A convergence theorem for Iterative Feedback Tuning *

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Abstract

Iterative Feedback Tuning (IFT) is a widely used procedure for controller tuning. It is a sequence of iteratively performed special experiments on the plant interlaced with periods of data collection under normal operating conditions. In this note we prove a rigorous result on the convergence of the IFT procedure for disturbance rejection, which is one of the main fields of application.

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1 Introduction

Iterative Feedback Tuning (IFT) is a data based method for the tuning of restricted complexity controllers. It has proved to be very effective in practice and is now widely used in process control, often for disturbance rejection. Following the original formulation of the method in [5], [6] many improvements and modifications of IFT have been suggested and partially implemented in practice. The reader is referred to [4] for a recent overview.

However, surprisingly little attention was paid to the theoretical properties of IFT. A quantitative study of the asymptotic aspects of convergence was undertaken in [2]. The only proof of convergence of the algorithm appeared in [3]. Unfortunately this proof contains a flaw, so that strictly speaking the convergence of the method is not yet proven. The goal of this note is to even out this discrepancy between the wide practical use of IFT and the state of theoretical knowledge about it. In this note we focus on IFT for disturbance rejection.

The objective of IFT is to minimize a quadratic performance criterion. IFT is a stochastic gradient descent scheme in a finitely parameterized controller space. The gradient of the cost function at each step is estimated from data. These data are collected with the actual controller in the loop. One of the advantages of IFT is that most data are collected while the process runs under normal operating conditions. These data are then used to design a special experiment, which yields a noisy, but unbiased, estimate of the cost function gradient. This gradient estimate is used to perform the next descent step in controller space. For more details of the procedure see [5].

It will be shown that under suitable assumptions the algorithm converges to a stationary point of the performance criterion. The proof is based on the proof provided in [3], but contains an additional proposition which is necessary for mathematical correctness.

The remainder of the note is structured as follows. In the next section we summarize the details of the IFT algorithm for disturbance rejection. In Section 3 we state and prove the convergence theorem and establish conditions for its validity.

2 IFT for disturbance rejection

In this section we review the IFT method for the disturbance rejection problem with a classical LQ criterion. For a more general and detailed presentation of IFT the reader is referred to [5], [6].

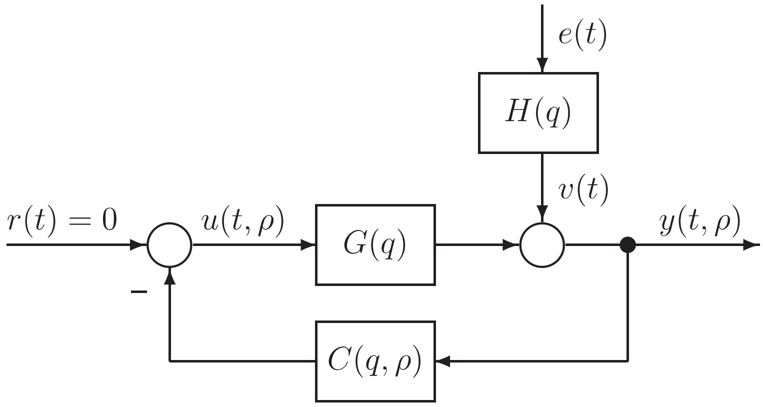

Figure 1: The control system under normal operating conditions.

Consider a SISO discrete time system described by

$$y(t) = G(q)u(t) + v(t), \tag{1}$$

where $y(t)$ is the output, $u(t)$ is the input, $G(q)$ is a linear time-invariant transfer function, with q being the shift operator, and $v(t) = H(q)e(t)$ is the process disturbance. Here $H(q)$ is a monic, stable and inversely stable transfer function and $e(t)$ is zero mean white noise with variance σ^2.

