Semidefinite Representations of Sets Delineated by Plane Quartics

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Outline

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1. Semi-definite representability
   - Semi-algebraic sets
   - Semi-definite representations
2. Representation of planar quartic sets
   - Planar sets and their homogenizations
   - Lasserre construction and its homogenization
   - Representation of planar quartic sets

Semi-algebraic sets

Definition

A subset

$$S = \{x \in \mathbb{R}^n \,|\, p_k(x) = 0,\ k = 1,\dots,m;\ q_l(x) > 0,\ l = 1,\dots,m'\}$$

given by a finite number of polynomial equalities and inequalities is called basic semi-algebraic.

Definition

A set $S \subset \mathbb{R}^n$ which is a finite union of basic-semi-algebraic sets is called semi-algebraic.

Conic semi-algebraic sets

Lemma

Let $S \subset \mathbb{R}^n$ be a semi-algebraic set. Then the set

$$\tilde{S} = \{\tilde{x} = (\lambda, \lambda x^T)^T \mid \lambda \geq 0,\ x \in S\} \subset \mathbb{R}^{n+1}$$

is also semi-algebraic.

- homogenize polynomials
- add constraint $x_0 > 0$
- unite with $\{0\}$

Semi-definite representability

Definition

A cone K is called semi-definite representable if it is linearly isomorphic to a linear projection of a linear section of $S_+(n)$ for some n.

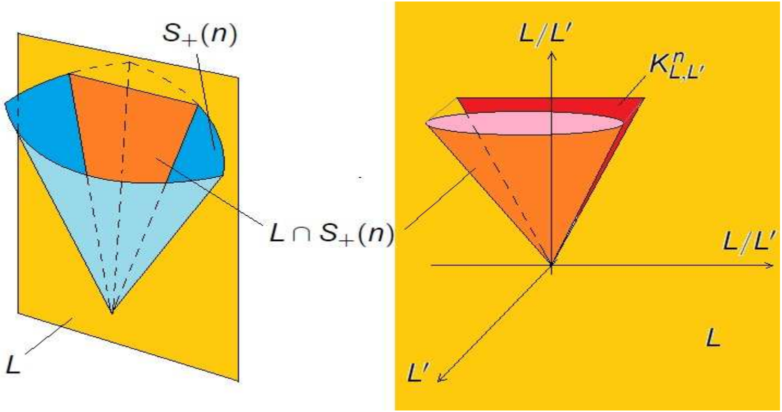
- linear intersection with subspace $L \subset \mathcal{S}(n)$
- linear projection along subspace $L' \subset L$

assume $L \cap S_{++}(n) \neq \emptyset$

K linearly isomorphic to

$$K^n_{L,L'} = \{x \in L/L' \mid \exists\, y \in x : \quad y \in L \cap S_+(n)\}$$

Semi-definite representable cones



Example

Nonnegative ternary quartics [Hilbert, 1888]:

$$\sum_{\alpha+\beta\le 4} c_{\alpha\beta}x^\alpha y^\beta \ge 0 \quad \forall\, x,y\in\mathbb{R}$$

$$\Leftrightarrow \qquad \exists\, a_{02},a_{20},a_{22},a_{21},a_{12},a_{11}\in\mathbb{R}:$$

$$\begin{pmatrix} 2c_{40} & c_{22}-a_{22} & c_{20}-a_{20} & c_{21}-a_{21} & c_{30} & c_{31} \\ c_{22}-a_{22} & 2c_{04} & c_{02}-a_{02} & c_{03} & c_{12}-a_{12} & c_{13} \\ c_{20}-a_{20} & c_{02}-a_{02} & 2c_{00} & c_{01} & c_{10} & c_{11}-a_{11} \\ c_{21}-a_{21} & c_{03} & c_{01} & 2a_{02} & a_{11} & a_{12} \\ c_{30} & c_{12}-a_{12} & c_{10} & a_{11} & 2a_{20} & a_{21} \\ c_{31} & c_{13} & c_{11}-a_{11} & a_{12} & a_{21} & 2a_{22} \end{pmatrix} \succeq 0$$

