ON THE ALGEBRAIC STRUCTURE OF THE COPOSITIVE CONE

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A closed convex cone can be decomposed into a disjoint union of interiors of its faces. This well-known facial decomposition yields a lot of information on the structure of the cone. However, in general there are infinitely many faces, and for some purposes this decomposition is too fine. Some cones admit a coarser, finite decomposition which unites faces which are of the same type. For example, the cone of positive semi-definite matrices of size n decomposes into $n+1$ relatively open manifolds, each of which contains positive semi-definite matrices of constant rank and which are themselves unions of interiors of similar faces.

We propose such a finite decomposition for the copositive cone $\mathcal{COP}^n$. The components of the decomposition are parameterized by the *extended minimal zero support set*. This means that each component $S_{\mathcal{E}}$ is composed of copositive matrices A with the same extended minimal zero support set $\mathcal{E}$. This set is a collection of pairs $\mathcal{E} = (I_\alpha, J_\alpha)_{\alpha=1,\dots,|\mathcal{E}|}$, where α enumerates the minimal zeros u_α of A, I_α is the support of the minimal zero u_α, and the index set $J_\alpha \supset I_\alpha$ consists of those indices $j \in \{1,\dots,n\}$ such that $(Au_\alpha)_j = 0$.

The set $S_{\mathcal{E}}$ lies in a real-algebraic variety $Z_{\mathcal{E}}$ which is given by a finite number of polynomial equalities, namely those equivalent to the rank-deficiency of the submatrix $A_{I_\alpha \times J_\alpha}$. Our main result states that for every $A \in \mathcal{COP}^n$ with extended minimal zero support set $\mathcal{E}$, there exists a neighbourhood U of A in the space of real symmetric matrices such that $U \cap Z_{\mathcal{E}} \subset S_{\mathcal{E}}$, i.e., $S_{\mathcal{E}}$ is open in $Z_{\mathcal{E}}$. Thus the polynomial equalities cited above fully determine the local structure of $S_{\mathcal{E}}$.

References

[1] Roland Hildebrand. On the algebraic structure of the copositive cone. *Optim. Lett.* 14(8):2007–2019, (2020).

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