{I}|}$ and $a_{ij} < 1$ for all $i,j \in \mathcal{I}$ with $i \neq j$.*

Corollary 1.3. *Consider $A \in \mathcal{COP}^n$ such that $a_{ii} = 1$ for all i and A is irreducible with respect to $\mathcal{N}^n$. Then $A \in \mathcal{S}^n_1$.*

Due to this result, from now on we will often limit ourselves to matrices in $\mathcal{S}^n_1$. We have the following two results on matrices in this set.

Lemma 1.4. *Consider $A \in \mathcal{S}^n_1$. Then for all i,j there exists a unique $\theta_{ij} \in [0,\pi]$ such that $a_{ij} = \cos\theta_{ij}$.*

Lemma 1.5. *For $a,b \in [-1,1]$ consider the following matrix:*

$$A = \begin{pmatrix} 1 & -1 & b \\ -1 & 1 & a \\ b & a & 1 \end{pmatrix}$$

Then we have $A \in \mathcal{COP}^3$ if and only if $a + b \geq 0$.

Proof. To prove the reverse implication we note that if $a + b \geq 0$ then from

$$A = \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix}^\mathsf{T} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a+b \\ 0 & a+b & 1-b^2 \end{pmatrix}$$

we have $A \in \mathcal{COP}^3$.

To prove the forward implication we note that if $A \in \mathcal{COP}^3$ then for all $\varepsilon > 0$ we have

$$0 \leq \begin{pmatrix} 1 \\ 1 \\ \varepsilon \end{pmatrix}^\mathsf{T} A \begin{pmatrix} 1 \\ 1 \\ \varepsilon \end{pmatrix} = 2\varepsilon(a+b) + \varepsilon^2,$$

which implies that $a + b \geq 0$. ☐

Lemma 1.6. *Let $A \in \mathcal{COP}^n \cap \mathcal{S}^n_1$ and $i,j,k \in \{1,\ldots,n\}$ such that $i \neq j \neq k \neq i$ and $-1 = a_{ij} = a_{jk}$. Then we have $a_{ik} = 1$.*

Proof. We have $0 \leq (\mathbf{e}_i + 2\mathbf{e}_j + \mathbf{e}_k)^\mathsf{T} A(\mathbf{e}_i + 2\mathbf{e}_j + \mathbf{e}_k) = 2a_{ik} - 2$. ☐

1.2 Set of zeros of A

We now consider the set of zeros of a matrix, starting with the following basic results.

Lemma 1.7. *For $A \in \mathcal{COP}^n$ and $\mathbf{v} \in \mathcal{V}^A$, letting $\mathcal{I} = \operatorname{supp}(\mathbf{v})$, we have*

$$\begin{aligned} A_{\mathcal{I}} &\in \mathcal{S}^n_+, \\ (A\mathbf{v})_{\mathcal{I}} &= A_{\mathcal{I}}\mathbf{v}_{\mathcal{I}} = \mathbf{0}, \\ (A\mathbf{v})_i &\geq 0 \qquad \text{for all } i \notin \mathcal{I}. \end{aligned}$$

Lemma 1.8. *For all $A \in \mathcal{S}^n$ we have $\mathcal{V}^A = \mathbb{R}_{++}\mathcal{V}^A$.*

Lemma 1.9. *Consider $A \in \mathcal{S}^n$ and a permutation matrix $P \in \mathbb{R}^{n\times n}$. Then $\mathcal{V}^{PAP^\mathsf{T}} = P\mathcal{V}^A$.*

We now consider the following result on the minimal zero support sets.

Lemma 1.10. *Let $A \in \mathcal{COP}^n \cap \mathcal{S}^n_1$ and $\mathbf{v} \in \mathcal{V}^A_{\min}$ such that $|\operatorname{supp}(\mathbf{v})| \geq 3$. Then $|a_{ij}| < 1$ for all $i, j \in \operatorname{supp}(\mathbf{v})$ such that $i \neq j$.*

Proof. Suppose for the sake of contradiction there exists $i, j \in \operatorname{supp}(\mathbf{v})$ with $i \neq j$ such that $a_{ij} = -1$. Then $(\mathbf{e}_i + \mathbf{e}_j) \in \mathcal{V}^A$ and $\operatorname{supp}(\mathbf{e}_i + \mathbf{e}_j)$ is strictly contained in $\mathcal{I}$.

Now suppose for the sake of contradiction there exists $i, j \in \operatorname{supp}(\mathbf{v})$ with $i \neq j$ such that $a_{ij} = 1$. Now letting $\mathbf{u} = \big(\mathbf{v} + v_j(\mathbf{e}_i - \mathbf{e}_j)\big)$, we have that $\mathbf{u} \neq \mathbf{0}$ and $\operatorname{supp}(\mathbf{u})$ is strictly contained in $\operatorname{supp}(\mathbf{v})$. By the minimality of $\mathbf{v}$ this implies the contradiction

$$0 < \mathbf{u}^\mathsf{T}A\mathbf{u} = \mathbf{v}^\mathsf{T}A\mathbf{v} + 2v_j(\mathbf{e}_i - \mathbf{e}_j)^\mathsf{T}A\mathbf{v} + v_j^2(\mathbf{e}_i - \mathbf{e}_j)^\mathsf{T}A(\mathbf{e}_i - \mathbf{e}_j) = 0. \qquad \square$$

Lemma 1.11. *Let $A \in \mathcal{COP}^n \cap \mathcal{S}^n_1$ and $\mathbf{u} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ such that $\operatorname{supp}(\mathbf{u}) = \{1, 2\}$. Then we have $\mathbf{u} \in \mathcal{V}^A_{\min}$ if and only if $a_{12} = -1$ and $u_1 = u_2$.*

