Self-similar trajectories in multi-input systems

Roland Hildebrand

Abstract

Self-similar trajectories play an important role in deterministic feedback control systems that possess a symmetry group of Fuller type. We consider self-similar trajectories in multi-input systems and the linear part of the associated Poincaré maps in orbit space with respect to the symmetry group. We show that Fuller groups contract the symplectic structure of the system's phase space and derive some properties of the spectrum and the eigenvectors of the Poincaré map.

Introduction

We consider self-similar trajectories in deterministic feedback control systems, which give rise to Hamiltonian dynamics via Pontryagins maximum principle. Self-similar trajectories were discovered by Fuller in 1960 [1]. Fuller considered the optimal control system

$$\dot{x} = y, \quad \dot{y} = u \in [-1, 1], \quad x(0) = x_0, \quad y(0) = y_0, \quad J(u(\cdot)) = \frac{1}{2}\int_0^\infty x^2(t)\,dt \to \inf\,. \quad (1)$$

Here u is a bounded scalar control; x, y parametrize the state space $\mathbb{R}^2$; and the cost functional J is to be minimized over all measurable control functions $u(t)$. Fuller has found that outside the origin the optimal feedback control is bang-bang. The origin itself is a singular stationary trajectory of second order. The ratio of successive intervals of time between successive control switchings turned out to be constant. Therefore Fuller named the corresponding trajectories *constant-ratio trajectories*. They are the prototype of self-similar trajectories. In system (1) the origin is an accumulation point of control switchings. This phenomenon is called *chattering* and is closely related to self-similar trajectories.

Fuller discovered [2] that system (1) possesses a one-parametric symmetry group $\mathcal{G}_F$. Namely, if $(x(t), y(t), u(t))$ is an optimal trajectory of the system, then $(\lambda^2 x(\frac{t}{\lambda}), \lambda y(\frac{t}{\lambda}), u(\frac{t}{\lambda}))$ is also an optimal trajectory for any positive number λ. We say that a trajectory is *self-similar* if it is invariant with respect to a nontrivial subgroup of $\mathcal{G}_F$. Consider an orbit of $\mathcal{G}_F$ in state space. The phase velocity of the optimal synthesis has the same direction at any point of this orbit, and determines a direction field on orbit space. Without the origin $(0, 0)$, the orbit space will be homeomorphic to S^1. It consists of a single periodic orbit of the direction field and corresponds to a one-parametric family of self-similar trajectories.

This example demonstrates that self-similar trajectories are important elements in the structure of the optimal synthesis of systems with a symmetry of Fuller type. Namely, they correspond to periodic orbits and stationary points of the direction field that is defined on orbit space with respect to the symmetry group. Fuller [3] developped a detailed algorithm for calculating self-similar trajectories in the case of a symmetrically bounded scalar input. He considered the case where the control jumps between the two extremal values and the intervals between successive control switchings form a decreasing geometric progression.

The techniques presented in this paper allow to find self-similar trajectories in the general case and to reveal the behaviour of the system in the neighbourhood of these trajectories.

Applications of self-similar trajectories are not restricted to systems with a Fuller group. Zelikin and Borisov developped a theory of chattering [7]. In particular, they showed that, provided a certain approximate Fuller symmetry holds, singular manifolds of codimension 2 and order 2 serve as base of a fibration in state space with fibres orbitally equivalent to the optimal synthesis of system (1). The global structure of the synthesis is retained in the fibre, and the self-similar trajectories of the non-perturbed system possess equivalents in the perturbed system, although the latter are no more self-similar.

The paper is organized as follows. In section 1 we define self-similar trajectories and Fuller groups and describe the treated class of optimal control systems. In sections 2 and 3 we develop algorithms for finding self-similar trajectories and calculating the linear part of the Poincaré map associated with the corresponding periodic orbits, or alternatively, the linearization in the neighbourhood of corresponding stationary points (Algorithms 1 and 2, Theorem 2.1). We provide a criterion of optimality of a self-similar trajectory (Theorem 2.4) and investigate integral manifolds consisting of optimal trajectories in a neighbourhood of a hyperbolic periodic orbit (Theorem 3.3). In the next section we show that Fuller groups contract the sympletic structure of phase space and derive some properties of the spectrum and the eigenvalues of the Poincaré map (Theorems 4.10 and 4.11). Finally we draw some conclusions.

1 Definitions and preliminaries

We look for self-similar trajectories in systems given by the following generalization of (1).

$$x = (x_1, \dots, x_l) \in \mathbb{R}^l, \ \sum_{i=1}^{l} m_i x_i^{(j)} = 0 \ \forall\, j = 1, \dots, n-1; \ J(u(\cdot)) = \frac{1}{2}\int_0^\infty \sum_{i=1}^{l} m_i x_i^2 \, dt \to \inf;$$

$$x^{(n)} = u, \quad u \in U = \left\{ (u_1, \cdots, u_l) \in \mathbb{R}^l \,|\, u_i \le 1 \ \forall\, i = 1, \dots, l; \ \sum_{i=1}^{l} m_i u_i = 0 \right\}. \qquad (2)$$

Here $m_i > 0$; the state space M is parametrized by the l components of the vector x and its time derivatives $x_i^{(j)}$ up to order $n-1$. Since there are restrictions $\sum_{i=1}^{l} x_i^{(j)} = 0$, the state space has $n \times (l-1)$ dimensions. The cost functional J is to be minimized over all measurable inputs $u(t)$. The set U of admissible control values is an $(l-1)$-dimensional simplex embedded in the space $\mathbb{R}^l$ parametrized by the coordinates $u_1, \dots, u_l$. Suppose the initial point $(x(0), \dot{x}(0), \dots, x^{(n-1)}(0))$ in state space is fixed; then the functional J is strictly convex on the convex set of all admissible functions $u(\cdot)$. Hence by Kuhn-Tucker's theorem, the optimal solution exists and is unique for any fixed initial point in state space.

Let us apply Pontryagin's maximum principle [6] to system (2). We obtain the Pontryagin function $H = -\frac{1}{2}\sum_{i=1}^{l} m_i x_i^2 + \sum_{j=1}^{n-1}\sum_{i=1}^{l}\psi_i^j x_i^{(j)} + \sum_{i=1}^{l} u_i\psi_i^n$. Here the vector $\psi^j = (\psi_1^j, \dots, \psi_l^j)$ is conjugated to the vector of $(j-1)$-th order derivatives $x^{(j-1)}$. The conjugated variables satisfy the equations $\dot\psi_i^1 = m_i x_i \; \forall\, i = 1, \dots, l$; $\dot\psi^j = -\psi^{j-1} \; \forall\, j = 2, \dots, n$. The control u is determined by ψ^n according to the maximum principle. Suppose there exists a $i \in \{1, \dots, l\}$ such that $\frac{\psi_i^n}{m_i} < \frac{\psi_{i'}^n}{m_{i'}}$ for any $i' \neq i$, then the control is given by

$$u_{i'} = 1 \;\forall\, i' \neq i, \qquad u_i = -\frac{\sum_{i' \neq i} m_{i'}}{m_i}. \tag{3}$$

Without loss of generality we can put $\sum_{i=1}^{l}\psi_i^j = 0$ for all $j = 1, \dots, n$ [4]. We consider only regular trajectories in system (2). By [4, Remark on p.3] we then cover also the case of singular trajectories, anyway.

Let us introduce coordinates y_i^j in phase space; here $i \in \{1, \dots, l\}, j \in \{1, \dots, 2n\}$:

$$y_i^j = (-1)^{j-1}\frac{\psi_i^{n-j+1}}{m_i}, \quad y_i^{n+j} = (-1)^{n-1}x_i^{(j-1)}, \quad \forall\, i = 1, \dots, l; j = 1, \dots, n.$$

Let us join coordinates with equal upper indices to vectors $y^j \in \mathbb{R}^l$. Then the restrictions and Hamiltonian dynamics are given by

$$\dot y^j = y^{j+1} \;\forall\, j = 1, \dots, 2n-1; \quad \dot y^{2n} = (-1)^{n-1}u; \qquad \sum_{i=1}^{l} m_i y_i^j = 0 \;\forall\, j = 1, \dots, 2n. \tag{4}$$

The control u is given by (3); here i is the index for which y_i^1 is smallest. The phase space of the system is parametrized by the $2nl$ components of the vectors $y^1, \dots, y^{2n}$ with $2n$ restrictions. Hence it has $2n(l-1)$ dimensions. The phase space can be considered as the cotangent fibration T^*M over the state space M. The latter is parametrized by the components of the vectors $y^{n+1}, \dots, y^{2n}$ with n restrictions. It has $n(l-1)$ dimensions.