$n=6$, $\dim L=\dim\mathcal{S}(6)=21$, $\dim L'=6$,
$\dim K=\dim L/L'=15$

Duality

Definition

Let $K \subset \mathbb{R}^n$ be a convex cone. The dual cone to K is given by

$$K^* = \{y \in \mathbb{R}_n \,|\, \langle x, y\rangle \geq 0 \quad \forall\, x \in K\}.$$

Theorem

Let $L' \subset L \subset \mathcal{S}(n)$ be linear subspaces, $L \cap S_{++}(n) \neq \emptyset$. Then

$$(K^n_{L,L'})^* = K^n_{L'^\perp, L^\perp}$$

Here $L'^\perp, L^\perp$ are the orthogonal complements of L', L.

Problem formulation

necessary conditions for semi-definite representability of K

- K is convex
- K is semi-algebraic

Is every regular convex semi-algebraic cone semi-definite representable?

this talk

- $\dim K = 3$
- defining polynomials are quartics

Known results

- semi-definite representability local property of the boundary [Helton, Nie 2009]
- smooth boundary patches with positive curvature are not an obstacle [Helton, Nie 2010]
- more explicit construction in [Nie, 2010] with convexity assumptions on the defining polynomials

- the semi-definite representation is a member of the Lasserre hierarchy [Lasserre, 2009]
- degree of the LMI can be arbitrarily high

[Henrion, 2009]: semi-definite representation of convex hulls of certain rational algebraic varieties and of zero sets of convex polynomials with fixed block size

Planar semi-algebraic sets

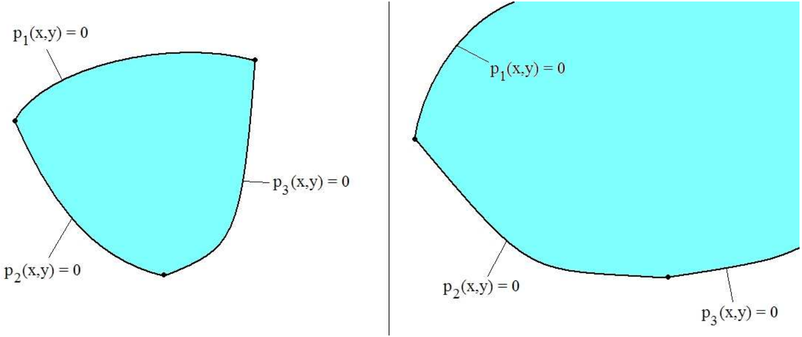
Lemma

A closed convex semi-algebraic set $S \subset \mathbb{R}^2$ is bounded by a finite number of arcs. The Zariski closure of each arc is a plane algebraic curve, which is the zero set of some nonzero irreducible polynomial.

w.r.o.g.

- the interior of each arc consists of nonsingular points
- in the interior of each arc the curvature is nonzero
- defining polynomials are positive on the inside of S

Compact and noncompact case



Homogenization

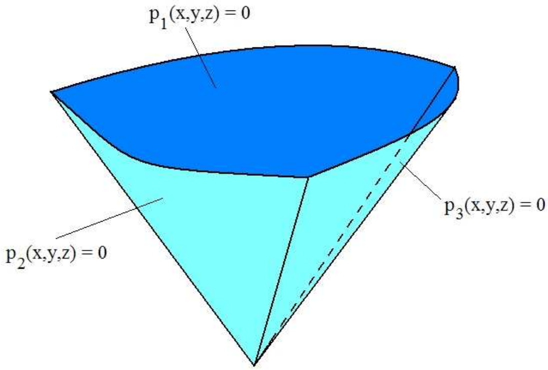
Lemma

A regular convex semi-algebraic cone $K \subset \mathbb{R}^3$ is bounded by a finite number of conic surface patches. The Zariski closure of each patch corresponds to a plane projective algebraic curve, which is the zero set of some nonzero irreducible polynomial.

w.r.o.g.

