Case 11

Andrey Afonin, Roland Hildebrand

February 1, 2019

Basic structure

The minimal zero supports are given by $\{1,2\},\{1,3,4\},\{1,3,5\},\{1,4,6\},\{2,5,6\},\{3,5,6\}$. We may write a copositive matrix with this minimal zero support set as

$$A=\begin{pmatrix} 1 & -1 & -\cos(\phi_2) & -\cos(\phi_1) & \cos(\phi_2+\phi_3) & -\cos(\phi_4) \\ -1 & 1 & b_1 & b_2 & \cos(\phi_5+\phi_6) & -\cos(\phi_6) \\ -\cos(\phi_2) & b_1 & 1 & \cos(\phi_1+\phi_2) & -\cos(\phi_3) & \cos(\phi_3+\phi_5) \\ -\cos(\phi_1) & b_2 & \cos(\phi_1+\phi_2) & 1 & b_3 & \cos(\phi_1+\phi_4) \\ \cos(\phi_2+\phi_3) & \cos(\phi_5+\phi_6) & -\cos(\phi_3) & b_3 & 1 & -\cos(\phi_5) \\ -\cos(\phi_4) & -\cos(\phi_6) & \cos(\phi_3+\phi_5) & \cos(\phi_1+\phi_4) & -\cos(\phi_5) & 1 \end{pmatrix},$$

where $\phi_j \in (0,\pi)$, $j=1,\dots,6$; $\phi_i+\phi_j<\pi$, for the terms $\cos(\phi_i+\phi_j)$ appearing in the matrix.

The minimal zeros of A are given by

$$(u_1,u_2,u_3,u_4,u_5,u_6)=\begin{pmatrix}1\\1\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}\sin(\phi_1+\phi_2)\\0\\\sin(\phi_1)\\\sin(\phi_2)\\0\\0\end{pmatrix}\begin{pmatrix}\sin(\phi_3)\\0\\\sin(\phi_2+\phi_3)\\0\\\sin(\phi_2)\\0\end{pmatrix}\begin{pmatrix}\sin(\phi_1+\phi_4)\\0\\0\\\sin(\phi_4)\\0\\\sin(\phi_1)\end{pmatrix}\begin{pmatrix}0\\\sin(\phi_5)\\0\\0\\\sin(\phi_6)\\\sin(\phi_5+\phi_6)\end{pmatrix}\begin{pmatrix}0\\0\\\sin(\phi_5)\\0\\\sin(\phi_3+\phi_5)\\\sin(\phi_3)\end{pmatrix}$$

First order conditions

The conditions $(Au_i)_j \geq 0$ amount to

1. $$Au_1=\begin{pmatrix}0\\0\\b_1-\cos(\phi_2)\\b_2-\cos(\phi_1)\\\cos(\phi_2+\phi_3)+\cos(\phi_5+\phi_6)\\-\cos(\phi_4)-\cos(\phi_6)\end{pmatrix}\Rightarrow\begin{cases}b_1\geq\cos(\phi_2)\\b_2\geq\cos(\phi_1)\\\cos(\phi_2+\phi_3)+\cos(\phi_5+\phi_6)\geq 0\\-\cos(\phi_4)-\cos(\phi_6)\geq 0\end{cases}$$

2. $$Au_2=\begin{pmatrix}0\\b_1*\sin(\phi_1)+b_2*\sin(\phi_2)-\sin(\phi_1+\phi_2)\\0\\0\\b_3*\sin(\phi_2)+\sin(\phi_2)\cos(\phi_1+\phi_2+\phi_3)\\\sin(\phi_1)(\cos(\phi_3+\phi_5)-\cos(\phi_2-\phi_4))\end{pmatrix}\Rightarrow\begin{cases}b_1*\sin(\phi_1)+b_2*\sin(\phi_2)-\sin(\phi_1+\phi_2)\geq 0\\b_3\geq-\cos(\phi_1+\phi_2+\phi_3)\\\cos(\phi_3+\phi_5)-\cos(\phi_2-\phi_4)\geq 0\end{cases}$$