Lemma 1.12. *Consider $A \in \mathcal{COP}^n \cap \mathcal{S}^n_1$ and $\mathbf{v} \in \mathbb{R}^n_+$ with $\operatorname{supp}(\mathbf{v}) = \{1, 2, 3\}$. Then we have the following:*

1. *If $\mathbf{v} \in \mathcal{V}^A$ then $2v_i \leq \sum_{j=1}^3 v_j$ for all $i = 1, 2, 3$.*
2. *If $\mathbf{v} \in \mathcal{V}^A_{\min}$ then $2v_i < \sum_{j=1}^3 v_j$ for all $i = 1, 2, 3$.*

Proof. If $\mathbf{v} \in \mathcal{V}^A$ then from Lemma 1.7, for all $i = 1, 2, 3$ we have

$$0 = (A\mathbf{v})_i = \sum_{j=1}^{3} a_{ij}v_j = v_i + \sum_{\substack{j=1,2,3 \\ j\neq i}} a_{ij}v_j \geq v_i - \sum_{\substack{j=1,2,3 \\ j\neq i}} v_j = 2v_i - \sum_{j=1}^{3} v_j.$$

This proves the first part.

Furthermore we note that if there exists $i \in \{1, 2, 3\}$ such that $2v_i = \sum_{j=1}^3 v_j$ then from the above inequality we get $a_{ij} = -1$ for all $j \in \mathcal{I}\setminus\{i\}$. In such a case, for all $j \in \{1, 2, 3\}\setminus\{i\}$ we have $(\mathbf{e}_i + \mathbf{e}_j) \in \mathcal{V}^A$ and $\operatorname{supp}(\mathbf{e}_i + \mathbf{e}_j)$ is strictly contained in $\{1, 2, 3\}$, and thus $\mathbf{v} \notin \mathcal{V}^A_{\min}$ $\square$

Lemma 1.13. *Consider $\mathbf{v} \in \mathbb{R}^n_+$ with $\operatorname{supp}(\mathbf{v}) = \{1, 2, 3\}$ and $2v_i < \sum_{j=1}^3 v_j$ for all $i = 1, 2, 3$. Then we have*

$$\left| \left\{ (\nu, \theta_1, \theta_2) \in \mathbb{R}^3_{++} \;\middle|\; \theta_1 + \theta_2 < \pi, \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \nu \begin{pmatrix} \sin(\theta_1) \\ \sin(\theta_2) \\ \sin(\theta_1 + \theta_2) \end{pmatrix} \right\} \right| = 1.$$

Proof. Consider arbitrary $\theta_1, \theta_2 > 0$ such that $\theta_1 + \theta_2 < \pi$. Letting $\theta_3 = \pi - \theta_1 - \theta_2 > 0$ we have $\theta_i + \theta_3 < \pi$ for all $i = 1, 2$ and

$$\begin{pmatrix} \sin(\theta_1) \\ \sin(\theta_2) \\ \sin(\theta_1 + \theta_2) \end{pmatrix} = \begin{pmatrix} \sin(\theta_2 + \theta_3) \\ \sin(\theta_2) \\ \sin(\theta_3) \end{pmatrix} = \begin{pmatrix} \sin(\theta_1) \\ \sin(\theta_1 + \theta_3) \\ \sin(\theta_3) \end{pmatrix}.$$

Therefore the result is independent of permutations and without loss of generality we may assume that $\mathcal{I} = \{1, 2, 3\}$ and $0 < v_1 \leq v_2 \leq v_3 < v_1 + v_2$.

If for some $\nu, \theta_1, \theta_2 > 0$ with $\theta_1 + \theta_2 < \pi$ we have

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \nu \begin{pmatrix} \sin(\theta_1) \\ \sin(\theta_2) \\ \sin(\theta_1 + \theta_2) \end{pmatrix},$$

Then we must have $\theta_1, \theta_2 \leq \pi/2$, otherwise we get the contradiction that $v_3 \leq \max\{v_1, v_2\}$.

For all $\nu \geq v_2$ and $i = 1, 2$ we let $\theta_i(\nu) \in (0, \pi/2]$ be such that $v_i = \nu \sin(\theta_i(\nu))$. Note that:

- the value of $\theta_i(\nu)$ is uniquely determined by the value of ν,
- the value of $\theta_i(\nu)$ varies continuously with ν,
- $\theta_i(\nu)$ is a strictly decreasing function with $\lim_{\nu\to\infty} \theta_i(\nu) = 0$,
- $\theta_2(v_2) = \pi/2$.

We now let

$$\begin{aligned} f(\nu) &= \nu \sin(\theta_1(\nu) + \theta_2(\nu)) \\ &= \nu \sin(\theta_1(\nu)) \cos(\theta_2(\nu)) + \nu \sin(\theta_2(\nu)) \cos(\theta_1(\nu)) \\ &= v_1 \cos(\theta_2(\nu)) + v_2 \cos(\theta_1(\nu)). \end{aligned}$$

We have that $f(\nu)$ is a strictly increasing function with $f(v_2) = v_2 \cos(\theta_1(v_2)) < v_2 \leq v_3$ and $\lim_{\nu\to\infty} f(\nu) = v_1 + v_2 > v_3$.