- the interior of each patch consists of nonsingular points
- in the interior of each patch the curvature is nonzero
- defining polynomials are positive on the inside of K

3-dimensional cones



Regularity condition

$\Delta \subset \partial S$ — boundary patch defined by $p(x,y,z) = 0$

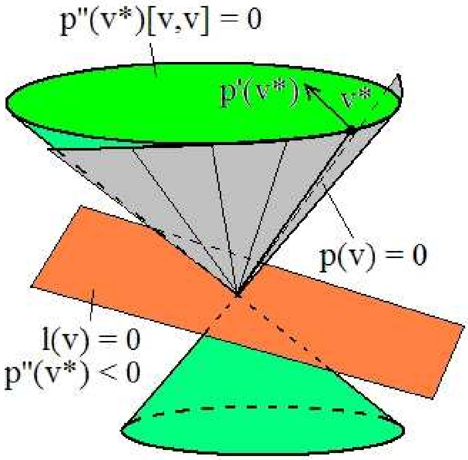
$v^* = (x^*, y^*, z^*) \in \Delta^o$ — interior point

v^* nonsingular: $p'(v^*) \neq 0$

curvature at v^* nonzero: $p''(v^*) < 0$ on the direction $v^* \times p'(v^*)$
$\Leftrightarrow$ $\det p''(v^*) > 0$

$p''(v^*)$ of signature $(+ - -)$

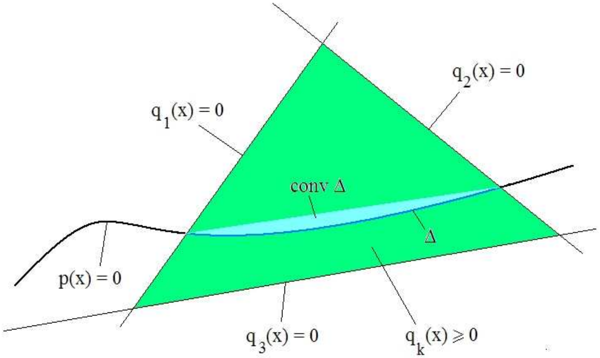
Local geometry of $p''(v^*)$



Problem formulation

describe by LMI's the convex hull of the set

$$\Delta = \{x \in \mathbb{R}^n \,|\, p(x) = 0,\ q_k(x) \geq 0,\ k = 1, \dots, m\}$$



Moment vectors

Definition

For $x \in \mathbb{R}^n$, let $\mathcal{X}_N(x)$ be the vector of monomials x^α with $|\alpha| \leq N$.

the PSD rank 1 matrix $\mathcal{X}_N(x)\mathcal{X}_N^T(x)$ contains the elements of $\mathcal{X}_{2N}(x)$

$\Rightarrow$ the vector $\mathcal{X}_{2N}(x)$ has to satisfy the corresponding LMI

LMI constraints on moment vectors

$$\Delta = \{x \in \mathbb{R}^n \,|\, p(x) = 0,\ q_k(x) \geq 0,\ k = 1,\dots,m\}$$

for every $x \in \Delta$, $N \in \mathbb{N}$ sufficiently large, $\mathcal{X}_{2N}(x)$ satisfies the linear constraint

$$p(x)b(x) = 0$$

for every polynomial b with $\deg b + \deg p \leq 2N$

for every $I \subset \{1,\dots,m\}$, $\mathcal{X}_{2N}(x)$ satisfies the LMI

$$\prod_{k\in I} q_k(x)\mathcal{X}_{N'}(x)\mathcal{X}_{N'}^T(x) \succeq 0$$

with $N' \leq N - \frac{1}{2}\sum_{k\in I} \deg q_k$

Recovery of the original point

suppose we have a semi-definite representable set S in moment space

the map $\Pi : \mathcal{X}_{2N}(x) \mapsto x$ is a linear projection

Lemma

Let $\Delta \subset \mathbb{R}^n$ be a set and

$$S \supset \{\mathcal{X}_{2N}(x) \,|\, x \in \Delta\}$$

a convex outer approximation of the corresponding moment set. Then $\Pi[S]$ is an outer approximation of convΔ.