3. $$Au_3=\begin{pmatrix}0\\-\sin(\phi_3)+b_1*\sin(\phi_2+\phi_3)+\sin(\phi_2)\cos(\phi_5+\phi_6)\\0\\b_3*\sin(\phi_2)+\sin(\phi_2)\cos(\phi_1+\phi_2+\phi_3)\\0\\-\sin(\phi_3)\cos(\phi_4)+\sin(\phi_3)\cos(\phi_2+\phi_3+\phi_5)\end{pmatrix}\Rightarrow$$
$$\begin{cases}-\sin(\phi_3)+b_1*\sin(\phi_2+\phi_3)+\sin(\phi_2)\cos(\phi_5+\phi_6)\geq 0\\b_3\geq-\cos(\phi_1+\phi_2+\phi_3)\\-\cos(\phi_4)+\cos(\phi_2+\phi_3+\phi_5)\geq 0\end{cases}$$

4. $Au_4 = \begin{pmatrix} 0 \\ b_2 * \sin(\phi_4) - \sin(\phi_1+\phi_4) - \sin(\phi_1)\cos(\phi_6) \\ -\sin(\phi_1)\cos(\phi_2-\phi_4) + \sin(\phi_1)\cos(\phi_3+\phi_5) \\ 0 \\ b_3 * \sin(\phi_4) + \sin(\phi_1+\phi_4)\cos(\phi_2+\phi_3) - \sin(\phi_1)\cos(\phi_5) \\ 0 \end{pmatrix} \Rightarrow$

$$\begin{cases} b_2 * \sin(\phi_4) - \sin(\phi_1+\phi_4) - \sin(\phi_1)\cos(\phi_6) \geq 0 \\ -\cos(\phi_2-\phi_4) + \cos(\phi_3+\phi_5) \geq 0 \\ b_3 * \sin(\phi_4) + \sin(\phi_1+\phi_4)\cos(\phi_2+\phi_3) - \sin(\phi_1)\cos(\phi_5) \geq 0 \end{cases}$$

5. $Au_5 = \begin{pmatrix} -\sin(\phi_5) + \sin(\phi_6)\cos(\phi_2+\phi_3) - \sin(\phi_5+\phi_6)\cos(\phi_4) \\ 0 \\ b_1 * \sin(\phi_5) + \cos(\phi_3+\phi_5+\phi_6)\sin(\phi_5) \\ b_2 * \sin(\phi_5) + b_3 * \sin(\phi_6) + \sin(\phi_5+\phi_6)\cos(\phi_1+\phi_4) \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$$\begin{cases} -\sin(\phi_5) + \sin(\phi_6)\cos(\phi_2+\phi_3) - \sin(\phi_5+\phi_6)\cos(\phi_4) \geq 0 \\ b_1 \geq -\cos(\phi_3+\phi_5+\phi_6) \\ b_2 * \sin(\phi_5) + b_3 * \sin(\phi_6) + \sin(\phi_5+\phi_6)\cos(\phi_1+\phi_4) \geq 0 \end{cases}$$

6. $Au_6 = \begin{pmatrix} -\sin(\phi_3)\cos(\phi_4) + \sin(\phi_3)\cos(\phi_2+\phi_3+\phi_5) \\ b_1 * \sin(\phi_5) + \sin(\phi_5)\cos(\phi_3+\phi_5+\phi_6) \\ 0 \\ \sin(\phi_3)\cos(\phi_1+\phi_4) + \sin(\phi_5)\cos(\phi_1+\phi_2) + b_3\sin(\phi_3+\phi_5) \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$$\begin{cases} -\cos(\phi_4) + \cos(\phi_2+\phi_3+\phi_5) \geq 0 \\ b_1 \geq -\cos(\phi_3+\phi_5+\phi_6) \\ \sin(\phi_3)\cos(\phi_1+\phi_4) + \sin(\phi_5)\cos(\phi_1+\phi_2) + b_3\sin(\phi_3+\phi_5) \geq 0 \end{cases}$$

from inequalities on ϕ_i get $\begin{cases} \phi_2+\phi_3 \leq \pi - \phi_5 - \phi_6 \\ \pi - \phi_6 \leq \phi_4 \end{cases}$. This implies $\phi_2+\phi_3+\phi_5 \leq \phi_4$ and all inequalities $\phi_i + \phi_j < \pi$ above except $\phi_1 + \phi_4 < \pi$.