Therefore there exists a unique $\widehat{\nu} \in (v_2, \infty)$ such that $f(\widehat{\nu}) = v_3$, which completes the proof. □

Lemma 1.14. *Consider $n \geq 3$ and $A \in \mathcal{COP}^n \cap \mathcal{S}_1^n$ and $\mathcal{I} = \{1, 2, 3\}$ and $\nu, \theta_1, \theta_2 > 0$ such that $\theta_1 + \theta_2 < \pi$. Then the following are equivalent:*

1. $A_{\mathcal{I}} = \begin{pmatrix} 1 & \cos(\theta_1 + \theta_2) & -\cos(\theta_2) \\ \cos(\theta_1 + \theta_2) & 1 & -\cos(\theta_1) \\ -\cos(\theta_2) & -\cos(\theta_1) & 1 \end{pmatrix}$
2. $\nu \begin{pmatrix} \sin(\theta_1) \\ \sin(\theta_2) \\ \sin(\theta_1 + \theta_2) \\ \mathbf{0} \end{pmatrix} \in \mathcal{V}^A$
3. $\left\{\mathbf{v} \in \mathcal{V}^A \;\middle|\; \operatorname{supp}(\mathbf{v}) \subseteq \mathcal{I}\right\} = \mathbb{R}_{++} \begin{pmatrix} \sin(\theta_1) \\ \sin(\theta_2) \\ \sin(\theta_1 + \theta_2) \\ \mathbf{0} \end{pmatrix}.$

Proof. We now split this proof into three parts:

1⇒3: Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3_+$ such that

$$\mathbf{a} = \begin{pmatrix} -\cos(\theta_2) \\ -\cos(\theta_1) \\ 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} \sin(\theta_2) \\ -\sin(\theta_1) \\ 0 \end{pmatrix}.$$

These are linearly independent vectors and we have

$$\{\mathbf{u} \in \mathbb{R} \mid 0 = \mathbf{a}^\mathsf{T}\mathbf{u} = \mathbf{b}^\mathsf{T}\mathbf{u}\} = \mathbb{R} \begin{pmatrix} \sin(\theta_1) \\ \sin(\theta_2) \\ \sin(\theta_1 + \theta_2) \end{pmatrix}.$$

If $A_{\mathcal{I}}$ is as given in 1 then we have $A_{\mathcal{I}} = \mathbf{a}\mathbf{a}^\mathsf{T} + \mathbf{b}\mathbf{b}^\mathsf{T}$ and thus for $\mathbf{v} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ such that $\operatorname{supp}(\mathbf{v}) \subseteq \mathcal{I}$ we have

$$0 = \mathbf{v}^\mathsf{T} A \mathbf{v} \quad \Leftrightarrow \quad 0 = (\mathbf{a}^\mathsf{T}\mathbf{v}_{\mathcal{I}})^2 + (\mathbf{b}^\mathsf{T}\mathbf{v}_{\mathcal{I}})^2 \quad \Leftrightarrow \quad \mathbf{v}_{\mathcal{I}} \in \mathbb{R}_{++} \begin{pmatrix} \sin(\theta_1) \\ \sin(\theta_2) \\ \sin(\theta_1 + \theta_2) \end{pmatrix}$$

3⇒2: This comes directly from the definitions.

2⇒1: From Lemma 1.10 we have $|a_{ij}| < 1$ for all $i, j = 1, 2, 3$ such that $i \neq j$. Therefore there exists $\alpha_1, \alpha_2, \alpha_3 \in (0, \pi)$ such that

$$A_{\mathcal{I}} = \begin{pmatrix} 1 & \cos(\alpha_3) & -\cos(\alpha_2) \\ \cos(\alpha_3) & 1 & -\cos(\alpha_1) \\ -\cos(\alpha_2) & -\cos(\alpha_1) & 1 \end{pmatrix}.$$

From Lemma 1.7, $A_{\mathcal{I}}$ is a singular and thus

$$\begin{aligned} 0 = \det A_{\mathcal{I}} &= \sin^2(\alpha_1)\sin^2(\alpha_2) - (\cos(\alpha_3) - \cos(\alpha_1)\cos(\alpha_2))^2, \\ \cos(\alpha_3) &= \cos(\alpha_1)\cos(\alpha_2) \mp \sin(\alpha_1)\sin(\alpha_2) = \cos(\alpha_1 \pm \alpha_2). \end{aligned}$$

Suppose for the sake of contradiction that $\cos(\alpha_3) = \cos(\alpha_1 - \alpha_2)$ then

$$A_{\mathcal{I}} = \begin{pmatrix} -\cos(\alpha_2) \\ -\cos(\alpha_1) \\ 1 \end{pmatrix} \begin{pmatrix} -\cos(\alpha_2) \\ -\cos(\alpha_1) \\ 1 \end{pmatrix}^{\mathsf{T}} + \begin{pmatrix} \sin(\alpha_2) \\ \sin(\alpha_1) \\ 0 \end{pmatrix} \begin{pmatrix} \sin(\alpha_2) \\ \sin(\alpha_1) \\ 0 \end{pmatrix}^{\mathsf{T}}$$

We then we have the contradiction $\mathbf{u}^{\mathsf{T}} A_{\mathcal{I}} \mathbf{u} > 0$ for all $\mathbf{u} \in \mathbb{R}^3_{++}$.