Proposition 1.1. *[4, Proposition 1] Suppose ρ is a trajectory of system (4). Then ρ is a lifting to T^*M of an optimal trajectory if and only if it tends to the origin of T^*M.*

In the sequel, we shall call any trajectory in phase space T^*M optimal, if it is a lift of an optimal trajectory in state space M.

On phase space there acts a one-parametric group $\mathcal{G}$ of linear transformations. It is parametrized by a multiplicative parameter $\lambda \in \mathbb{R}_+$. The element $\mathcal{G}_\lambda$ corresponding to the number λ multiplies the coordinate y_i^j by λ^{2n-j+1}, $\mathcal{G}_\lambda : (y^1, \dots, y^{2n}) \mapsto (\lambda^{2n}y^1, \dots, \lambda y^{2n})$.

Definition 1.2. We call the group $\mathcal{G}$ *Fuller group*.

The action of $\mathcal{G}$ on T^*M induces an action on state space M, because the action on the variables $y^{n+1}, \dots, y^{2n}$ does not depend on the values of the other variables. It is readily seen that $\mathcal{G}_\lambda$ takes any trajectory of system (4) to another trajectory with a change of time scale by the factor λ. Optimal trajectories are taken to optimal trajectories.

Corollary 1.3. *Suppose $y(t)$ is a trajectory of system (4); then for any $\lambda > 0$ the trajectory $y'(t) = \mathcal{G}_\lambda(y(\frac{t}{\lambda}))$ is also a solution of system (4). If $y(t)$ is optimal, then so is $y'(t)$.* □

Thus system (4) induces a direction field on orbit space $(T^*M)/\mathcal{G}$. If we remove the origin, which is a separate orbit, orbit space will be homeomorphic to the sphere $S^{2n(l-1)-1}$. Denote the space $((T^*M)/\mathcal{G}) \setminus \{0\} \cong S^{2n(l-1)-1}$ by Σ^*. Since at any point of an orbit the same control is applied, the control function u given by (3) is defined also on Σ^*. We have the same situation in state space. The velocity field defined on M by the optimal synthesis points in the same direction at any point of a given orbit with respect to $\mathcal{G}$. Hence the optimal synthesis induces a direction field on orbit space $M/\mathcal{G}$. If we remove the origin, then this orbit space will be homeomorphic to the sphere $S^{n(l-1)-1}$. Denote the space $(M/\mathcal{G}) \setminus \{0\} \cong S^{n(l-1)-1}$ by Σ. The optimal synthesis defines a section $M \to T^*M$ of the cotangent fibration. This section induces an embedding $\Sigma \to \Sigma^*$. In the sequel, we will identify Σ with its image in Σ^*. In view of the above, we call a trajectory in Σ^* an optimal trajectory if it belongs to Σ.

Definition 1.4. We call any trajectory of system (4) *self-similar*, if it is invariant with respect to a non-trivial subgroup of $\mathcal{G}$.

Corollary 1.5. *The preimage of a self-similar trajectory in Σ^* consists of stationary points and/or periodic orbits.* □

Definition 1.6. We call a periodic orbit ζ in Σ^* an *s-chain*, if the restriction of the control function to ζ is piecewise constant and has s points of discontinuity.

It can be shown that any periodic orbit in Σ is an s-chain for some $s > 1$. Hence we can restrict our study to stationary points and s-chains consisting of regular arcs.

2 An algorithm for finding self-similar trajectories

Beside the group $\mathcal{G}$, there exists a subgroup of the permutation group S_l that acts on state space of system (4). This group permutes components y_i^j of the vectors y^j that correspond to equal weights m_i. Let us denote this subgroup by $\mathcal{S}$. Clearly it takes optimal trajectories to optimal trajectories. Moreover, the action of $\mathcal{S}$ commutes with the action of $\mathcal{G}$. Hence, $\mathcal{S}$ acts also on the spaces Σ and Σ^*.

There exists an involution $\mathcal{T}$ of system (4), which multiplies the coordinate y_i^j by $(-1)^{j+1}$. Suppose $y(t)$ is a trajectory of system (4); then $y'(t) = \mathcal{T}(y(-t))$ is also a trajectory. However, $\mathcal{T}$ does not take optimal trajectories to optimal ones.

We look for periodic orbits in Σ^* that have a finite number of control switchings and consist of regular arcs. Let us index the l vertices of the simplex U of admissible controls. We denote control (3) by *control* i, $i = 1, \ldots, l$.

Theorem 2.1. *Self-similar trajectories with constant control exist only for odd n. In this case for any $i \in \{1, \ldots, l\}$ there is a unique self-similar trajectory with constant control i. This trajectory passes through the origin and consists of three orbits of the group $\mathcal{G}$. One of them corresponds to a stationary point on Σ.*

The proof is by integrating the system dynamics and applying Proposition 1.1.

Proposition 2.2. *If n is even, then there are no stationary points on Σ. If n is odd, then there exist exactly $2^l - 2$ stationary points on Σ. l of them correspond to regular trajectories.*
Proof. There is a bijection between stationary points on Σ and optimal self-similar trajectories with constant control. By Theorem 2.1, there are no such trajectories for even n. Suppose n is odd. It is not hard to deduce from the maximum principle that any proper face of the simplex U contains a unique control that is realized on an optimal trajectory with constant control. But there are $2^l - 2$ such faces. The regular trajectories correspond to the l vertices of the simplex. □

Consider an s'-chain ζ ($s' > 1$), where the controls $i_1, \dots, i_{s'}$ are used successively, thereafter the chain closes. Denote the initial switching point of ζ by $\tilde{q}_0$. Denote the next switching point, which is attained after using control i_1, by $\tilde{q}_1$, the next by $\tilde{q}_2$ and so on, up to the terminal point $\tilde{q}_{s'}$, which coincides with $\tilde{q}_0$. Consider a point $q_0 \in T^*M$ on the orbit $\tilde{q}_0$. Denote the trajectory of system (4) that goes through q_0 by ρ. Denote the next switching point on ρ by q_1, the next by q_2 and so on, up to $q_{s'}$. The point $q_{s'}$ lies again on the orbit $\tilde{q}_{s'} = \tilde{q}_0$, but in general it does not coincide with q_0. There exists exactly one positive number λ' such that the element $\mathcal{G}_{\lambda'} \in \mathcal{G}$ takes q_0 to $q_{s'}$.

Definition 2.3. We call the number λ' the *contraction coefficient* of the s'-chain ζ.

The definition is correct, because λ' does not depend on the initial point $q_0 \in \tilde{q}_0$. By Proposition 1.1, the following assertions hold.

Theorem 2.4. *An s-chain with contraction coefficient λ is optimal if and only if $\lambda < 1$.* □

Corollary 2.5. *Trajectories in Σ^* that converge to on optimal s'-chain ζ are optimal and lie in the space Σ.* □

The complexity of the equations determining $\tilde{q}_0$ and λ' quickly increases with s'. But if we restrict our investigations to chains that are invariant with respect to an element of the discrete symmetry group $\mathcal{S}$, then the number of relevant switchings can be reduced.

Suppose there exists a divisor s of s' and a permutation $\sigma \in \mathcal{S}$ such that σ takes $\tilde{q}_0$ to $\tilde{q}_s$. There exists a number λ such that $\mathcal{G}_\lambda(\sigma(q_0)) = q_s$. Clearly $\lambda' = \lambda^{s'/s}$, and the condition $\lambda' < 1$ is equivalent to $\lambda < 1$. Instead of the equation $\mathcal{G}_{\lambda'}(q_0) = q_{s'}$ we can now consider the simpler equation $\mathcal{G}_\lambda(\sigma(q_0)) = q_s$, for fixed σ and $i_1, \dots, i_s$.