We focus on the feedback loop around $G(q)$ depicted in Figure 1, where $C(q, \rho)$ is a one-degree-of-freedom controller belonging to a parameterized set of controllers with parameter $\rho \in \mathbf{R}^n$. We assume that in the control system of Figure 1 the reference signal $r(t)$ is set at zero under normal operating conditions. Our goal is to tune the controller $C(q, \rho)$ so that the variance of the noise-driven closed loop output

$$y(t, \rho) = \frac{1}{1 + G(q)C(q, \rho)} v(t) = S(q, \rho)v(t)$$

is as small as possible. Here the transfer function $S(q, \rho)$ is the sensitivity function.
In order to avoid a large control effort, it is common to include a penalty also on the variance of the input signal

$$u(t, \rho) = -C(q, \rho)S(q, \rho)v(t).$$

Thus we have to find a minimizer for the cost function

$$J(\rho) = \frac{1}{2} \mathbf{E} \left[y(t, \rho)^2 + \lambda u(t, \rho)^2 \right], \tag{2}$$

where $\lambda \geq 0$ is a scalar expressing the importance of the penalty on the control effort.

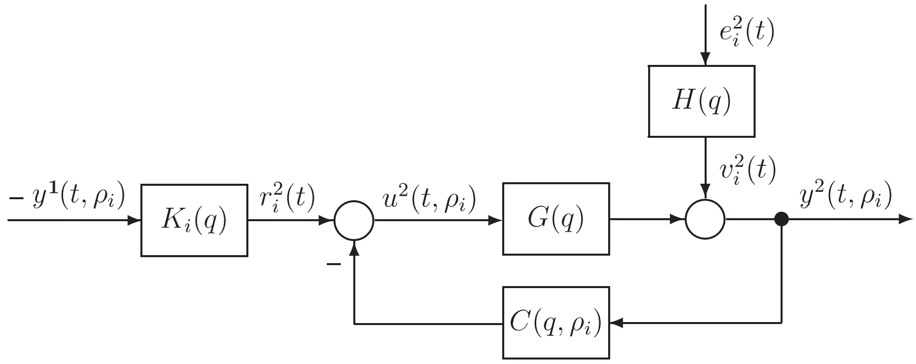

Figure 2: The setting of the second experiment.

The IFT method yields an approximate solution to the above problem. IFT is based on the possibility of obtaining an unbiased estimate of the gradient $\frac{\partial J}{\partial \rho}(\rho)$ of the cost function at $\rho = \rho_i$ from data collected from the closed-loop system with the controller $C(\rho_i)$ operating on the loop. The cost function $J(\rho)$ can then be minimized with an iterative stochastic gradient descent scheme of Robbins-Monro type [1]. In the scheme a sequence of controllers $C(q,\rho_i)$ is computed and applied to the plant. In the i-th iteration step data obtained from the system with the controller $C(\rho_i)$ operating on the loop are used to construct the next parameter vector ρ_{i+1} according to

$$\rho_{i+1} = \rho_i - \gamma_i R_i^{-1} est_N \left[\frac{\partial J}{\partial \rho}(\rho_i) \right]. \tag{3}$$

Here γ_i is a nonnegative scalar sequence of step lengths, R_i is a sequence of positive definite matrices and $est_N \left[\frac{\partial J}{\partial \rho}(\rho_i) \right]$ is an unbiased estimate of the gradient $\frac{\partial J}{\partial \rho}(\rho)$ obtained from data.

In the sequel we describe the construction of the unbiased gradient estimate. The exact expression of the gradient of $J(\rho)$ is given by

$$\frac{\partial J}{\partial \rho}(\rho_i) = \mathbf{E} \left[y(t,\rho_i) \frac{\partial y}{\partial \rho}(t,\rho_i) + \lambda u(t,\rho_i) \frac{\partial u}{\partial \rho}(t,\rho_i) \right]. \tag{4}$$

Its unbiased estimate $est_N \left[\frac{\partial J}{\partial \rho}(\rho_i) \right]$ is obtained from two data sets collected from the closed loop as follows. First, a sequence of N input-output data are collected under normal operating conditions, i.e. without reference signal:

$$\begin{aligned} u^1(t,\rho_i) &= -C(q,\rho_i) S(q,\rho_i) v_i^1(t), \\ y^1(t,\rho_i) &= S(q,\rho_i) v_i^1(t). \end{aligned}$$