Therefore we have $\cos(\alpha_3) = \cos(\alpha_1 + \alpha_2)$ and

$$\begin{aligned} A_{\mathcal{I}} &= \begin{pmatrix} -\cos(\alpha_2) \\ -\cos(\alpha_1) \\ 1 \end{pmatrix} \begin{pmatrix} -\cos(\alpha_2) \\ -\cos(\alpha_1) \\ 1 \end{pmatrix}^{\mathsf{T}} + \begin{pmatrix} \sin(\alpha_2) \\ -\sin(\alpha_1) \\ 0 \end{pmatrix} \begin{pmatrix} \sin(\alpha_2) \\ -\sin(\alpha_1) \\ 0 \end{pmatrix}^{\mathsf{T}}, \\ \{\mathbf{u} \in \mathbb{R}^3 \mid \mathbf{u}^{\mathsf{T}} A_{\mathcal{I}} \mathbf{u} = 0\} &= \mathbb{R} \begin{pmatrix} \sin(\alpha_1) \\ \sin(\alpha_2) \\ \sin(\alpha_1 + \alpha_2) \end{pmatrix}. \end{aligned}$$

Therefore we must have $\alpha_1 + \alpha_2 < \pi$ and from Lemma 1.13 we have $\alpha_i = \theta_i$ for $i = 1, 2$. □

Corollary 1.15. *Let $A \in \mathcal{COP}^n \cap \mathcal{S}^n_1$ and $\mathbf{u} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ and $\mathcal{I} = \operatorname{supp}(\mathbf{u})$ such that $|\mathcal{I}| = 3$. Then we have $\mathbf{u} \in \mathcal{V}^A_{\min}$ if and only if there exists $\nu, \theta_1, \theta_2 > 0$ such that $\theta_1 + \theta_2 < \pi$ and*

$$A_{\mathcal{I}} = \begin{pmatrix} 1 & \cos(\theta_1 + \theta_2) & -\cos(\theta_2) \\ \cos(\theta_1 + \theta_2) & 1 & -\cos(\theta_1) \\ -\cos(\theta_2) & -\cos(\theta_1) & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{u}_{\mathcal{I}} = \nu \begin{pmatrix} \sin(\theta_1) \\ \sin(\theta_2) \\ \sin(\theta_1 + \theta_2) \end{pmatrix}.$$

Corollary 1.16. *For $n \geq 4$ let $A \in \mathcal{COP}^n \cap \mathcal{S}^n_1$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ such that* $\operatorname{supp}(\mathbf{u}) = \{1, 3, 4\}$ *and* $\operatorname{supp}(\mathbf{v}) = \{2, 3, 4\}$.

Then we have $\mathbf{u}, \mathbf{v} \in \mathcal{V}^A_{\min}$ if and only if there exists $\nu, \lambda, \theta_0, \theta_1, \theta_2 > 0$ and $a \in \mathbb{R}$ such that $\theta_0 + \theta_i < \pi$ for $i = 1, 2$ and

$$A_{\{1,2,3,4\}} = \begin{pmatrix} 1 & a & \cos(\theta_0 + \theta_1) & -\cos(\theta_1) \\ a & 1 & \cos(\theta_0 + \theta_2) & -\cos(\theta_2) \\ \cos(\theta_0 + \theta_1) & \cos(\theta_0 + \theta_2) & 1 & -\cos(\theta_0) \\ -\cos(\theta_1) & -\cos(\theta_2) & -\cos(\theta_0) & 1 \end{pmatrix},$$

$$\mathbf{u}_{\{1,2,3,4\}} = \nu \begin{pmatrix} \sin(\theta_0) \\ 0 \\ \sin(\theta_1) \\ \sin(\theta_0 + \theta_1) \end{pmatrix}, \qquad \mathbf{v}_{\{1,2,3,4\}} = \lambda \begin{pmatrix} 0 \\ \sin(\theta_0) \\ \sin(\theta_2) \\ \sin(\theta_0 + \theta_2) \end{pmatrix}$$

1.3 Irreducibility of A

Lemma 1.17. *For $A \in \mathcal{COP}^n \cap \mathcal{S}^n_1$ we have the following:*

1. *A is irreducible with respect to $\mathcal{N}^n$ if and only if for all i, j there exists $\mathbf{u} \in \mathcal{V}^A_{\min}$ such that $u_i + u_j > 0$ and $(A\mathbf{u})_i = (A\mathbf{u})_j = 0$.*

2. *A is irreducible with respect to $\mathcal{S}^n_+$ if and only if* $\operatorname{span} \mathcal{V}^A_{\min} = \mathbb{R}^n$.

1.4 Technical results

Lemma 1.18. *For $a, b \in \mathbb{R}$ and $\theta_1, \theta_2 > 0$ such that $\theta_1 + \theta_2 < \pi$, consider the following matrix $A \in \mathcal{S}^4$:*

$$A = \begin{pmatrix} 1 & -1 & a & b \\ -1 & 1 & -\cos(\theta_2) & \cos(\theta_1+\theta_2) \\ a & -\cos(\theta_2) & 1 & -\cos(\theta_1) \\ b & \cos(\theta_1+\theta_2) & -\cos(\theta_1) & 1 \end{pmatrix}.$$

Then $A \in \mathcal{COP}^4$ if and only if $a \geq \cos(\theta_2)$ and $b \geq -\cos(\theta_1 + \theta_2)$.

Proof. In order to prove the forward implication we note that $\begin{pmatrix}1 & 1 & 0 & 0\end{pmatrix}^\mathsf{T} \in \mathcal{V}^A$ and thus if $A \in \mathcal{COP}^4$ then

$$\mathbf{0} \leq A \begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a - \cos(\theta_2) \\ b + \cos(\theta_1+\theta_2)\end{pmatrix}.$$

In order to prove the reverse implication we note that $\mathcal{COP}^4 = \mathcal{S}^4_+ + \mathcal{N}^4$ and

$$A = \begin{pmatrix} -1 \\ 1 \\ -\cos(\theta_2) \\ \cos(\theta_1+\theta_2)\end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ -\cos(\theta_2) \\ \cos(\theta_1+\theta_2)\end{pmatrix}^\mathsf{T} + \begin{pmatrix} 0 \\ 0 \\ \sin(\theta_2) \\ -\sin(\theta_1+\theta_2)\end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \sin(\theta_2) \\ -\sin(\theta_1+\theta_2)\end{pmatrix}^\mathsf{T}$$
$$+ \begin{pmatrix} 0 & 0 & a-\cos(\theta_2) & b+\cos(\theta_1+\theta_2) \\ 0 & 0 & 0 & 0 \\ a-\cos(\theta_2) & 0 & 0 & 0 \\ b+\cos(\theta_1+\theta_2) & 0 & 0 & 0\end{pmatrix}.$$