By definition, put $i_0 = \sigma^{-1}(i_s)$ and $i_{s+1} = \sigma(i_1)$. For any $k \in \{1, \dots, s\}$ the point q_k lies on the orbit $\tilde{q}_k$. On the arc that connects q_{k-1} and q_k the control i_k is applied. On the interior of this arc the condition of optimality of control i_k holds:

$$y^1_{i_k} < y^1_{i_{k'}} \ \forall \ i_{k'} \neq i_k. \tag{5}$$

Denote the time that system (4) needs to pass from q_{k-1} to q_k by t_k. Let us compute q_k as a function of q_{k-1} by integrating (4) with control i_k. Assemble the coordinates of phase space in a vector $y = (y_1^{2n}, \dots, y_l^{2n}, y_1^{2n-1}, \dots, y_l^1)^T \in \mathbb{R}^{2nl}$. Then we can rewrite system (4) as $\dot{y} = Ay + b_{i_k}$. Here A is a constant matrix. The diagonal that is obtained by shifting the main diagonal by l positions downwards is filled with 1's, the rest with zeros. The constant vector b_i depends only on the control i. Its first l elements form the control i multiplied by

$(-1)^{n-1}$, the other elements are zero. The matrix exponent $F(t) = e^{tA}$ is

$$F(t) = \begin{pmatrix} D_0 & 0 & \dots & 0 \\ D_1 & D_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ D_{2n-1} & D_{2n-2} & \dots & D_0 \end{pmatrix}.$$

Here $D_i = \frac{t^i}{i!} I_l$ and I_l is the $l \times l$ identity matrix. By integrating the system with initial value q_{k-1} we obtain $y(q_k) = F(t_k)y(q_{k-1}) + \bar{B}_k$. Here $\bar{B}_k$ depends only on t_k and b_{i_k},

$$\bar{B}_k = \int_0^{t_k} F(t_k - \tau) b_{i_k}\, d\tau = C(t_k) \int_0^1 F(1-\tau) b_{i_k}\, d\tau = C(t_k) B_{i_k}.$$

Here $C(t) = \operatorname{diag}(tI_l, t^2 I_l, \dots, t^{2n} I_l)$ is a diagonal $2nl \times 2nl$-matrix, and $B_i = \int_0^1 F(1-\tau) b_i\, d\tau$ is a constant vector, which depends only on control i.

By iterating the equation $y(q_k) = F(t_k)y(q_{k-1}) + \bar{B}_k$ we finally obtain

$$y(q_k) = F(\sum_{i=1}^{k} t_i) y(q_0) + \sum_{j=1}^{k} F(\sum_{i=j+1}^{k} t_i) \bar{B}_j. \tag{6}$$

By definition empty sums are zero. Denote the linear operator corresponding to the permutation σ by H_σ. The action of the element $\mathcal{G}_\lambda$ can be described as multiplication by the matrix $C(\lambda)$. Hence we get the polynomial system of equations

$$y(q_s) = y(\mathcal{G}_\lambda(\sigma(q_0))) = C(\lambda) H_\sigma y(q_0) = F(\sum_{i=1}^{s} t_i) y(q_0) + \sum_{j=1}^{s} F(\sum_{i=j+1}^{s} t_i) \bar{B}_j. \tag{7}$$

Further, $F(t)$ is block-triangular with identity matrices on the diagonal. Since eigenvalues of permutation matrices have absolute value 1, we get

Proposition 2.6. *Suppose* $|\lambda| \neq 1$, *then* $\det(C(\lambda)H_\sigma - F(\sum_{i=1}^{s} t_i)) \neq 0$. □

For optimal chains we have $|\lambda| \neq 1$, and equations (7) can be resolved with respect to $y(q_0)$:

$$y(q_0) = \left(C(\lambda) H_\sigma - F(\sum_{i=1}^{s} t_i) \right)^{-1} \left(\sum_{j=1}^{s} F(\sum_{i=j+1}^{s} t_i) \bar{B}_j \right). \tag{8}$$

The point q_k lies on the switching hypersurface from control i_k to control i_{k+1}. This implies $y^1_{i_k}(q_k) = y^1_{i_{k+1}}(q_k)$. By v_k denote the $2nl$-dimensional row vector that has a 1 on position $((2n-1)l + i_k)$ and a -1 on position $((2n-1)l + i_{k+1})$, all other elements being zero. v_k depends only on the controls i_k and i_{k+1}. Then $y^1_{i_k}(q_k) = y^1_{i_{k+1}}(q_k)$ transforms into

$$v_k \left[F(\sum_{i=1}^{k} t_i) y(q_0) + \sum_{j=1}^{k} F(\sum_{i=j+1}^{k} t_i) \bar{B}_j \right] = 0. \tag{9}$$

For $k = 0, \dots, s-1$ we obtain a system of s polynomial equations. It is easily checked that equation (9) for $k = s$ follows from equation (9) for $k = 0$ and equation (7).

Equations (7) and (9) form an overdetermined linear system of $2nl+s$ equations with respect to the $2nl$ unknown components of the vector $y(q_0)$. Let us join the coefficient matrix of this system and its right-hand side to the $(2nl+s)\times(2nl+1)$-matrix

$$L = \begin{pmatrix} C(\lambda)H_\sigma - F(\sum_{i=1}^{s} t_i) & \sum_{j=1}^{s} F(\sum_{i=j+1}^{s} t_i)\bar{B}_j \\ v_0 & 0 \\ v_1 F(t_1) & -v_1\bar{B}_1 \\ \vdots & \vdots \\ v_{s-1}F(\sum_{i=1}^{s-1} t_i) & -v_{s-1}\left(\sum_{j=1}^{s-1} F(\sum_{i=j+1}^{s-1} t_i)\bar{B}_j\right) \end{pmatrix}. \tag{10}$$

System (7), (9) has a solution if and only if all minors of dimension $2nl+1$ of L vanish. By Proposition 2.6, this is equivalent to vanishing of all s minors that contain the first $2nl$ rows. Thus we obtain a system of s polynomial equations on $s+1$ unknown quantities $t_1,\dots,t_s;\lambda$. Since the point q_0 on the orbit $\tilde{q}_0$ can be chosen arbitrarily, we can introduce an additional condition, e.g. $\sum_{i=1}^{s} t_i = 1$. Then the number of unknowns will be equal to the number of equations. Any solution that satisfies conditions (5) and the inequalities $t_i > 0\ \forall\ i = 1,\dots,s;\ \lambda > 0, \lambda \neq 1$ determines an s'-chain on Σ^*. The coordinates of the point q_0 are to be found from (8). It can easily be seen that the restrictions on y_i^j are fulfilled. On ζ the a priori chosen sequence of controls $i_1,\dots,i_s$ is realized and ζ is invariant with respect to the symmetry σ. Theorem 2.4 yields that ζ is optimal if $\lambda < 1$.

Any minor of L that contains the first $2nl$ rows is a homogeneous polynomial of degree $2n$ with respect to $t_1,\dots,t_s$ and a polynomial of degree $nl(2n+1)$ with respect to λ. Hence the degree of the polynomials does not depend on the parameters s, σ, $i_1,\dots,i_s$.

Thus we obtain the following algorithm for detection of chains with a given sequence of controls and invariance with respect to a given symmetry $\sigma \in \mathcal{S}$. Any solution corresponds to a one-parametric family of self-similar trajectories related by the action of $\mathcal{G}$.

Algorithm 1: *(compare also [4, Section 3]) Calculation of optimal self-similar trajectories.*

1. Calculate the matrix L according to (10).

2. Compose a system of polynomial equations by putting all minors of dimension $2nl+1$ of L to zero that contain the first $2nl$ rows.

3. Solve the system with an additional constraint eliminating homogeneity in the t_i.

4. Calculate the coordinates of the initial point q_0 of the self-similar trajectory according to (8), and of the other switching points according to (6).