Here $v_i^1(t)$ denotes the corresponding realization of the noise $v(t)$ for the first batch of collected data at iteration i. Secondly, a special experiment of the same length N is performed. During this experiment the loop is fed with the reference signal

$$r_i^2(t) = -K_i(q)y^1(t,\rho_i),$$

where $K_i(q)$ is a suitable prefilter (see Figure 2). The obtained input and output data are given by

$$\begin{aligned} u^2(t,\rho_i) &= -S(q,\rho_i)\left[K_i(q)y^1(t,\rho_i) + C(q,\rho_i)v_i^2(t)\right] , \\ y^2(t,\rho_i) &= -G(q)S(q,\rho_i)K_i(q)y^1(t,\rho_i) + S(q,\rho_i)v_i^2(t) , \end{aligned}$$

where $v_i^2(t)$ is the corresponding realization of the noise, i.e. for the second batch of data at iteration i.

We assume the two experiments of iteration step i to be sufficiently separated in time, so that the realization $v_i^2(t)$ of the noise can be considered as being independent of the realization $v_i^1(t)$. The obtained data are used to form the following estimates of the gradients of $u^1(t,\rho_i)$ and $y^1(t,\rho_i)$:

$$est\left[\frac{\partial u^1}{\partial \rho}(t,\rho_i)\right] = \frac{1}{K_i(q)}\frac{\partial C}{\partial \rho}(q,\rho_i)\, u^2(t,\rho_i) \; , \tag{5}$$

$$est\left[\frac{\partial y^1}{\partial \rho}(t,\rho_i)\right] = \frac{1}{K_i(q)}\frac{\partial C}{\partial \rho}(q,\rho_i)\, y^2(t,\rho_i) \, . \tag{6}$$

These estimates are corrupted by the noise $v_i^2(t)$ of the second experiment as follows:

$$\begin{aligned} est\left[\frac{\partial u^1}{\partial \rho}(t,\rho_i)\right] &= \frac{\partial u^1}{\partial \rho}(t,\rho_i) - \frac{S(q,\rho_i)}{K_i(q)}C(q,\rho_i)\frac{\partial C}{\partial \rho}(q,\rho_i)\, v_i^2(t) , \\ est\left[\frac{\partial y^1}{\partial \rho}(t,\rho_i)\right] &= \frac{\partial y^1}{\partial \rho}(t,\rho_i) + \frac{S(q,\rho_i)}{K_i(q)}\frac{\partial C}{\partial \rho}(q,\rho_i)\, v_i^2(t) \, . \end{aligned} \tag{7}$$

Using (5) and (6), an estimate of the gradient $\frac{\partial J}{\partial \rho}(\rho_i)$ is then obtained as

$$est_N\left[\frac{\partial J}{\partial \rho}(\rho_i)\right] = \frac{1}{N}\sum_{t=1}^{N}\left[y^1(t,\rho_i)est\left[\frac{\partial y^1}{\partial \rho}(t,\rho_i)\right] + \lambda u^1(t,\rho_i)est\left[\frac{\partial u^1}{\partial \rho}(t,\rho_i)\right]\right] . \tag{8}$$

The estimate is unbiased because independency between the disturbance realizations in the first and second experiments was assumed.

Thus the IFT procedure for disturbance rejection amounts to the iterative scheme (3) with the gradient estimate $est_N\left[\frac{\partial J}{\partial \rho}(\rho_i)\right]$ given by (8), (5–6). The sequences γ_i and R_i are basically left to the choice of the user, but have to fulfill some requirements for the algorithm to converge, which will be specified below. The consistency of the algorithm and its convergence properties are studied in the next section.

3 Convergence analysis of IFT

This section contains the main result of the present note. It is largely based on the results presented in the Appendix of [3], but Proposition 3.4 is new. Further we state exact conditions under which the IFT algorithm for disturbance rejection is guaranteed to converge to a subset of the set of stationary points of the cost function. We reformulate the convergence theorem and fill a gap in its original proof as it is found in [3].

The proof of convergence is based on the following proposition stated in [7].