□

Lemma 1.19. *Let $A \in \mathcal{COP}^n \cap \mathcal{S}^n_1$. Then there does not exist $i_1, i_2, j_1, j_2, k \in \{1, \dots, n\}$, all mutually different, such that the minimal zero support set of A contains*

$$\{i_1, i_2\}, \qquad \{j_1, j_2\}, \qquad \{i_1, j_1, k\}, \qquad \{i_2, j_2, k\}$$

Proof. Suppose for the sake of contradiction that A does contain such a minimal zero support set.

Without loss of generality $i_1 = 1$, $i_2 = 2$, $j_1 = 3$, $j_2 = 4$, $k = 5$, and we have $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x} \in \mathcal{V}^A$ where

$$\mathbf{u} = \begin{pmatrix}1\\1\\0\\0\\0\\\mathbf{0}\end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix}0\\0\\1\\1\\0\\\mathbf{0}\end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix}\sin(\theta_1)\\0\\\sin(\theta_2)\\0\\\sin(\theta_1+\theta_2)\\\mathbf{0}\end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix}0\\\sin(\theta_3)\\0\\\sin(\theta_4)\\\sin(\theta_3+\theta_4)\\\mathbf{0}\end{pmatrix},$$

with $\theta_1, \theta_2, \theta_3, \theta_4 > 0$ and $\theta_1 + \theta_2 < \pi$ and $\theta_3 + \theta_4 < \pi$.

There exist $a, b \in \mathbb{R}$ such that

$$A_{\{1,2,3,4,5\}} = \begin{pmatrix} 1 & -1 & \cos(\theta_1+\theta_2) & a & -\cos(\theta_2) \\ -1 & 1 & b & \cos(\theta_3+\theta_4) & -\cos(\theta_4) \\ \cos(\theta_1+\theta_2) & b & 1 & -1 & -\cos(\theta_1) \\ a & \cos(\theta_3+\theta_4) & -1 & 1 & -\cos(\theta_3) \\ -\cos(\theta_2) & -\cos(\theta_4) & -\cos(\theta_1) & -\cos(\theta_3) & 1 \end{pmatrix}.$$

We must have $0 \leq (A\mathbf{u})_5 = -\cos(\theta_2) - \cos(\theta_4) = \cos(\pi - \theta_2) - \cos(\theta_4)$, which is equivalent to $\pi - \theta_2 \leq \theta_4$, which is in turn equivalent to $\theta_2 + \theta_4 \geq \pi$. Similarly $0 \leq (A\mathbf{v})_5$ holds if and only if $\theta_1 + \theta_3 \geq \pi$.

This then gives the contradiction $2\pi \leq (\theta_2 + \theta_4) + (\theta_1 + \theta_3) = (\theta_1 + \theta_2) + (\theta_3 + \theta_4) < 2\pi$. □

2 Cases 5, 7, 12, 13, 14, 15, 16, 56

From Lemma 1.19, there does not exist a copositive matrix with these minimal zero support sets.

3 Case 3

This is the case when we consider a matrix A with minimal zero support set:

$$\{1,2\}, \quad \{1,3\}, \quad \{1,4\}, \quad \{2,5\}, \quad \{3,5,6\}, \quad \{4,5,6\}. \tag{1}$$

From Corollary 1.16 without loss of generality we have $\mathcal{V}^A_{\min} = \mathbb{R}_{++}\mathcal{W}$, where

$$\mathcal{W} = \left\{ \begin{pmatrix}1\\1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\\sin(\theta_0)\\0\\\sin(\theta_1)\\\sin(\theta_0+\theta_1)\end{pmatrix}, \begin{pmatrix}0\\0\\0\\\sin(\theta_0)\\\sin(\theta_2)\\\sin(\theta_0+\theta_2)\end{pmatrix} \right\} \tag{2}$$

and $\theta_0 > 0$ and $0 < \theta_1 \le \theta_2 < \pi - \theta_0$.

We shall also consider the following matrix:

$$B = \begin{pmatrix} 1 & -1 & -1 & -1 & 1 & \cos(\theta_1) \\ -1 & 1 & 1 & 1 & -1 & \cos(\theta_0) \\ -1 & 1 & 1 & 1 & \cos(\theta_0+\theta_1) & -\cos(\theta_1) \\ -1 & 1 & 1 & 1 & \cos(\theta_0+\theta_2) & -\cos(\theta_2) \\ 1 & -1 & \cos(\theta_0+\theta_1) & \cos(\theta_0+\theta_2) & 1 & -\cos(\theta_0) \\ \cos(\theta_1) & \cos(\theta_0) & -\cos(\theta_1) & -\cos(\theta_2) & -\cos(\theta_0) & 1 \end{pmatrix}$$

Lemma 3.1. *For $A \in \mathcal{S}^6_1$ the following are equivalent:*

1. *$A \in \mathcal{COP}^6$ with $\mathcal{V}^A \supseteq \mathcal{W}$,*
2. *For all $i, j = 1, \dots, n$ with $i \le j$ we have*

$$\begin{aligned} a_{ij} &= b_{ij} && \text{if } (i,j) \ne (1,6),\ (2,6) \\ a_{ij} &\ge b_{ij} && \text{if } (i,j) = (1,6),\ (2,6) \end{aligned}$$

Proof. This follows directly from Lemmas 1.2, 1.5, 1.11 and 1.14, noting that we have $\cos(\theta_0)+\cos(\theta_1) > \cos(\theta_0) + \cos(\pi - \theta_0) = 0$. □

Lemma 3.2. *For $A \in \mathcal{S}^6_1$ the following are equivalent:*

1. *$A \in \mathcal{COP}^6$ and $\mathcal{W} \subseteq \mathcal{V}^A$ and A is irreducible with respect to $\mathcal{N}^6$,*
2. *We have $A = B$.*

Proof. The implication 1⇒2 follows directly from Lemma 4.1.