5. Check the conditions $\lambda \in (0,1), t_i > 0$ and (5).

A similar algorithm was given in [3] for the case $l = 2$, $m_1 = m_2 = 1$, $s = 1$, $\sigma = (12)$, n arbitrary, and in [5] for the case $l = 2$, $n = 2$, $s = s' = 2$, m_i arbitrary.

3 The linear part of the Poincaré map

Let ζ be an s'-chain, invariant with respect to the permutation $\sigma \in \mathcal{S}$, with parameters $t_1,\dots,t_s;\lambda$ and switching points $\tilde{q}_0,\dots,\tilde{q}_s$. Let us investigate the behaviour of the dynamical system on Σ^* in a neighbourhood of ζ. We consider only the generic case, when ζ transversally intersects the switching hypersurfaces at the switching points and the switching points do not lie on the intersection of more than one switching hypersurface. Then

$$y^1_{i_k}(q_k) = y^1_{i_{k+1}}(q_k) < y^1_{i_{k'}}(q_k) \;\forall\; i_{k'} \notin \{i_k, i_{k+1}\};\; \dot y^1_{i_k}(q_k) - \dot y^1_{i_{k+1}}(q_k) = y^2_{i_k}(q_k) - y^2_{i_{k+1}}(q_k) > 0. \tag{11}$$

Let us consider a neighbourhood $\tilde Q$ of $\tilde q_0$ on the switching hypersurface $\tilde\Gamma_{i_0 i_1}$ from control i_0 to control i_1. If $\tilde q \in \tilde Q$ is sufficiently close to $\tilde q_0$, then the trajectory that goes through $\tilde q$ will intersect $\tilde\Gamma_{i_0 i_1}$ again in some point $\tilde P'(\tilde q) \in \tilde Q$ after s' control switchings. The mapping $\tilde P'$ that takes $\tilde q$ to $\tilde P'(\tilde q)$ is called the Poincaré map associated with the periodic orbit ζ. The point $\tilde q_0$ is a fixed point of $\tilde P'$. It is not hard to prove that conditions (11) are sufficient for nondegeneracy of the Poincaré map at $\tilde q_0$.

Let us define a mapping $\tilde P$ on a neighbourhood of $\tilde q_0$. Suppose the point $\tilde q \in \tilde Q$ is sufficiently close to $\tilde q_0$. Consider the trajectory that goes through $\tilde q$. After s control switchings on this trajectory we obtain a point $\tilde q'$. By definition, put $\tilde P(\tilde q) = \sigma^{-1}(\tilde q')$. Clearly $\tilde q_0$ is a fixed point of the map $\tilde P$, and $\tilde P' \equiv \tilde P^{s'/s}$. Conditions (11) are sufficient for nondegeneracy of $\tilde P$ at $\tilde q_0$. The map $\tilde P$ is the Poincaré map associated with the image of ζ in orbit space Σ^*/σ.

Let us compute the maps $\tilde P', \tilde P$. Define operators $F_i(t)$ $(i = 1, \ldots, l)$ acting on T^*M by $F_i(t) : y \mapsto F(t)y + C(t)B_i$. The operator $F_i(t)$ is given by the transition matrix of the dynamical system defined by the control i.

Let $Q_k \subset T^*M$ be a neighbourhood of $q_k \in \tilde q_k$. Then q_k lies on the switching hypersurface $\Gamma_{i_k i_{k+1}}$ from control i_k to control i_{k+1}. Let q be a point in Q_k. By ρ_k (resp. ρ_{k+1}) denote the trajectory through q of the dynamical system on T^*M defined by control i_k (resp. i_{k+1}). The trajectories ρ_k, ρ_{k+1} intersect the hypersurface $\Gamma_{i_k i_{k+1}}$ in some points q', q''. By τ_k (resp. τ_{k+1}) denote the time that is needed to get from q to q' (resp. q'') along the trajectory ρ_k (resp. ρ_{k+1}). The functions $\tau_k(q), \tau_{k+1}(q)$ are smooth in the neighbourhood Q_k, provided Q_k is sufficiently small. These functions are zero on $\Gamma_{i_k i_{k+1}}$, specifically at q_k.

Let us define the mapping $T_{i_k i_{k+1}} = F_{i_{k+1}}(\tau_k) \circ F_{i_k}(-\tau_k)$ on Q_k. Any point on $\Gamma_{i_k i_{k+1}}$ is a fixed point of $T_{i_k i_{k+1}}$, specifically q_k. We can define the following mappings on a sufficiently small neighbourhood Q_0 of q_0: $P = \sigma^{-1} \circ \mathcal{G}_{\lambda^{-1}} \circ T_{i_s i_{s+1}} \circ F_{i_s}(t_s) \circ \cdots \circ T_{i_1 i_2} \circ F_{i_1}(t_1)$, $P' = \mathcal{G}_{(\lambda')^{-1}} \circ T_{i_{s'} i_{s'+1}} \circ F_{i_{s'}}(t_{s'}) \circ \cdots \circ T_{i_1 i_2} \circ F_{i_1}(t_1)$. The point q_0 is a fixed point of the mappings P, P'. These mappings commute with the action of $\mathcal{G}$. The mapping $F_{i_1}(\tau_1)$, which projects Q_0 on the switching surface $\Gamma_{i_0 i_1}$ along the trajectories of the system defined by control i_1, also commutes with $\mathcal{G}$. Hence the compositions $F_{i_1}(\tau_1) \circ P$, $F_{i_1}(\tau_1) \circ P'$ induce mappings of a neighbourhood of $\tilde q_0$ on the switching surface $\tilde\Gamma_{i_0 i_1}$ in Σ^*. It is not hard to see that these induced mappings coincide with the Poincaré maps $\tilde P$, $\tilde P'$. Therefore the linear part of $\tilde P$, $\tilde P'$ can be computed from the linear part of the mappings P, P' at q_0.

Note that the phase velocity vector $v_t = Ay(q_0) + b_{i_1}$ is an eigenvector of the Jacobi matrices $\frac{\partial P(y)}{\partial y}|_{y=y(q_0)}$, $\frac{\partial P'(y)}{\partial y}|_{y=y(q_0)}$ with eigenvalues $\lambda^{-1}, (\lambda')^{-1}$ respectively. Hence the differentials of P and P' induce linear operators on the quotient space $T_{q_0}(T^*M)/\mathrm{span}\{v_t\}$.

Since any point of the switching surface $\Gamma_{i_0 i_1}$ is invariant with respect to the projection operator $F_{i_1}(\tau_1)$, the differential of the restriction of the mapping $F_{i_1}(\tau_1) \circ P$ (resp. $F_{i_1}(\tau_1) \circ P'$) on $\Gamma_{i_0 i_1}$ coincides with the differential of P (resp. P'). Hence the action of the differentials $D(F_{i_1}(\tau_1) \circ P)$, $D(F_{i_1}(\tau_1) \circ P')$ on the tangent space $T_{q_0}\Gamma_{i_0 i_1}$ can be identified with the action of the differentials of P, P' on the quotient space $T_{q_0}(T^*M)/\mathrm{span}\{v_t\}$.

The vector v_λ given by $\frac{\partial}{\partial \lambda}\mathcal{G}_\lambda(y(q_0))|_{\lambda=1} = \frac{dC(\lambda)}{d\lambda}|_{\lambda=1}y(q_0)$ in the tangent space to q_0 is tangent to the orbit $\tilde{q}_0$. Since the mappings $F_{i_1}(\tau_1) \circ P$, $F_{i_1}(\tau_1) \circ P'$ commute with the action of $\mathcal{G}$, v_λ is an eigenvector of the differentials $D(F_{i_1}(\tau_1) \circ P)$, $D(F_{i_1}(\tau_1) \circ P')$ with eigenvalue 1. The vector v_λ consists of the components of the vectors $y^{2n}(q_0), 2y^{2n-1}(q_0), \dots, 2ny^1(q_0)$ written one after the other. Hence the differentials $D(F_{i_1}(\tau_1) \circ P)$, $D(F_{i_1}(\tau_1) \circ P')$ induce an action on the quotient space $T_{q_0}\Gamma_{i_0 i_1}/\mathrm{span}\{v_\lambda\}$. This action can be identified with the action of the differentials of the Poincaré maps $\tilde{P}$, $\tilde{P}'$ on the tangent space $T_{\tilde{q}_0}\tilde{\Gamma}_{i_0 i_1}$. We proved the following assertion.