Scheme of relaxation

$$\begin{array}{ccc} \{\mathcal{X}_{2N}(x) \mid x \in \Delta\} & \subset & S \\ \Pi \downarrow & & \Pi \downarrow \\ \Delta & \subset & \Pi[S] \end{array}$$

- S defined by LMIs which $\mathcal{X}_{2N}(x)$ satisfy
- $S, \Pi[S]$ semi-definite representable
- $\Pi[S]$ is the Lasserre relaxation of $\operatorname{conv} \Delta$

Homogeneous moments

Definition

For $x \in \mathbb{R}^{n+1}$, let $\mathcal{X}_N(x)$ be the vector of monomials x^α with $|\alpha| = N$.

the PSD rank 1 matrix $\mathcal{X}_N(x)\mathcal{X}_N^T(x)$ contains the elements of $\mathcal{X}_{2N}(x)$

$\Rightarrow$ the vector $\mathcal{X}_{2N}(x)$ has to satisfy the corresponding LMI

other LMIs carry over from inhomogeneous case

Recovery of the original point

the linear projection $\Pi : \mathcal{X}_{2N}(x) \mapsto x$ is no longer available

- choose polynomial $f(x)$ of degree $2N - 1$
- define linear projection $\Pi : \mathcal{X}_{2N}(x) \mapsto (f(x)x_k)_{k=0,\dots,n}$

$x \mapsto \mathcal{X}_{2N}(x) \mapsto f(x) \cdot x$ is "pointwise homothety"

recovery of original point up to scalar factor $f(x)$

does not matter since we are in homogeneous setting

Recovery cont'd

Lemma

Let $\Delta \subset \mathbb{R}^{n+1}$ be a conic set and

$$S \supset \{\mathcal{X}_{2N}(x) \,|\, x \in \Delta\}$$

a closed convex outer approximation of the moment set.
Let f of degree $2N-1$ be such that $f(x) > 0$ a.e. on Δ. Then $K = \Pi[S]$ is an outer approximation of $\operatorname{conv}\Delta$.

many degrees of freedom to construct relaxations

Form of relaxation parameters

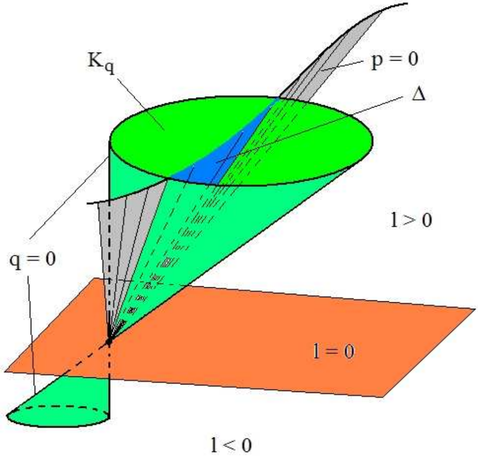
p defining boundary patch Δ is a quartic polynomial