We will now prove the reverse implication. We trivially have $\mathcal{W} \subseteq \mathcal{V}^B$ and from Lemma 4.1 we have $B \in \mathcal{COP}^n$ and B is irreducible with respect to E_{16}, E_{26}.

We note from Lemmas 1.7 and 1.17 and the fact that $\mathcal{W} \subseteq \mathcal{V}^B$, we have that B is irreducible with respect to E_{11}, E_{12}, E_{13}, E_{14}, E_{22}, E_{25}, E_{33}, E_{35}, E_{36}, E_{44}, E_{45}, E_{46}, E_{55}, E_{56}, E_{66}.

We are thus left to show that B is irreducible with respect to $E_{15}, E_{23}, E_{24}, E_{34}$.

Considering $\begin{pmatrix}1, & 1, & 0, & 0, & 0, & 0\end{pmatrix}^\mathsf{T} \in \mathcal{V}^B$ we have

$$B \begin{pmatrix}1\\1\\0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\\cos(\theta_0)+\cos(\theta_1)\end{pmatrix}.$$

Therefore B is irreducible with respect to E_{15}, E_{23}, E_{24}.

Considering $\begin{pmatrix}1, & 0, & 0, & 1, & 0, & 0\end{pmatrix}^\mathsf{T} \in \mathcal{V}^B$ we have

$$B\begin{pmatrix}1\\0\\0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\1+\cos(\theta_0+\theta_2)\\\cos(\theta_1)-\cos(\theta_2)\end{pmatrix}.$$

Therefore B is irreducible with respect to E_{34}, which completes the proof. □

Lemma 3.3. *We have* $\mathcal{V}^B_{\min} = \mathbb{R}_{++}\mathcal{W}$.

Proof. There are trivially no zeros of B with support of cardinality one.

From Lemma 1.11, up to multiplication by a positive scalar, the zeros of B with support of cardinality two are exactly those given in $\mathcal{W}$.

From Lemma 1.10, if we wish to find minimal zeros of B whose support have cardinality strictly greater than two, we need only consider the maximal principle submatrices of B of order strictly greater than two and with no off-diagonal entries equal to plus or minus one. These are the principle submatrices

$$B_{\{3,5,6\}} = \begin{pmatrix}1 & \cos(\theta_0+\theta_1) & -\cos(\theta 1)\\ \cos(\theta_0+\theta_1) & 1 & -\cos(\theta_0)\\ -\cos(\theta_1) & -\cos(\theta_0) & 1\end{pmatrix},$$
$$B_{\{4,5,6\}} = \begin{pmatrix}1 & \cos(\theta_0+\theta_2) & -\cos(\theta 2)\\ \cos(\theta_0+\theta_2) & 1 & -\cos(\theta_0)\\ -\cos(\theta_2) & -\cos(\theta_0) & 1\end{pmatrix}.$$

The required result then immediately follows. □

Theorem 3.4. *We have B is irreducible with respect to $(\mathcal{S}^n_+ + \mathcal{N}^n)$ if and only if $\theta_1 < \theta_2$.*

Proof. From Lemma 3.2 we have that B is irreducible with respect to $\mathcal{N}^n$. We are thus left to show that B is irreducible with respect to $\mathcal{S}^n_+$ if and only if $\theta_1 < \theta_2$.

From Lemmas 1.17 and 3.3, B is reducible with respect to $\mathcal{S}^n_+$ if and only if

$$\mathbb{R}^6 \neq \operatorname{span}\mathcal{W}$$
$$= \left\{\begin{pmatrix}1\\1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\\sin(\theta_0)\\\sin(\theta_1)\\\sin(\theta_0+\theta_1))\end{pmatrix}, \begin{pmatrix}0\\0\\0\\\sin(\theta_0)\\\sin(\theta_2)\\\sin(\theta_0+\theta_2))\end{pmatrix}\right\},$$

which in turn holds if and only if

$$0 = \begin{vmatrix}1 & \sin(\theta_0) & \sin(\theta_0)\\ 1 & \sin(\theta_1) & \sin(\theta_2)\\ 0 & \sin(\theta_0+\theta_1) & \sin(\theta_0+\theta_2)\end{vmatrix} = \sin(\theta_0)\left(\sin(\theta_1)+\sin(\theta_2-\theta_1)-\sin(\theta_2)\right).$$

For a fixed $\theta_2 \in (0,\pi)$ we now let $f(\theta_1) = \sin(\theta_1)+\sin(\theta_2-\theta_1)-\sin(\theta_2)$.