Corollary 3.1. *The action of the differentials of the Poincaré maps $\tilde{P}$, $\tilde{P}'$ can be canonically identified with the action induced by the differentials of the mappings P, P' on the quotient space $T_{q_0}(T^*M)/\mathrm{span}\{v_t, v_\lambda\}$.* □

Let us compute the derivatives $\frac{\partial P}{\partial y}$, $\frac{\partial P'}{\partial y}$. Consider the differential $DT_{i_k i_{k+1}}$ at the point q_k. Note that any vector in $T_{q_k}(T^*M)$ tangent to $\Gamma_{i_k i_{k+1}}$ is invariant with respect to $DT_{i_k i_{k+1}}$. Furthermore, $DT_{i_k i_{k+1}}$ takes the phase velocity vector $Ay(q_k) + b_{i_k}$ induced by the control i_k to the phase velocity vector $Ay(q_k) + b_{i_{k+1}}$ induced by the control i_{k+1}.

Denote the base vectors in the tangent fibration $T(T^*M)$ corresponding to differentiation with respect to y_i^j by e_i^j. Let us introduce another system of base vectors e'^j_i, where $e^1_{i_k}$ is replaced by $e'^1_{i_k} = e^1_{i_k} + e^1_{i_{k+1}}$, $e^1_{i_{k+1}}$ is replaced by $e'^1_{i_{k+1}} = Ay(q_k) + b_{i_k}$, and the other vectors e'^j_i coincide with the corresponding vectors e^j_i. Denote the image of e'^j_i under the action of $DT_{i_k i_{k+1}}$ by e''^j_i. Then $e''^1_{i_{k+1}}$ is given by $e''^1_{i_{k+1}} = Ay(q_k) + b_{i_{k+1}}$, and the other vectors e''^j_i coincide with the corresponding vectors e'^j_i.

The switching surface $\Gamma_{i_k i_{k+1}}$ is given by $y^1_{i_k}(q_k) = y^1_{i_{k+1}}(q_k)$. It is easily shown that all vectors e'^j_i, e''^j_i except $e'^1_{i_{k+1}}$, $e''^1_{i_{k+1}}$ are tangent to $\Gamma_{i_k i_{k+1}}$. By (11), the vectors $Ay(q_k) + b_{i_k}$, $Ay(q_k) + b_{i_{k+1}}$ are transversal to $\Gamma_{i_k i_{k+1}}$. Hence the matrix E' (resp. E'') consisting of the column vectors e'^j_i (resp. e''^j_i) is nonsingular. Note that the vectors b_{i_k}, $b_{i_{k+1}}$ differ only at the i_k-th and i_{k+1}-th positions. By substituting these vectors we obtain $e''^1_{i_{k+1}} - e'^1_{i_{k+1}} = (-1)^{n-1}\frac{1}{m_{i_k}}(\sum_{i=1}^l m_i)e^{2n}_{i_k} + (-1)^n \frac{1}{m_{i_{k+1}}}(\sum_{i=1}^l m_i)e^{2n}_{i_{k+1}}$. At the $((2n-1)l + i_k)$-th position of $e'^1_{i_{k+1}}$, $e''^1_{i_{k+1}}$, which corresponds to the base vector $e^1_{i_k}$, we have the term $y^2_{i_k}(q_k)$; at the $((2n-1)l + i_{k+1})$-th position, which corresponds to the base vector $e^1_{i_{k+1}}$, we have the term $y^2_{i_{k+1}}(q_k)$. The differential $DT_{i_k i_{k+1}}$ is given by $E''(E')^{-1}$. By substituting we obtain

$$DT_{i_k i_{k+1}} = I_{2nl} + \frac{(-1)^{n-1}\frac{1}{m_{i_k}}\sum_{i=1}^l m_i}{y^2_{i_k}(q_k) - y^2_{i_{k+1}}(q_k)} e_{i_k,(2n-1)l+i_k} + \frac{(-1)^n \frac{1}{m_{i_k}}\sum_{i=1}^l m_i}{y^2_{i_k}(q_k) - y^2_{i_{k+1}}(q_k)} e_{i_k,(2n-1)l+i_{k+1}}$$
$$+ \frac{(-1)^n \frac{1}{m_{i_{k+1}}}\sum_{i=1}^l m_i}{y^2_{i_k}(q_k) - y^2_{i_{k+1}}(q_k)} e_{i_{k+1},(2n-1)l+i_k} + \frac{(-1)^{n-1}\frac{1}{m_{i_{k+1}}}\sum_{i=1}^l m_i}{y^2_{i_k}(q_k) - y^2_{i_{k+1}}(q_k)} e_{i_{k+1},(2n-1)l+i_{k+1}}, \quad (12)$$

where $e_{r,s}$ is a matrix which has a 1 at position (r, s) and is elsewhere filled with zeros. Thus the matrix $DT_{i_k i_{k+1}}$ differs from I_{2nl} at four positions, which are located at rows i_k, i_{k+1}, which correspond to the base vectors $e^{2n}_{i_k}, e^{2n}_{i_{k+1}}$, and columns $(2n-1)l+i_k$, $(2n-1)l+i_{k+1}$, which correspond to the base vectors $e^1_{i_k}, e^1_{i_{k+1}}$.

The mappings $F_{i_k}(t_k)$ are affine and their differential is given by $F(t_k)$. Since every element of the groups $\mathcal{S}$, $\mathcal{G}$ is a linear transformation, it coincides with its differential. Thus

$$\begin{aligned} DP &= H_{\sigma^{-1}}C(\lambda^{-1})DT_{i_s i_{s+1}}F(t_s)\dots DT_{i_1 i_2}F(t_1), \\ DP' &= C((\lambda')^{-1})DT_{i_{s'} i_{s'+1}}F(t_{s'})\dots DT_{i_1 i_2}F(t_1). \end{aligned} \quad (13)$$

We obtain the following algorithm for calculating the differentials of the Poincaré maps $\tilde{P}, \tilde{P}'$ of a given s'-chain ζ that is invariant with respect to a given permutation σ.

Algorithm 2: *Calculus of the linear part of the Poincaré map of a given periodic orbit.*

1. Compute the matrices DP, DP' according to (12),(13).

2. Compute the vector $v_t = Ay(q_0) + b_{i_1}$ and the vector v_λ, which consists of the components of the vectors $y^{2n}(q_0), 2y^{2n-1}(q_0), \dots, 2ny^1(q_0)$ written one after the other. The vectors v_t, v_λ span a subspace, which is invariant with respect to the transformations DP, DP'.

3. Compute the linear transformations that are induced by DP, DP' on the quotient space $T_{q_0}(T^*M)/\mathrm{span}\{v_t, v_\lambda\}$.

The space $T_{q_0}(T^*M)$ has dimension $2n(l-1)$, whereas the matrices of the differentials DP, DP' given by (13) have size $2nl \times 2nl$. The space $T_{q_0}(T^*M)$ is an invariant subspace of these matrices. Now we shall investigate the linear transformations induced by matrices (13) on the quotient space $\mathrm{span}\{e^{2n}_1, \dots, e^1_l\}/T_{q_0}(T^*M)$.

Firstly, consider how the maps P, P' act on the quotient space of the $2nl$-dimensional space parametrized by the coordinates $y^{2n}_1, \dots, y^1_l$ with respect to the subspace T^*M. This quotient space can be parametrized by the coordinates $Y^j = \sum_{i=1}^l m_i y^j_i,\ j = 1, \dots, 2n$. Since $\sum_{i=1}^l m_i u_i = 0$, the derivatives $\dot{Y}^j$ do not depend on the control. They are given by $\dot{Y}^{2n} = 0, \dot{Y}^j = Y^{j+1},\ j = 1, \dots, 2n-1$. Hence transition by time t is a linear operator given by a triangular matrix with 1's on the diagonal. The sums Y^j are invariant with respect to the action of $\mathcal{S}$, whereas $\mathcal{G}_\lambda$ multiplies Y^j by λ^{2n+1-j}. Therefore the mapping P (resp. P') acts linearly on Y^j, and this action is given by a triangular $2n \times 2n$-matrix with $\lambda^{-1}, \dots, \lambda^{-2n}$ (resp. $\lambda'^{-1}, \dots, \lambda'^{-2n}$) on the diagonal. Obviously the induced mappings coincide with their differentials. Thus we proved the following assertion.