- $N = 2 \Rightarrow$ monomials up to 4th order
- q isolating Δ is a quadric with signature $(+--)$
- f defining the recovering projection Π is l^3

l linear functional s.t. $q \prec 0$, $p''(v) \prec 0$ on ker l for all $v \in \Delta$

$$\Delta = \{v = (x, y, z) \,|\, p(v) = 0,\ q(v) \geq 0,\ l(v) \geq 0\}$$

Geometric interpretation



Explicit description

$$\Sigma = S^* = \{p \cdot b + \sigma_1 q + \sigma_2 \,|\, b \in \mathbb{R},\ \sigma_1, \sigma_2 \text{ SOS}\}$$

is dual to the cone of moment vectors satisfying the LMI

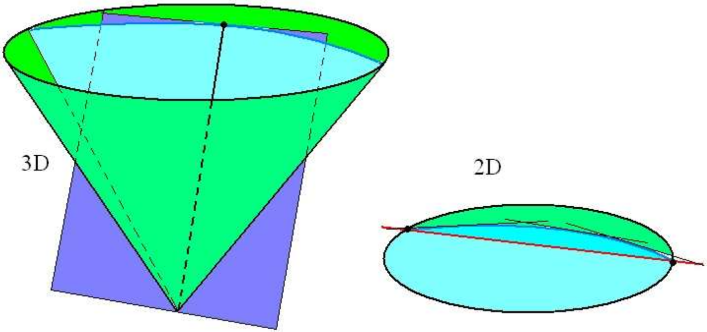
σ_1, σ_2 SOS of degree 2,4 $\Leftrightarrow$ $\sigma_1 \succeq 0$, $\sigma_2 \succeq 0$

$$K^* = (\Pi[S])^* = \{y \in \mathbb{R}_n \,|\, l^3 \cdot y(\cdot) \in \Sigma\}$$

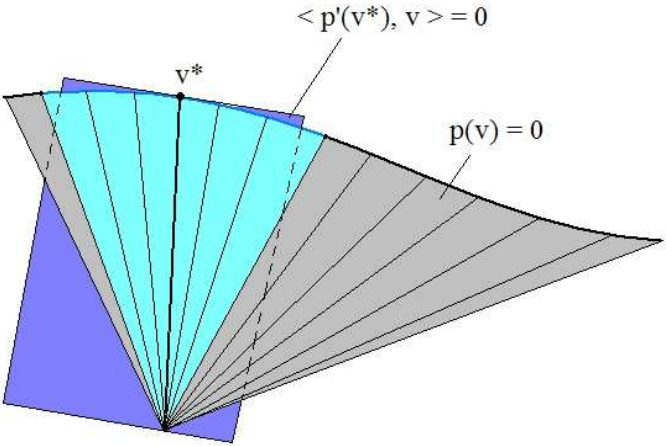
is the dual to the convex cone K approximating conv Δ

linear functionals y such that $l^3 y \in \Sigma$ are supporting the semi-definite approximation K

Supporting planes



Gradient functional



Reduction to polynomial inequality

to show: $l^3 \cdot \langle p'(v^*), \cdot \rangle = p \cdot b + \sigma_1 \cdot q + \sigma_2$, $b \in \mathbb{R}$, $\sigma_1, \sigma_2 \geq 0$

let v^0 be the centre of Δ normalized s.t. $l(v^0) = 1$

with $c > 0$ set

- $b = l^3(v^*)$
- $q = \varepsilon^2 l^2(v) + p''(v^0)[v - l(v)v^0, v - l(v)v^0]$
- $\sigma_1 = -c \cdot l(v^*) \cdot p''(v^0)[l(v^*)v - l(v)v^*, l(v^*)v - l(v)v^*]$

Reduction to polynomial inequality

to show: $l^3 \cdot \langle p'(v^*), \cdot \rangle = p \cdot b + \sigma_1 \cdot q + \sigma_2, \; b \in \mathbb{R}, \sigma_1, \sigma_2 \geq 0$

set $b = l^3(v^*)$, for v^0 with $l(v^0) = 1$
$q = \varepsilon^2 l^2(v) + p''(v^0)[v - l(v)v^0, v - l(v)v^0]$,
$\sigma_1 = -c \cdot l(v^*) \cdot p''(v^0)[l(v^*)v - l(v)v^*, l(v^*)v - l(v)v^*]$, $c > 0$