We have $f(0) = f(\theta_2) = 0$, and $f''(\theta_1) = -\sin(\theta_1)-\sin(\theta_2-\theta_1) < 0$ for all $\theta_1 \in [0,\theta_2]$. Therefore $f(\theta_1) = 0$ if and only if $\theta_1 = \theta_2$, which completes the proof. □

4 Case 4

This is the case when we consider a matrix A with minimal zero support set:

$$\{1,2\}, \quad \{1,3\}, \quad \{1,4\}, \quad \{2,5,6\}, \quad \{3,5,6\}, \quad \{4,5,6\} \tag{3}$$

From Corollary 1.16 without loss of generality we have $\mathcal{V}^A_{\min} = \mathbb{R}_{++}\mathcal{W}$, where

$$\mathcal{W} = \left\{ \begin{pmatrix} 1\\1\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 1\\0\\1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 1\\0\\0\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\ \sin(\theta_0)\\0\\0\\ \sin(\theta_1)\\ \sin(\theta_0+\theta_1) \end{pmatrix}, \begin{pmatrix} 0\\0\\ \sin(\theta_0)\\0\\ \sin(\theta_2)\\ \sin(\theta_0+\theta_2)) \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\ \sin(\theta_0)\\ \sin(\theta_3)\\ \sin(\theta_0+\theta_3)) \end{pmatrix} \right\} \tag{4}$$

and $\theta_0 > 0$ and $\theta_1 \geq \theta_2 \geq \theta_3 > 0$ and $\theta_0 + \theta_1 < \pi$.

We shall also consider the following matrix:

$$B = \begin{pmatrix} 1 & -1 & -1 & -1 & -\cos(\theta_0+\theta_1) & \cos(\theta_3) \\ -1 & 1 & 1 & 1 & \cos(\theta_0+\theta_1) & -\cos(\theta_1) \\ -1 & 1 & 1 & 1 & \cos(\theta_0+\theta_2) & -\cos(\theta_2) \\ -1 & 1 & 1 & 1 & \cos(\theta_0+\theta_3) & -\cos(\theta_3) \\ -\cos(\theta_0+\theta_1) & \cos(\theta_0+\theta_1) & \cos(\theta_0+\theta_2) & \cos(\theta_0+\theta_3) & 1 & -\cos(\theta_0) \\ \cos(\theta_3) & -\cos(\theta_1) & -\cos(\theta_2) & -\cos(\theta_3) & -\cos(\theta_0) & 1 \end{pmatrix}. \tag{5}$$

Lemma 4.1. *For $A \in \mathcal{S}^6_1$ the following are equivalent:*

1. *$A \in \mathcal{COP}^n$ and $\mathcal{W} \subseteq \mathcal{V}^A$,*
2. *For all $i, j = 1, \ldots, n$ with $i \leq j$ we have*

$$\begin{aligned} a_{ij} &= b_{ij} && \text{if } (i,j) \neq (1,5), (1,6), \\ a_{ij} &\geq b_{ij} && \text{if } (i,j) = (1,5), (1,6). \end{aligned}$$

Proof. 1⇔2 follows directly from Lemmas 1.2, 1.11, 1.14 and 1.18. □

Lemma 4.2. *For $A \in \mathcal{S}^6_1$ the following are equivalent:*

1. *$A \in \mathcal{COP}^6$ and $\mathcal{W} \subseteq \mathcal{V}^A$ and A is irreducible with respect to $\mathcal{N}^6$,*
2. *We have $A = B$.*

Proof. The implication 1⇒2 follows directly from Lemma 4.1.

We will now prove the reverse implication. We trivially have $\mathcal{W} \subseteq \mathcal{V}^B$ and from Lemma 4.1 we have $B \in \mathcal{COP}^n$ and B is irreducible with respect to E_{15}, E_{16}.

We note from Lemmas 1.7 and 1.17 and the fact that $\mathcal{W} \subseteq \mathcal{V}^B$, we have that B is irreducible with respect to E_{11}, E_{12}, E_{13}, E_{14}, E_{22}, E_{25}, E_{26}, E_{33}, E_{35}, E_{36}, E_{44}, E_{45}, E_{46}, E_{55}, E_{56}, E_{66}.

We are thus left to show that B is irreducible with respect to E_{23}, E_{24}, E_{34}.

Considering $\begin{pmatrix}1, & 1, & 0, & 0, & 0, & 0\end{pmatrix}^\mathsf{T} \in \mathcal{V}^B$ we have

$$B \begin{pmatrix} 1\\1\\0\\0\\0\\0 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\ \cos(\theta_3) - \cos(\theta_1) \end{pmatrix}.$$

Therefore B is irreducible with respect to E_{23}, E_{24}.

Considering $\begin{pmatrix}1, & 0, & 0, & 1, & 0, & 0\end{pmatrix}^\mathsf{T} \in \mathcal{V}^B$ we have

$$B \begin{pmatrix} 1\\0\\0\\1\\0\\0 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\ \cos(\theta_0+\theta_3) - \cos(\theta_0+\theta_1) \\ 0 \end{pmatrix}.$$

Therefore B is irreducible with respect to E_{34}, which completes the proof. □

Lemma 4.3. *If $\theta_1 = \theta_2 = \theta_3$ then $B \in \mathcal{S}^n_+$.*

Proof. In such a case we have

$$B = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \\ -\cos(\theta_0+\theta_1) \\ \cos(\theta_1) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \\ -\cos(\theta_0+\theta_1) \\ \cos(\theta_1) \end{pmatrix}^\mathsf{T} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\sin(\theta_0+\theta_1) \\ \sin(\theta_1) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\sin(\theta_0+\theta_1) \\ \sin(\theta_1) \end{pmatrix}^\mathsf{T}. \qquad \square$$

Lemma 4.4. *If $\theta_1 > \theta_3$ then $\mathcal{V}^B_{\min} = \mathbb{R}_{++}\mathcal{W}$.*

Proof. There are trivially no zeros of B with support of cardinality one.

From Lemma 1.11, up to multiplication by a positive scalar, the zeros of B with support of cardinality two are exactly those given in $\mathcal{W}$.