Corollary 3.2. *The action induced by matrices (13) on the quotient space $\mathrm{span}\{e^{2n}_1, \dots, e^1_l\}/T_{q_0}(T^*M)$ is given by diagonizable operators with simple eigenvalues $\lambda^{-1}, \dots, \lambda^{-2n}$ (resp. $\lambda'^{-1}, \dots, \lambda'^{-2n}$).* □

From the spectrum of DP, DP' we can conclude on the behaviour of the optimal synthesis in a neighbourhood of optimal self-similar trajectories. By Corollary 2.5, we have

Theorem 3.3. *Suppose ζ is an optimal s'-chain with control sequence $i_1, \dots, i_{s'}$, satisfying conditions (11). Let its Poincaré map $\tilde{P}'$ be hyperbolic, and m_- be the dimension of its stable invariant manifold. Then ζ is embedded in an $(m_- + 1)$-dimensional integral submanifold* *of Σ. The preimage of this submanifold in state space is an $(m_- + 2)$-dimensional integral submanifold, which consists of optimal trajectories. These trajectories undergo chattering with periodic sequence $\overline{i_1, \dots, i_{s'}}$ of controls when approaching the origin.* $\square$

4 Contracting groups and Poincaré map

In this section we prove that the action of $\mathcal{G}$ contracts the symplectic structure on T^*M. We will deduce some properties of the linear part of the Poincaré maps associated with optimal periodic orbits in Σ^*. We depart from a generalization of the Theorem of Lyapunov-Poincaré, whose proof is purely algebraic and omitted here for space limitation reasons.

Proposition 4.1. *Suppose B is a regular complex $n \times n$ matrix and $\Lambda \neq 0$ is a complex number. Suppose W is a complex $n \times n$ matrix such that $\Lambda B = W^T B W$. Then for any complex number $\lambda \neq 0$ and any natural number $m \in \mathbb{N}$ the equations $B[\mathrm{Ker}(W - \lambda)^m] = \mathrm{Ker}(W^T - \frac{\Lambda}{\lambda})^m$, $B^T[\mathrm{Ker}(W - \lambda)^m] = \mathrm{Ker}(W^T - \frac{\Lambda}{\lambda})^m$ are satisfied.*

Corollary 4.2. *Suppose λ is an eigenvalue of W; then $\frac{\Lambda}{\lambda}$ is also an eigenvalue of W. The dimensions of their root subspaces and their proper subspaces coincide.* $\square$

Suppose the matrices B, W and the number Λ are real, and W has only simple eigenvalues. The space $\mathbb{R}^n$ decomposes into a direct sum of minimal invariant subspaces of the operator W. A minimal invariant subspace has dimension 1 if the corresponding eigenvalue is real, and it has dimension 2, if it corresponds to a complex-conjugated pair of eigenvalues. Denote the minimal invariant subspaces of W by $V_1, \dots, V_r$. Let us put in correspondence to each subspace V_i a number λ_i. If V_i is onedimensional, then define λ_i as the corresponding real eigenvalue. If V_i is twodimensional, then define λ_i as the corresponding complex eigenvalue that has positive imaginary part. By $\bar{}$ denote complex conjugation.

Definition 4.3. We call a minimal invariant subspace V_j conjugated to the subspace V_i, if $\lambda_j \bar{\lambda}_i = \Lambda$.

By Corollary 4.2, for any subspace V_i there exists a conjugated subspace V_j. Suppose V_i, V_j are conjugated subspaces. Then we have $V_i = V_j$ if and only if $|\lambda_i|^2 = \Lambda$.

Proposition 4.4. *Suppose the assumptions made above are satisfied. Then*
a) For any minimal invariant subspace V_i of W there exist vectors $w_i \in V_i$, $w_j \in V_j$ such that $w_i^T B w_j \neq 0$. Here V_j is the subspace conjugated to V_i.
b) Suppose the minimal invariant subspaces V_i, V_j are not conjugated. Then for any vectors $w_i \in V_i$, $w_j \in V_j$ we have $w_i^T B w_j = 0$.

The proof uses the theorem on the Jordan structure of a matrix and is omitted here.

Consider a differentiable manifold V of even dimension $2m$. Let ω be a closed nondegenerate differential 2-form inducing a symplectic structure on V. By $T_z V$ denote the tangent space to V at the point $z \in V$. Then to any vector $v \in T_z V$ a 1-form $\theta_v \in T_z^* V$ in the cotangent space at the point z is assigned. The form θ_v takes any vector $u \in T_z V$ to $\theta_v(u) = \omega(v, u)$ and is the convolution $i_v \omega$ of the form ω with the vector v.

Definition 4.5. A one-parametric group $\mathbf{G}$ of diffeomorphisms g_γ of V, where $\gamma \in \mathbb{R}$ is an additive parameter, is called a *contracting group*, if the following condition holds. By the action of g_γ the form ω is multiplied by $e^{-\gamma}$, i.e. for any point $z \in V$ we have $\omega(z) = e^{-\gamma} g_\gamma^* \omega(g(z))$.

Suppose L is a Lagrange submanifold of V, i.e. a submanifold of dimension m on whose tangent space ω is zero. Then for any $z \in L$ and $v \in T_z L$ we have $T_z L \subset \operatorname{Ker} \theta_v$. Obviously any element of a contracting group takes Lagrange manifolds to Lagrange manifolds.

Suppose v_G is a smooth vector field on V. It generates a one-parametric group $\mathbf{G}$ of diffeomorphisms of the manifold V.

Proposition 4.6. *The following conditions are equivalent:*
(i) the group $\mathbf{G}$ *is a contracting group,*
(ii) the form $\theta_{v_G} = i_{v_G}\omega$ *satisfies the equation* $d\theta_{v_G} = \omega$.

Proof. By $\mathcal{L}_v$ denote the Lie derivative with respect to the vector field v. By definition, a group is contracting iff for any point $z \in V$ and any two vectors $u, w \in T_z V$ we have $\omega(g_\gamma(z))(Dg_\gamma(u), Dg_\gamma(w)) = e^\gamma \omega(z)(u, w)$. This equation is satisfied iff the equation $\mathcal{L}_{v_G}\omega = \omega$ holds. Since ω is closed, we obtain $\mathcal{L}_{v_G}\omega = d(i_{v_G}\omega) = d\theta_{v_G}$. □

Suppose $v_t(z)$ is a vector field on V. Parametrize the trajectories of the corresponding flow by time t. It is well-known that the form $\theta_{v_t}(z)$ is closed if and only if the symplectic form ω is invariant with respect to transitions along the trajectories of v_t. In this case v_t is a Hamiltonian flow. If θ_{v_t} is exact, then there exists a Hamiltonian $H(z)$ such that $\theta_{v_t} = -dH$.

Let L be a continuously differentiable integral Lagrange manifold, which is invariant with respect to the action of a contracting group $\mathbf{G}$. Then the generating vector field v_G of $\mathbf{G}$ and the vector field v_t are tangent to L. Suppose there exists a point $z_0 \in L$ and numbers $T > 0$, $\gamma \neq 0$ such that $g_\gamma \circ \Phi_T(z_0) = z_0$, where g_γ is an element of the group $\mathbf{G}$ and Φ_T is a shift by time T along the trajectories of the Hamiltonian system. The differential of the mapping $g_\gamma \circ \Phi_T$ is an automorphism of the tangent space $T_{z_0}V$. Suppose u, v are tangent vectors at the point z_0. Since the form ω is invariant under shifts in time, we have $\omega(u, v) = \omega(D\Phi_T(u), D\Phi_T(v))$, where $D\Phi_T(u), D\Phi_T(v)$ are tangent vectors at the point $\Phi_T(z_0)$. On the other hand, the action of the differential Dg_γ multiplies the form ω by e^γ. Therefore we have

$$\omega(D(g_\gamma \circ \Phi_T)(u), D(g_\gamma \circ \Phi_T)(v)) = e^\gamma \omega(u, v). \tag{14}$$

By W denote the matrix of the linear mapping $D(g_\gamma \circ \Phi_T)$, by ω_{z_0} denote the skew-symmetric matrix that corresponds to the form $\omega(z_0)$. Then (14) becomes $e^\gamma \omega_{z_0} = W^T \omega_{z_0} W$.