Proposition 3.1 [7] *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let Z_n, β_n, ξ_n and ζ_n be finite nonnegative $\mathcal{F}_n$-measurable random variables, where $\mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n \subset \ldots$ is a sequence of sub-σ-algebras of $\mathcal{F}$. Suppose that $\mathbf{E}(Z_{n+1} \,|\, \mathcal{F}_n) \leq Z_n(1+\beta_n) + \xi_n - \zeta_n$ for all n. Then the sequence $\{Z_n\}$ and the sum $\sum_{n=1}^{\infty} \zeta_n$ converge with probability 1 conditioned on the event that the sums $\sum_{n=1}^{\infty} \beta_n$, $\sum_{n=1}^{\infty} \xi_n$ converge.* □

In order to apply Proposition 3.1 to the IFT algorithm we introduce the following assumptions.

Assumption 3.2 *Let $\mathcal{D}$ be a convex compact subset of the parameter space $\mathbf{R}^n$. Let the following conditions hold.*

1. *The process noise v is uniformly bounded for all experiments. Realizations of the noise in different experiments are mutually independent.*

2. *There exists a neighbourhood $\mathcal{O}$ of $\mathcal{D}$ such that the set of controllers $\{C(\rho) \,|\, \rho \in \mathcal{O}\}$ is two times continuously differentiable with respect to ρ.*

3. *The controllers $C(\rho)$ and their first and second derivatives have their poles uniformly bounded away from the unit circle for $\rho \in \mathcal{D}$.*

4. *The closed loop systems corresponding to the controllers $C(\rho)$ are stable and have their poles uniformly bounded away from the unit circle for $\rho \in \mathcal{D}$.*

5. *The sequence $\{\gamma_n\}$ of step lengths is nonnegative and satisfies $\sum_{n=1}^{\infty} \gamma_n = \infty$, $\sum_{n=1}^{\infty} \gamma_n^2 < \infty$.*

6. *The sequence $\{R_n\}$ of positive definite symmetric weighting matrices satisfies $\alpha I \leq R_n \leq \beta I$ for some positive constants α, β.*

6a. *The weighting matrix R_n may also be a random variable, but R_n^{-1} is uncorrelated with the noise realizations v_n^1, v_n^2 in experiment n.*

7. The event $\mathcal{A} = \{\rho_n \in \mathcal{D} \; \forall \, n\}$ has a non-zero probability.

The first six conditions are standard assumptions in the literature on IFT (see e.g. [3],[4]).
By condition 6a, the matrix R_n^{-1} can be considered as fixed during iteration step n. Conditions 6 and 6a assure that at non-stationary points of the cost function J the expected value of J at the next step is smaller than its value at the current step. Condition 6a is practically relevant only in the neighbourhood of stationary points of J, where the error in the gradient estimate is comparable to or larger than the gradient itself. If condition 6a is to hold, the choice of R_n can be based on data collected during previous iterations, but not in the current one.
Condition 7 is to make convergence analysis meaningful. A necessary condition for it to hold is e.g. that the set $\mathcal{D}$ contains stationary points of the cost function J.

We are now ready to apply Proposition 3.1 to IFT. Setting $Z_n = J(\rho_n)$, $\beta_n = 0$, $\xi_n = \mathbf{E}\left[\left|J(\rho_{n+1}) - J(\rho_n) + \gamma_n \left(\frac{\partial J}{\partial \rho}(\rho_n)\right)^T R_n^{-1} est_N\left[\frac{\partial J}{\partial \rho}(\rho_n)\right]\right|\right]$, $\zeta_n = \gamma_n \left(\frac{\partial J}{\partial \rho}(\rho_n)\right)^T R_n^{-1} \frac{\partial J}{\partial \rho}(\rho_n)$, and defining $\mathcal{F}_n$ as the σ-algebra generated by iteration steps $1, \ldots, n-1$, we obtain the following result.

Proposition 3.3 *Let Assumption 3.2 hold. Then the sum $\sum_{n=1}^{\infty} \gamma_n \left(\frac{\partial J}{\partial \rho}(\rho_n)\right)^T R_n^{-1} \frac{\partial J}{\partial \rho}(\rho_n)$ and the sequence $J(\rho_n)$ converge with probability 1 conditioned on the event $\mathcal{A}$.*

Proof. A detailed proof can be found in [3]. □

By conditions 5 and 6 of Assumption 3.2, Proposition 3.3 implies that the sequence $\{\rho_n\}$ accumulates to stationary points of J with probability 1 conditioned on $\mathcal{A}$. However, it does not follow immediately that $\{\rho_n\}$ cannot also accumulate to non-stationary points. We have to exclude this possibility explicitly by the following proposition.

