Self-associated three-dimensional cones

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Outline

1. Simon-Wang theory
   - Calabi hypothesis on affine spheres
   - Wang's equation
   - Associated cones

2. Self-associated cones
   - Analytic condition
   - Reduced Wang's equation and Painlevé III
   - Description of the cone boundary
   - Classification

Calabi theorem

Theorem

Let $K \subset \mathbb{R}^n$ be a regular convex cone. Then there exists a unique complete hyperbolic affine sphere with mean curvature -1 which is asymptotic to the boundary ∂K of the cone. Every complete hyperbolic affine sphere is asymptotic to the boundary of a regular convex cone.

- conjectured by E. Calabi in 1972
- proven by S.Y. Cheng, S.T. Yau, T. Sasaki, A.-M. Li 1977–92
- equips properly convex sets in $\mathbb{RP}^{n-1}$ with a complete metric h and a trace-less cubic form C
- K can be reconstructed from h, C up to $SL(n, \mathbb{R})$ action

Three-dimensional cones

$n = 3$: affine sphere is a simply connected *Riemann surface*

introduce a *global* isothermal coordinate $z = x + iy \in M \subset \mathbb{C}$

conformal class: $M \sim \mathbb{D}$ or $M = \mathbb{C}$

- metric can be written $h = e^u |dz|^2$ with u a real scalar function
- cubic form equals $C = 2\,Re\,(Udz^3)$ with U a *holomorphic* function

Wang's equation

the real scalar u and the holomorphic function U on M satisfy the compatibility condition

$$e^u = \frac{1}{2}\Delta u + 2|U|^2 e^{-2u}$$

with $\Delta u = u_{xx} + u_{yy} = 4u_{z\bar{z}}$ and $e^u|dz|^2$ complete

- u determines U up to a multiplicative unimodular constant
- if $M \sim \mathbb{D}$ and $U\,dz^3$ is bounded in the uniformizing metric [Benoist, Hulin 2014] or $M = \mathbb{C}$ and U is polynomial [Dumas, Wolf 2015], then U determines u uniquely

Simon-Wang theorem

Theorem (Simon, Wang 1993)

Let (u, U) be a solution of Wang's equation on a simply connected domain $M \subset \mathbb{C}$ such that $h = e^u|dz|^2$ is complete and U is holomorphic. Then there exists a regular convex cone $K \subset \mathbb{R}^3$ such that the affine sphere which is asymptotic to ∂K has metric h and cubic form $C = 2\,Re(U\,dz^3)$. The $SL(3, \mathbb{R})$-orbit of K is uniquely determined.
A regular convex cone $K \subset \mathbb{R}^3$ defines a solution of Wang's equation via the affine sphere which is asymptotic to ∂K. This solution is determined up to conformal isomorphisms of the domain M.

Moving frame

to reconstruct the affine sphere $f : M \to \mathbb{R}^3$ and the cone K from a solution (u, U) one integrates the *frame equations*

$$F_x = F \begin{pmatrix} -e^{-u} Re\, U & \frac{u_y}{2} + e^{-u} Im\, U & e^{u/2} \\ -\frac{u_y}{2} + e^{-u} Im\, U & e^{-u} Re\, U & 0 \\ e^{u/2} & 0 & 0 \end{pmatrix},$$

$$F_y = F \begin{pmatrix} e^{-u} Im\, U & -\frac{u_x}{2} + e^{-u} Re\, U & 0 \\ \frac{u_x}{2} + e^{-u} Re\, U & -e^{-u} Im\, U & e^{u/2} \\ 0 & e^{u/2} & 0 \end{pmatrix}.$$

here $F = (e^{-u/2} f_x, e^{-u/2} f_y, f) \in SL(3, \mathbb{R})$ is the *moving frame*

Associated cones

let K be a convex cone and (u, U) a corresponding solution of Wang's equation

multiplying U by $e^{i\varphi}$ yields another solution and a corresponding $SL(3, \mathbb{R})$-orbit of convex cones

cones obtained this way are called *associated* with K

the set of $SL(3, \mathbb{R})$-orbits of all associated cones is called *associated family*

- on an associated family acts the circle group S^1
- the action of -1 yields the orbit of *dual* cones
- the associated families of semi-homogeneous cones have been computed in [Z. Lin, E. Wang 2016]

Orientation-reversing isomorphisms

Definition

A regular convex cone $K \subset \mathbb{R}^3$ is *self-associated* if all its associated cones are linearly isomorphic to K.