Since L is invariant under any shift Φ_T and any diffeomorhism g_γ, the subspace $T_{z_0}L$ of the tangent space $T_{z_0}V$ is invariant under the mapping W.

Suppose W has only simple eigenvalues. Then the space $T_{z_0}V$ decomposes into a direct sum of minimal invariant subspaces of W. Denote these subspaces by $V_1, \dots, V_r$. The tangent space $T_{z_0}L$ is an invariant subspace of W. Hence there exists a subset $V_S \subset \{V_1, \dots, V_r\}$ such that $T_{z_0}L = \operatorname{span}\{v \in V_i \mid V_i \in V_S\}$. Since $T_{z_0}L$ is isotropic, for any $V_i, V_j \subset T_{z_0}L$ and $v_i \in V_i, v_j \in V_j$ we have $\omega(v_i, v_j) = 2v_i^T \omega_{z_0} v_j = 0$.

Since the matrix ω_{z_0} is nonsingular, the matrix W satisfies the assumptions of Propositions 4.1 and 4.4. By Corollary 4.2, the set of eigenvalues of W breaks up into pairs. The product of the eigenvalues in each pair equals e^γ.

By Proposition 4.4, conjugated subspaces V_i, V_j cannot at the same time be contained in the set V_S. Since $\dim T_{z_0}L = \frac{1}{2}\dim T_{z_0}V$, there cannot exist any subspace V_i that coincides with its conjugated subspace. Hence the number of minimal invariant subspaces is even. There exist exactly $2^{\frac{r}{2}}$ Lagrange subspaces of $T_{z_0}V$ that are invariant under W.

Let us summarize these results.

Proposition 4.7. *Suppose the matrix $W = D(g_\gamma \circ \Phi_T)$ has only simple eigenvalues. Then the minimal invariant subspaces $V_1, \dots, V_{2r}$ of W and the corresponding eigenvalues $\lambda_1, \dots, \lambda_{2r}$ with nonnegative imaginary part can be arranged in a manner such that the following conditions hold.*
a) $T_{z_0}L = V_1 \oplus V_2 \oplus \cdots \oplus V_r$.
b) $\lambda_i\,\bar\lambda_{r+i} = \bar\lambda_i\,\lambda_{r+i} = e^\gamma$ *for any* $i = 1, \dots, r$.
c) For any $i = 1, \dots, r$ *there exist* $v_i \in V_i, v_{r+i} \in V_{r+i}$ *such that* $\omega(v_i, v_{r+i}) \neq 0$.
d) For any $v_i \in V_i, v_j \in V_j$ *such that* $(j - i) \not\equiv 0 \bmod(r)$ *we have* $\omega(v_i, v_j) = 0$.
e) $\dim V_i = \dim V_{r+i}$ *for any* $i = 1, \dots, r$. □

Suppose the differential of the diffeomorphism g_γ multiplies the vector field v_t by $e^{\kappa\gamma}$, $\kappa > 0$, i.e. at any point $z \in V$ we have $Dg_\gamma(v_t(z)) = e^{\kappa\gamma}v_t(g_\gamma(z))$. This equation holds iff $\mathcal{L}_{v_G}v_t = -\kappa v_t$, i.e.

$$[v_t, v_G] = \kappa v_t. \tag{15}$$

Let us compute the images $W(v_t), W(v_G)$. We have $D\Phi_T(v_t(z_0)) = v_t(\Phi_T(z_0))$. Now we shall compute $D\Phi_T(v_G(z_0))$. We have $\mathcal{L}_{v_t}v_G = \kappa v_t$, therefore $v_G(\Phi_T(z_0)) = D\Phi_T(v_G(z_0)) + \kappa T v_t(\Phi_T(z_0))$. Hence we obtain $D\Phi_T(v_G(z_0)) = (v_G - \kappa T v_t)(\Phi_T(z_0))$. The differential Dg_γ multiplies the vector field v_t by $e^{\kappa\gamma}$ and leaves v_G invariant. This yields

$$W(v_t) = e^{\kappa\gamma}v_t, \qquad W(v_G) = v_G - \kappa e^{\kappa\gamma}Tv_t.$$

Thus the vectors v_t and $v_G + \frac{\kappa T}{1-e^{-\kappa\gamma}}v_t$ are eigenvectors of the matrix W with eigenvalues $e^{\kappa\gamma}$ and 1, respectively.

Suppose v_t, v_G are linearly independent at z_0. Denote the $(2m-2)$-dimensional quotient space $T_{z_0}V/\mathrm{span}\{v_t, v_G\}$ by $\tilde V$. The linear operator W induces an automorphism $\tilde W$ of $\tilde V$. Since $v_t, v_G \in T_{z_0}L$, the quotient space $\tilde L = T_{z_0}L/\mathrm{span}\{v_t, v_G\}$ is well-defined. It is a $(m-2)$-dimensional subspace of $\tilde V$. By Proposition 4.7, the following assertion holds.

Proposition 4.8. *Suppose W has only simple eigenvalues, and the assumptions made above are satisfied. Then the $2r-2$ minimal subspaces $\tilde V_1, \dots, \tilde V_{2r-2}$ of $\tilde V$ that are invariant under the action of $\tilde W$, and the corresponding eigenvalues $\lambda_1, \dots, \lambda_{2r-2}$ with nonnegative imaginary part can be arranged in a manner such that the following conditions hold.*
a) $\tilde L = \tilde V_1 \oplus \tilde V_2 \oplus \cdots \oplus \tilde V_{r-2}$.
b) $\lambda_i\,\bar\lambda_{r-2+i} = \bar\lambda_i\,\lambda_{r-2+i} = e^\gamma$ *for any* $i = 1, \dots, r-2$.
c) $\dim \tilde V_{2r-3} = \dim \tilde V_{2r-2} = 1, \lambda_{2r-3} = e^{(1-\kappa)\gamma}, \lambda_{2r-2} = e^\gamma$.
d) $\dim \tilde V_i = \dim \tilde V_{r-2+i}$ *for any* $i = 1, \dots, r-2$. □

Now we apply these results to the self-similar trajectories of system (4). In our case V is the phase space T^*M with its canonical symplectic structure. The section of T^*M corresponding to the optimal synthesis is a Lagrange submanifold.

Proposition 4.9. *The Fuller group $\mathcal{G}$ is a contracting group. The parameters γ and λ are related to each other by the equation $\lambda^{2n+1} = e^{\gamma}$.*
Proof. We have $v_G(q) = \frac{d\mathcal{G}_\lambda(q)}{d\lambda}(\frac{d((2n+1)\ln\lambda)}{d\lambda})^{-1}|_{\lambda=1} = \frac{1}{2n+1}\frac{dC(\lambda)}{d\lambda}y(q)$. Hence the relation between the generating vector field v_G and the vector field v_λ, which was defined in the previous section, is given by

$$v_G(q) = \frac{1}{2n+1}v_\lambda = \frac{1}{2n+1}\sum_{i=1}^{l}\sum_{j=1}^{2n}(2n+1-j)y_i^j(q)\frac{\partial}{\partial y_i^j}.$$

The phase velocity vector v_t is given by

$$v_t(q) = Ay(q) + b_k = \sum_{i=1}^{l}\sum_{j=2}^{2n} y_i^{j-1}(q)\frac{\partial}{\partial y_i^j} + \sum_{i=1}^{l} u_i\frac{\partial}{\partial y_i^1}.$$

Here k is the applied control. It is easily checked that the Lie bracket $[v_t, v_G]$ is equal to $\frac{1}{2n+1}v_t$. Therefore condition (15) with $\kappa = \frac{1}{2n+1}$ is satisfied.