Proposition 3.4 *Let Assumption 3.2 hold. Then the sequence $\{\rho_n\}$ converges to a closed connected subset of the set $D_c = \{\rho \in \mathcal{D} \,|\, \frac{\partial J}{\partial \rho}(\rho) = 0\}$ with probability 1, conditioned on the event $\mathcal{A}$.*

Before proving Proposition 3.4, we furnish an auxiliary result.

Let U be a subset of R^k. Let V be a subset of U such that the minimal distance between points in V and points in the complement of U is strictly positive:

$$\inf\{|x-y| \,|\, x \in V, y \in \mathrm{R}^k \setminus U\} = l > 0. \tag{9}$$

Consider a stochastic process

$$X_{n+1} = X_n + \gamma_n Y_n,$$

where X_n, Y_n are random variables that take values in R^k and $\{\gamma_n\}$ is a sequence of non-negative numbers such that $\sum_{n=1}^{\infty} \gamma_n = \infty$. Suppose that the expectations and variances of the variables Y_n are uniformly bounded:

$$|\mathbf{E}Y_n| < c_E, \qquad \mathrm{Trace}[\mathbf{Cov}Y_n] < c_\sigma^2 \qquad \forall\ n,$$

where $c_E, c_\sigma > 0$ are positive constants.
Now define events $\mathcal{V}_n$, $n = 1, 2, \ldots$ by $\mathcal{V}_n = \{X_n \in V\}$. Given the event $\mathcal{V}_n$, define the random number $\hat{N}$ as the least integer $N > n$ such that $X_N \notin U$. If such a number does not exist, i.e. $X_m \in U$ for all $m \geq n$, then let $\hat{N} = \infty$.
With these definitions we have the following proposition.

Proposition 3.5

$$Prob\left\{\sum_{m=n}^{\hat{N}} \gamma_m > \frac{l}{2(c_E + c_\sigma)} \mid \mathcal{V}_n\right\} > \frac{1}{2}.$$

Proof. Suppose $X_n \in V$, i.e. the event $\mathcal{V}_n$ has occurred. Define $\bar{N}$ as the least integer $N > n$ such that $\sum_{m=n}^{N} \gamma_m > \frac{l}{2(c_E+c_\sigma)}$. Then we have

$$\begin{aligned}
\mathbf{E}\left[\max_{n\leq m\leq \bar{N}} |X_m - X_n|\right] &\leq \mathbf{E}\left[\sum_{m=n}^{\bar{N}-1} |X_{m+1} - X_m|\right] = \sum_{m=n}^{\bar{N}-1} \gamma_m \mathbf{E}|Y_m| \leq \sum_{m=n}^{\bar{N}-1} \gamma_m \sqrt{\mathbf{E}|Y_m|^2} \\
&= \sum_{m=n}^{\bar{N}-1} \gamma_m \sqrt{|\mathbf{E}Y_m|^2 + \mathrm{Trace}[\mathbf{Cov}Y_m]} \leq \sqrt{c_E^2 + c_\sigma^2}\,\frac{l}{2(c_E + c_\sigma)} < \frac{l}{2}.
\end{aligned}$$

It follows that

$$Prob\left\{\max_{n\leq m\leq \bar{N}} |X_m - X_n| < l\right\} > \frac{1}{2}.$$

But we have

$$\begin{aligned}
Prob\left\{\sum_{m=n}^{\hat{N}} \gamma_m > \frac{l}{2(c_E + c_\sigma)}\right\} &= Prob\{\hat{N} \geq \bar{N}\} = Prob\{X_n, \ldots, X_{\bar{N}-1} \in U\} \\
&\geq Prob\left\{\max_{n\leq m\leq \bar{N}} |X_m - X_n| < l\right\}.
\end{aligned}$$

Combining these inequalities completes the proof. $\square$

Now we are ready to prove that non-stationary points of J cannot be accumulation points of the sequence $\{\rho_n\}$.