From Lemma 1.10, if we wish to find minimal zeros of B whose support have cardinality strictly greater than two, we need only consider the maximal principle submatrices of B with no off-diagonal entries equal to plus or minus one. For $i = 1, 2, 3$ these are the principle submatrices

$$\begin{aligned} B_{\{i+1,5,6\}} &= \begin{pmatrix} 1 & \cos(\theta_0+\theta_i) & -\cos(\theta_i) \\ \cos(\theta_0+\theta_i) & 1 & -\cos(\theta_0) \\ -\cos(\theta_i) & -\cos(\theta_0) & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ \cos(\theta_0+\theta_i) \\ -\cos(\theta_i) \end{pmatrix} \begin{pmatrix} 1 \\ \cos(\theta_0+\theta_i) \\ -\cos(\theta_i) \end{pmatrix}^\mathsf{T} + \begin{pmatrix} 0 \\ \sin(\theta_0+\theta_i) \\ -\sin(\theta_i) \end{pmatrix} \begin{pmatrix} 0 \\ \sin(\theta_0+\theta_i) \\ -\sin(\theta_i) \end{pmatrix}^\mathsf{T}. \end{aligned}$$

and the principle submatrix

$$\begin{aligned} B_{\{1,5,6\}} &= \begin{pmatrix} 1 & -\cos(\theta_0+\theta_1) & \cos(\theta_3) \\ -\cos(\theta_0+\theta_1) & 1 & -\cos(\theta_0) \\ \cos(\theta_3) & -\cos(\theta_0) & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ \cos(\theta_0+\theta_i) \\ -\cos(\theta_i) \end{pmatrix} \begin{pmatrix} -1 \\ \cos(\theta_0+\theta_i) \\ -\cos(\theta_i) \end{pmatrix}^\mathsf{T} + \begin{pmatrix} 0 \\ \sin(\theta_0+\theta_i) \\ -\sin(\theta_i) \end{pmatrix} \begin{pmatrix} 0 \\ \sin(\theta_0+\theta_i) \\ -\sin(\theta_i) \end{pmatrix}^\mathsf{T} \\ &\quad + \begin{pmatrix} 0 & \cos(\theta_0+\theta_i)-\cos(\theta_0+\theta_1) & \cos(\theta_3)-\cos(\theta_i) \\ \cos(\theta_0+\theta_i)-\cos(\theta_0+\theta_1) & 0 & 0 \\ \cos(\theta_3)-\cos(\theta_i) & 0 & 0 \end{pmatrix}. \end{aligned}$$

We have $\cos(\theta_0+\theta_i) \ge \cos(\theta_0+\theta_1)$ and $\cos(\theta_3) \ge \cos(\theta_i)$, and at least one of these inequality relations is strict (otherwise $\theta_1 = \theta_i = \theta_3$).

The required result then immediately follows. $\square$

Lemma 4.5. *We have B is irreducible with respect to $\mathcal{S}^6_+$ if and only if $\theta_1 > \theta_2 > \theta_3$.*

Proof. If $\theta_1 = \theta_2 = \theta_3$ then from Lemma 4.3 we have that B is reducible with respect to $\mathcal{S}^6_+$.

If $\theta_1 > \theta_3$ then from Lemma 4.4 we have $\mathcal{V}^B_{\min} = \mathbb{R}_{++}\mathcal{W}$. From Lemma 1.17 we then have that B is reducible with respect to $\mathcal{S}^6_+$ if and only if

$$\begin{aligned} \mathbb{R}^6 &\neq \operatorname{span} \mathcal{W} \\ &= \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\sin(\theta_0) \\ 0 \\ 0 \\ 0 \\ \sin(\theta_1) \\ \sin(\theta_0+\theta_1) \end{pmatrix}, \begin{pmatrix} -\sin(\theta_0) \\ 0 \\ 0 \\ 0 \\ \sin(\theta_2) \\ \sin(\theta_0+\theta_2)) \end{pmatrix}, \begin{pmatrix} -\sin(\theta_0) \\ 0 \\ 0 \\ 0 \\ \sin(\theta_3) \\ \sin(\theta_0+\theta_3)) \end{pmatrix} \right\}. \end{aligned}$$

This in turn holds if and only if

$$\begin{aligned} 0 &= \begin{vmatrix} -\sin(\theta_0) & -\sin(\theta_0) & -\sin(\theta_0) \\ \sin(\theta_1) & \sin(\theta_2) & \sin(\theta_3) \\ \sin(\theta_0+\theta_1) & \sin(\theta_0+\theta_2) & \sin(\theta_0+\theta_3) \end{vmatrix} \\ &= -\sin^2(\theta_0)\left(\sin(\theta_1-\theta_2)(1-\cos(\theta_2-\theta_3)) + \sin(\theta_2-\theta_3)(1-\cos(\theta_1-\theta_2))\right) \end{aligned}$$

Finally, as $0 \leq \theta_1 - \theta_2 < \pi$ and $0 \leq \theta_2 - \theta_3 < \pi$, this holds if and only if either $\theta_2 = \theta_1$ or $\theta_2 = \theta_3$. □

Combining these results together we have the following theorem.

Theorem 4.6. *For $A \in \mathcal{S}_1^6$ the following are equivalent:*

1. *$A \in \mathcal{COP}^6$ and $\mathcal{W} \subseteq \mathcal{V}^A$ and A is irreducible with respect to $\mathcal{S}_+^6 + \mathcal{N}^6$,*
2. *We have $\theta_1 > \theta_2 > \theta_3$ and $A = B$.*

References

[1] Alan J. Hoffman and Francisco Pereira, *On copositive matrices with -1,0,1 entries*, Journal of Combinatorial Theory **14** (1973), no. 3, 302–309.