Lemma

A regular convex cone $K \subset \mathbb{R}^3$ is self-associated if and only if all cones which are associated to K are in the $SL(3, \mathbb{R})$-orbit of K.

Killing vector field

Theorem

Let (u, U) be a solution of Wang's equation on $M \subset \mathbb{C}$ corresponding to a self-associated cone. Then there exists a Killing vector field on M, given by a holomorphic function ψ on M satisfying

$$\begin{aligned} iU(z) + U'(z)\psi(z) + 3U(z)\psi'(z) &= 0, \\ Re(u'(z)\psi(z) + \psi'(z)) &= 0, \end{aligned}$$

On the other hand, if such a vector field exists for some solution (u, U), then the corresponding cone K is self-associated.

Affine spheres with Killing vectors

Theorem

Let $K \subset \mathbb{R}^3$ be a regular convex cone such that its affine sphere possesses a continuous group of isometries. Then K is self-associated or semi-homogeneous.

only ellipsoidal and simplicial cones in $\mathbb{R}^3$ are both self-associated and semi-homogeneous

Classification of (ψ, U)

Lemma

Let (u, U) be a solution of Wang's equation on $M \subset \mathbb{C}$ corresponding to a self-associated cone and ψ a corresponding Killing vector field. Then by a conformal isomorphism (ψ, U) reduces to one of the cases

(0) $U \equiv 0$, *corresponds to ellipsoidal cones*

(R) $M = B_R$, $U = z^k$, $\psi = -\frac{iz}{k+3}$, $(k, R) \in \mathbb{N} \times (0, \infty]$,

(T) $M = (a, b) + i\mathbb{R}$, $U = e^z$, $\psi = -i$, $-\infty \le a < b \le +\infty$.

- ψ generates an automorphism subgroup of M
- $e^{2\pi\psi}$ generates an isomorphism $T \in SL(3, \mathbb{R})$ of K

Types of cones

Lemma

Let K be a self-associated cone and T the linear isomorphism generated by the corresponding isometry $e^{2\pi\psi}$. Then one of the three following mutually exclusive conditions holds:

(E) *There exists an integer $q \geq 3$ such that $T^q = I$.*

(P) *T has the eigenvalue 1 with algebraic multiplicity 3 and geometric multiplicity 1.*

(H) *The spectrum of T is given by $\{1, \lambda, \lambda^{-1}\}$ with real $\lambda > 1$.*

we shall call the corresponding cones of *elliptic, parabolic* and *hyperbolic type*

Reducing Wang's equation

(R): the radial symmetry of the metric leads to

$$\frac{d^2\upsilon}{dr^2} = 2e^{\upsilon} - \frac{1}{r}\frac{d\upsilon}{dr} - 4r^{2k}e^{-2\upsilon}$$

with υ even on $(-R, R)$ and $e^{\upsilon}\,dr^2$ complete

(T): the translational symmetry leads to

$$\frac{d^2\upsilon}{dx^2} = 2e^{\upsilon} - 4e^{2x}e^{-2\upsilon}$$

with $e^{\upsilon}\,dx^2$ complete on (a, b)

Lemma

For both equations the solution υ exists and is unique. Both equations are equivalent to Painlevé III of type D_7 with $\beta = 0$.