sufficient condition: for all $v \in \mathbb{R}^3$ and all $v^* \in K_q$

$$l(v^*) \left[l^3(v) \cdot \langle p'(v^*), v \rangle - p(v) \cdot b - \sigma_1 \cdot q \right] - 3l^4(v) \cdot p(v^*) \geq 0$$

homogeneous in each of v, v^* of degree 4

Reduction to polynomial inequality

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homogeneous in each of v, v^* of degree 4

for $l(v^*) = l(v) = 1$:

$$c\left(\varepsilon^2 + p''(v^0)[v - v^0, v - v^0]\right) \leq \frac{p(v) - p(v^*) - \langle p'(v^*), v - v^* \rangle}{p''(v^0)[v - v^*, v - v^*]}$$

Polar coordinates

pass to polar coordinates in $\ker l$ with scalar product $-p''(v^0)$

with $v - v^* = \delta \begin{pmatrix} \cos\zeta \\ \sin\zeta \end{pmatrix}$, $v^* - v^0 = \rho \begin{pmatrix} \cos\xi \\ \sin\xi \end{pmatrix}$

$$c\left(\varepsilon^2 + p''(v^0)[v - v^0, v - v^0]\right) = $$
$$= \quad c\left(\varepsilon^2 - ||v - v^0||^2\right) \leq c\left(\varepsilon^2 - (\delta - \rho)^2\right)$$

$$\frac{p(v) - p(v^*) - \langle p'(v^*), v - v^* \rangle}{p''(v^0)[v - v^*, v - v^*]} =$$
$$= \quad -\frac{\frac{1}{2}p''(v^*)[v - v^*] + \frac{1}{6}p'''(v^*)[v - v^*] + \frac{1}{24}p^{IV}[v - v^*]}{\delta^2}$$
$$= \quad \frac{1}{2} + \sum_{1 \leq k+l \leq 2} c_{kl}(\zeta, \xi)\delta^k \rho^l \geq \frac{1}{2} + \sum_{1 \leq k+l \leq 2} \min_{\zeta,\xi} c_{kl}\delta^k \rho^l$$

Reduction to copositivity condition

with $\tilde{c}_{kl} = \min_{\zeta,\xi} c_{kl}$:

$$c\left(\varepsilon^2 - (\delta - \rho)^2\right) \leq \frac{1}{2} + \sum_{1 \leq k+l \leq 2} \tilde{c}_{kl} \delta^k \rho^l$$

for all $\delta = ||v - v^*|| \in \mathbb{R}_+$, $\rho = ||v^* - v^0|| \in [0, \varepsilon]$

$$\Leftrightarrow$$

$$\begin{pmatrix} \frac{1}{2} - c\varepsilon^2 & \frac{\tilde{c}_{10}}{2} & \frac{1}{2} + \varepsilon\frac{\tilde{c}_{01}}{2} - c\varepsilon^2 \\ \frac{\tilde{c}_{10}}{2} & \tilde{c}_{20} + c & \frac{\tilde{c}_{10}}{2} + \varepsilon\frac{\tilde{c}_{11}}{2} - \varepsilon c \\ \frac{1}{2} + \varepsilon\frac{\tilde{c}_{01}}{2} - c\varepsilon^2 & \frac{\tilde{c}_{10}}{2} + \varepsilon\frac{\tilde{c}_{11}}{2} - \varepsilon c & \frac{1}{2} + \varepsilon\tilde{c}_{01} + \varepsilon^2\tilde{c}_{02} \end{pmatrix} \in \mathcal{C}_3$$

satisfied if c large and ε small

Main result

Theorem

Let p be a homogeneous ternary quartic, let $Z(p)$ be the zero set of p, and let $v^0 \in Z(p)$ be a regular point such that $\det p''(v^*) > 0$.
Then there exists a conic subset $\Delta \subset Z(p)$ containing v^0 in its interior such that $\operatorname{conv}\Delta$ has a semi-definite description with blocks of size 1,3 and 6 and with 11 lifting variables.

Thank you