We also have

$$\phi_1 + \phi_2 + \phi_3 + \phi_5 < \pi, \quad \phi_1 < \phi_6.$$

Now under these conditions

$$\cos\phi_2 \geq \max(\frac{\sin\phi_3 - \sin\phi_2\cos(\phi_5+\phi_6)}{\sin(\phi_2+\phi_3)}, -\cos(\phi_3+\phi_5+\phi_6))$$

and hence $b_1 = -\cos\phi_2$.

Further

$$\cos\phi_1 \geq \frac{\sin(\phi_1+\phi_4) + \sin\phi_1\cos\phi_6}{\sin\phi_4}, \quad \frac{\sin\phi_1\cos\phi_5 - \sin(\phi_1+\phi_4)\cos(\phi_2+\phi_3)}{\sin\phi_4} \geq -\cos(\phi_1+\phi_2+\phi_3),$$

$$-\frac{\cos\phi_1\sin\phi_5 + \sin(\phi_5+\phi_6)\cos(\phi_1+\phi_4)}{\sin\phi_6} \geq -\frac{\sin\phi_3\cos(\phi_1+\phi_4) + \sin\phi_5\cos(\phi_1+\phi_2)}{\sin(\phi_3+\phi_5)}.$$

We therefore get the following possibilities for the values of b_2, b_3.

Case 1: If $\frac{\sin\phi_1\cos\phi_5 - \sin(\phi_1+\phi_4)\cos(\phi_2+\phi_3)}{\sin\phi_4} \geq -\frac{\cos\phi_1\sin\phi_5 + \sin(\phi_5+\phi_6)\cos(\phi_1+\phi_4)}{\sin\phi_6} \geq -\frac{\sin\phi_3\cos(\phi_1+\phi_4) + \sin\phi_5\cos(\phi_1+\phi_2)}{\sin(\phi_3+\phi_5)}$, then

$$b_2 = \cos\phi_1, \quad b_3 = \frac{\sin\phi_1\cos\phi_5 - \sin(\phi_1+\phi_4)\cos(\phi_2+\phi_3)}{\sin\phi_4}.$$

If $-\frac{\cos\phi_1\sin\phi_5+\sin(\phi_5+\phi_6)\cos(\phi_1+\phi_4)}{\sin\phi_6} \geq \frac{\sin\phi_1\cos\phi_5-\sin(\phi_1+\phi_4)\cos(\phi_2+\phi_3)}{\sin\phi_4} \geq -\frac{\sin\phi_3\cos(\phi_1+\phi_4)+\sin\phi_5\cos(\phi_1+\phi_2)}{\sin(\phi_3+\phi_5)}$, then the constraint $(Au_5)_4 \geq 0$ which involves both b_2, b_3 is active and we have the possibilities

Case 2:

$$b_2 = \frac{-\sin\phi_1\cos\phi_5\sin\phi_6 + \sin(\phi_1+\phi_4)\cos(\phi_2+\phi_3)\sin\phi_6 - \sin(\phi_5+\phi_6)\cos(\phi_1+\phi_4)\sin\phi_4}{\sin\phi_4\sin\phi_5}$$

$$b_3 = \frac{\sin\phi_1\cos\phi_5 - \sin(\phi_1+\phi_4)\cos(\phi_2+\phi_3)}{\sin\phi_4},$$

Case 3:

$$b_2 = \cos\phi_1, \quad b_3 = -\frac{\cos\phi_1\sin\phi_5 + \sin(\phi_5+\phi_6)\cos(\phi_1+\phi_4)}{\sin\phi_6}.$$