Proof of Proposition 3.4

The proof is by *reductio ad absurdum.* Assume there is a non-zero probability, conditioned on $\mathcal{A}$, that $\{\rho_n\}$ accumulates to a non-stationary point of J. Denote this event by $\mathcal{A}_{NSt}$. Suppose $\mathcal{A}_{NSt}$ has occurred. Let $\hat{\rho} \in \mathcal{D}$ be an accumulation point of $\{\rho_n\}$ such that $\frac{\partial J}{\partial \rho}(\hat{\rho}) \neq 0$. By Proposition 3.3 there exists another accumulation point ρ^* of $\{\rho_n\}$ with $\frac{\partial J}{\partial \rho}(\rho^*) = 0$. Then there exist a positive number $c > 0$, a neighbourhood U of $\hat{\rho}$ and a neighbourhood U' of ρ^* such that $|\frac{\partial J}{\partial \rho}(\rho)| > c$ for all $\rho \in U$ and the intersection $U \cap U'$ is empty. Further there exist a positive number $l > 0$ and a neighbourhood V of $\hat{\rho}$ such that condition (9) holds.

Now observe that

$$\left(\frac{\partial J}{\partial \rho}(\rho)\right)^T R_n^{-1} \frac{\partial J}{\partial \rho}(\rho) \geq \beta^{-1} c^2 \qquad \forall\, \rho \in U. \tag{10}$$

Moreover, the quantities $\left|\mathbf{E}\left[R_n^{-1} est_N \left[\frac{\partial J}{\partial \rho}(\rho_n)\right]\right]\right|$, Trace $\left[\mathbf{Cov}\left[R_n^{-1} est_N \left[\frac{\partial J}{\partial \rho}(\rho_n)\right]\right]\right]$ are bounded uniformly by the positive numbers

$$c_E = \alpha^{-1} \max_{\rho \in \mathcal{D}} \left|\frac{\partial J}{\partial \rho}(\rho)\right|, \qquad c_\sigma^2 = \alpha^{-2} \max_{\rho \in \mathcal{D}} \operatorname{Trace} \mathbf{Cov}\left[est_N\left[\frac{\partial J}{\partial \rho}(\rho_n)\right]\right]$$

with $c_\sigma > 0$.

Now Proposition 3.5 can be applied. By combining it with (10) we get that for any pair of integers (n_1, n_2) such that $n_2 > n_1$ and $\rho_{n_1} \in V$, $\rho_{n_2} \in U'$, we have

$$Prob\left\{\sum_{n=n_1}^{n_2} \gamma_n \left(\frac{\partial J}{\partial \rho}(\rho_n)\right)^T R_n^{-1} \frac{\partial J}{\partial \rho}(\rho_n) > \frac{l\beta^{-1}c^2}{2(c_E + c_\sigma)}\right\} > \frac{1}{2}.$$

But both $\hat{\rho}$ and ρ^* are accumulation points of the sequence $\{\rho_n\}$. Hence there exist infinitely many consecutive pairs of such numbers n_1, n_2. Thus the sum $\sum_{n=1}^{\infty} \gamma_n \frac{\partial J}{\partial \rho}(\rho_n)^T R_n^{-1} \frac{\partial J}{\partial \rho}(\rho_n)$ diverges with probability 1, conditioned on $\mathcal{A}_{NSt}$. Hence it diverges with a non-zero probability conditioned on $\mathcal{A}$. This contradicts Proposition 3.3.

We have proven that with probability 1 $\{\rho_n\}$ accumulates only to a subset of D_c. This subset is closed by definition and is connected with probability 1 because the expectation of $|\rho_{n+1} - \rho_n|$ tends to zero as $n \to \infty$. The proof is complete. $\square$

Remark. Generically the stationary points of the cost function J will be isolated and non-degenerated. It seems clear that the algorithm cannot converge to a local maximum or a saddle point of J if the noise in the gradient estimate is exciting in unstable directions. Therefore the assumption of convergence to an isolated local minimum is justified.

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