Approximating frames

the boundary is determined by the asymptotics of the affine sphere f as $z \to \partial M$

to integrate the moving frame F we introduce an explicit *approximating frame* $V : M \to SL(3, \mathbb{R})$ such that $G = FV^{-1}$ is finite as $z \to \partial M$

recall $F = (e^{-u/2} f_x, e^{-u/2} f_y, f)$

find a scalar $\gamma > 0$ such that $\gamma \cdot Ve_3$ remains finite as $z \to \partial M$

then $\gamma \cdot f = G \cdot (\gamma Ve_3)$ tends to a point v on ∂K

Differential equation

the moving frame equation on F implies a similar eq. on G

for R finite in case (R) and b finite in case (T) this gives

$$v''' + \alpha v' + \beta \cdot \sin t \cdot v = 0$$

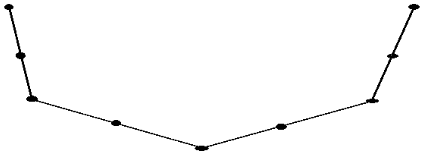
this is a linear third-order 2π-periodic ODE on v

the monodromy of the equation is adjoint to the isomorphism T

Polyhedral boundary

for $R = +\infty$ in case (R) and $b = +\infty$ in case (T) the vector function $v(t)$ is piece-wise constant

the values of v alternate between corners and edges $\Rightarrow$ the boundary ∂K is polyhedral



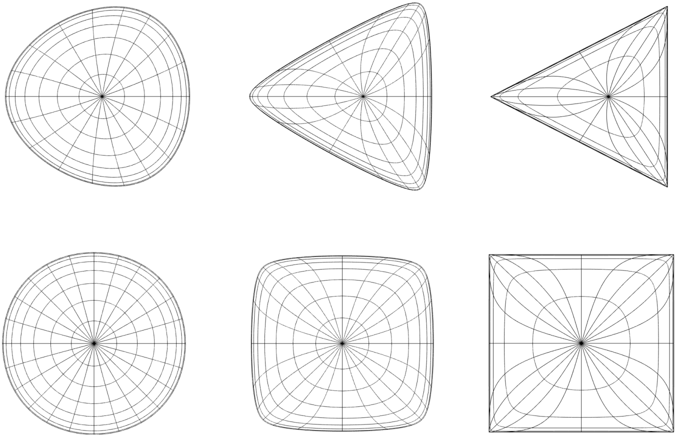
Classification

type	symmetry	domain M	parameters
(E)	(R)	B_R, $R \in (0, +\infty]$	$q = k + 3$, R
(P)	(T)	$(-\infty, b) + i\mathbb{R}$, $b \in (-\infty, +\infty]$	b
(H)	(T)	$(a, b) + i\mathbb{R}$, $-\infty < a < b \leq \infty$	a, b

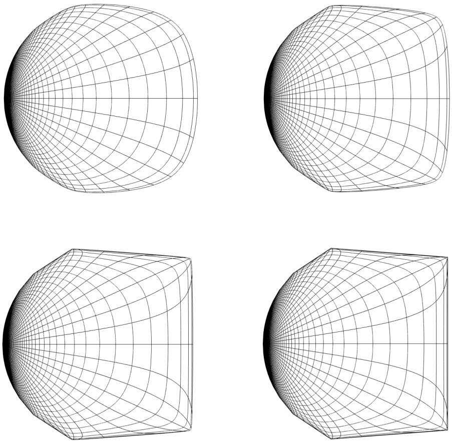
in addition to the isomorphism T there exists an orientation-reversing reflection Σ corresponding to the anti-conformal automorphism $z \mapsto \bar{z}$ of M

together these generate the dihedral group D_q in case (R) and the infinite dihedral group D_∞ in case (T)

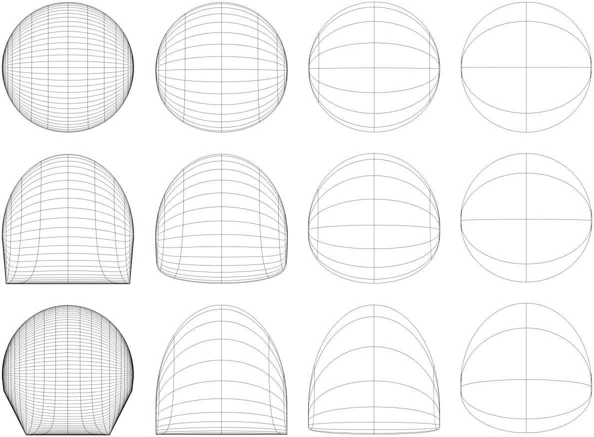
Cones of elliptic type



Cones of parabolic type



Cones of hyperbolic type



Thank you