If $-\frac{\cos\phi_1\sin\phi_5+\sin(\phi_5+\phi_6)\cos(\phi_1+\phi_4)}{\sin\phi_6} \geq -\frac{\sin\phi_3\cos(\phi_1+\phi_4)+\sin\phi_5\cos(\phi_1+\phi_2)}{\sin(\phi_3+\phi_5)} \geq \frac{\sin\phi_1\cos\phi_5-\sin(\phi_1+\phi_4)\cos(\phi_2+\phi_3)}{\sin\phi_4}$, then the constraint $(Au_5)_4 \geq 0$ is active and we have the possibilities

Case 4:

$$\begin{aligned} b_2 &= \frac{\sin\phi_3\cos(\phi_1+\phi_4)\sin\phi_6 + \sin\phi_5\cos(\phi_1+\phi_2)\sin\phi_6 - \sin(\phi_5+\phi_6)\cos(\phi_1+\phi_4)\sin(\phi_3+\phi_5)}{\sin\phi_5\sin(\phi_3+\phi_5)} \\ &= \frac{-\cos(\phi_1+\phi_4)\sin(\phi_3+\phi_5+\phi_6) + \sin\phi_6\cos(\phi_1+\phi_2)}{\sin(\phi_3+\phi_5)}, \end{aligned}$$

$$b_3 = -\frac{\sin\phi_3\cos(\phi_1+\phi_4) + \sin\phi_5\cos(\phi_1+\phi_2)}{\sin(\phi_3+\phi_5)}$$

or again Case 3.

Copositivity

Consider the submatrix $A_{\{456\}}$. Set $b_3 = -\cos\varphi$, $\varphi \in [0,\pi)$. Then we must have $\varphi+\phi_5+\pi-\phi_1-\phi_4 > \pi$, otherwise the submatrix is not copositive or (in case of an equality) we get another minimal zero with support a subset of $\{4,5,6\}$. This is equivalent to $\varphi > \phi_1+\phi_4-\phi_5$. But $\phi_1+\phi_4-\phi_5 \in (0,\pi)$, hence we get $b_3 > -\cos(\phi_1+\phi_4-\phi_5)$.

It needs to be checked whether we can have

$$-\cos(\phi_1+\phi_4-\phi_5) < \max\left(-\frac{\cos\phi_1\sin\phi_5 + \sin(\phi_5+\phi_6)\cos(\phi_1+\phi_4)}{\sin\phi_6}, \frac{\sin\phi_1\cos\phi_5 - \sin(\phi_1+\phi_4)\cos(\phi_2+\phi_3)}{\sin\phi_4}\right).$$

1. $-\cos(\phi_1+\phi_4-\phi_5)\sin(\phi_6) \leq -(\cos\phi_1\sin\phi_5+\sin(\phi_5+\phi_6)\cos(\phi_1+\phi_4)) => cos(\phi_1) \leq -cos(\phi_1+\phi_4+\phi_6), \phi_1+\phi_4+\phi_6 > \pi => \pi \geq \phi_4+\phi_6 \Rightarrow$ Performs only if $\pi = \phi_4+\phi_6, b_3 = -\cos(\phi_1+\phi_4-\phi_5)$ And there is additional zero with sup (4,5,6)

2. $-\cos(\phi_1+\phi_4-\phi_5)sin(\phi_4) < \sin\phi_1\cos\phi_5 - \sin(\phi_1+\phi_4)\cos(\phi_2+\phi_3), -cos(\phi_2+\phi_3) \leq -cos(\phi_4-\phi_5) => -\cos(\phi_1+\phi_4-\phi_5)sin(\phi_4) < \sin\phi_1\cos\phi_5 - \sin(\phi_1+\phi_4)\cos(\phi_4-\phi_5) => 0 < 0 \Rightarrow$ Performs only if $cos(\phi_2+\phi_3) = cos(\phi_4-\phi_5), b_3 = -\cos(\phi_1+\phi_4-\phi_5)$ And there is additional zero with sup (4,5,6).

Result

There are no extremal exceptional copositive matrices with such a minimal zero support set.