The symplectic form on $V = T^*M$ is given by $\omega = \frac{1}{2}\sum_{i=1}^{l}\sum_{j=1}^{2n}(-1)^{n+1-j}m_i dy_i^{2n+1-j}\wedge dy_i^j$. In coordinate representation we have

$$(\omega_{ij}) = \frac{1}{2}\begin{pmatrix} 0 & 0 & \dots & 0 & (-1)^n\Delta_m \\ 0 & 0 & \dots & (-1)^{n-1}\Delta_m & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & (-1)^{-n+2}\Delta_m & \dots & 0 & 0 \\ (-1)^{-n+1}\Delta_m & 0 & \dots & 0 & 0 \end{pmatrix}. \qquad (16)$$

Here $\Delta_m = \mathrm{diag}\{m_1, \dots, m_l\}$ is a diagonal $l\times l$-matrix.

Hence the form $\theta_{v_G} = i_{v_G}\omega$ is given by

$$\theta_{v_G}(q) = \frac{1}{2n+1}\sum_{i=1}^{l}\sum_{j=1}^{2n}(-1)^{n+1-j}m_i j y_i^{2n+1-j}(q)dy_i^j.$$

It is not hard to prove that the differential of this form after alternation is equal to (16). Proposition 4.6 concludes the proof. □

Since the mappings $\sigma^{-1}\circ T_{i_s i_{s+1}}\circ F_{i_s}(t_s)\circ\dots\circ T_{i_1 i_2}\circ F_{i_1}(t_1)$, $T_{i_{s'} i_{s'+1}}\circ F_{i_{s'}}(t_{s'})\circ\dots\circ T_{i_1 i_2}\circ F_{i_1}(t_1)$ are not transitions in time of system (4), Proposition 4.7 cannot be applied formally to the mappings P and P'. But recall that for proving Propositions 4.7 and 4.8 we used only that Φ_T preserves the form ω. The mappings $F_i(t)$ are transitions in time and preserve ω. Clearly the action of the group $\mathcal{S}$ preserves ω. The differential of the mapping $T_{i_k i_{k+1}}$ can be represented as a composition of differentials of the mappings $F_{i_{k+1}}(\tau_k)$ and $F_{i_k}(-\tau_k)$ with frozen argument τ_k and the differential of $T_{i_k i_{k+1}}$ at some point on the switching surface $\Gamma_{i_k i_{k+1}}$. The first two differentials are differentials of transitions in time and hence preserve ω. By multiplying (12) and (16), we obtain $(\omega_{ij}) = (DT_{i_k i_{k+1}})^T(\omega_{ij})DT_{i_k i_{k+1}}$. Therefore $T_{i_k i_{k+1}}$ also preserves ω.

Note that on any self-similar trajectory corresponding to an s'-chain with $s' > 1$ the vector fields v_G and v_t are linearly independent. Hence Propositions 4.7 and 4.8 remain valid also for system (4). By L denote the Lagrange section of the cotangent fibration T^*M that is induced by the optimal synthesis. We proved the following assertion on the differentials of the mappings P, P'.

Theorem 4.10. *Suppose the differential DP has only simple eigenvalues. Then the minimal subspaces $V_1, \ldots, V_{2r}$ of the space $T_{q_0}(T^*M)$ that are invariant under the action of DP, and the corresponding eigenvalues $\lambda_1, \ldots, \lambda_{2r}$ with nonnegative imaginary part can be arranged in a manner such that the following conditions hold.*
a) If L is differentiable at q_0, then $T_{q_0}L = V_1 \oplus V_2 \oplus \cdots \oplus V_r$.
b) $\lambda_i\,\bar{\lambda}_{r+i} = \bar{\lambda}_i\,\lambda_{r+i} = \lambda^{-(2n+1)}$ for any $i = 1, \ldots, r$.
c) For any $i = 1, \ldots, r$ there exist $v_i \in V_i, v_{r+i} \in V_{r+i}$ such that $\omega(v_i, v_{r+i}) \neq 0$.
d) For any $v_i \in V_i, v_j \in V_j$ such that $(j - i) \not\equiv 0 \bmod(r)$ we have $\omega(v_i, v_j) = 0$.
e) $\lambda_1 = \lambda^{-1},\ \lambda_2 = 1,\ V_1 = \operatorname{span}\{v_t\},\ V_2 = \operatorname{span}\{v_\lambda + \frac{\sum_{i=1}^{s} t_i}{1-\lambda} v_t\}$.
f) $\dim V_i = \dim V_{r+i}$ for any $i = 1, \ldots, r$.

Here the vector v_t is the phase velocity vector defined by control i_1, and the vector v_λ is tangent to the orbit $\tilde{q}_0$.

Analogous assertions hold for the differential DP', with λ, s replaced by λ', s'. □

By $\tilde{L}$ denote the image of the intersection $L \cap \Gamma_{i_0 i_1}$ in space Σ^*. The results on the differentials of the Poincaré maps $\tilde{P}$, $\tilde{P}'$ can be summarized as follows.

Theorem 4.11. *Suppose the differential DP has only simple eigenvalues. Then the minimal subspaces $\tilde{V}_1, \ldots, \tilde{V}_{2r-2}$ of the space $T_{\tilde{q}_0}\tilde{\Gamma}_{i_0 i_1}$ that are invariant under the action of the differential $D\tilde{P}$, and the corresponding eigenvalues $\lambda_1, \ldots, \lambda_{2r-2}$ with nonnegative imaginary part can be arranged in a manner such that the following conditions hold.*
a) If $\tilde{L}$ is differentiable at $\tilde{q}_0$, then $T_{\tilde{q}_0}\tilde{L} = \tilde{V}_1 \oplus \tilde{V}_2 \oplus \cdots \oplus \tilde{V}_{r-2}$.
b) $\lambda_i\,\bar{\lambda}_{r-2+i} = \bar{\lambda}_i\,\lambda_{r-2+i} = \lambda^{-(2n+1)}$ for any $i = 1, \ldots, r-2$.
c) $\dim \tilde{V}_{2r-3} = \dim \tilde{V}_{2r-2} = 1, \lambda_{2r-3} = \lambda^{-2n}, \lambda_{2r-2} = \lambda^{-(2n+1)}$.
d) $\dim \tilde{V}_i = \dim \tilde{V}_{r-2+i}$ for any $i = 1, \ldots, r-2$.

Analogous assertions hold for the differential $D\tilde{P}'$, with λ replaced by λ'. □

Note that the spectrum of the matrices (13) does not coincide with the spectrum of the differentials DP, DP'. By Corollary 3.2, the eigenvalues $\lambda^{-1}, \lambda^{-2n}$ (resp. $(\lambda')^{-1}, (\lambda')^{-2n}$) of matrices (13) are always multiple, whereas absence of multiple eigenvalues in the spectrum of the differentials DP, DP' is the generic case. Corollary 3.2 yields the following criterion of absence of multiple eigenvalues in the spectrum of the differentials DP, DP'.

Proposition 4.12. *The differential DP (resp. DP') has only simple eigenvalues if and only if the numbers $\lambda^{-1}, \ldots, \lambda^{-2n}$ (resp. $\lambda'^{-1}, \ldots, \lambda'^{-2n}$) are eigenvalues of the corresponding matrix (13) with multiplicity not greater than 2, and any other eigenvalue is simple.* □

5 Conclusions

When constructing an optimal synthesis in a deterministic control problem, one usually first considers singular trajectories and submanifolds provided by the maximum principle. These act as structuring elements in the phase portrait. In classical dynamical systems, however, the structuring elements are fixed points, periodic cycles and associated invariant submanifolds. If the optimal control problem possesses a certain symmetry group (a Fuller group), one can consider these classical objects in orbit space with respect to the group. Computing them yields valuable information on the global structure of optimal synthesis.

In this paper we provide tools and algorithms to compute such elements. To this end we expolited the interaction of the symplectic structure of the Hamiltonian dynamics emanating from the maximum principle on the one hand and the Fuller group on the other hand.

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Roland Hildebrand
LMC, IMAG
Université Joseph Fourier
roland.hildebrand@imag.fr