On the algebraic structure of the copositive cone

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ILAS 2022, Galway, Ireland

June 21, 2022

Faces of a convex cone

Definition

Let K be a convex cone. A set $F \subset K$ is called a *face* if for every line segment $I \subset K$ and every interior point $x \in I$ such that $x \in F$ we have $I \subset F$.
The *minimal face* of a point $x \in K$ is the smallest face of K which contains x.

as a consequence a face has the following description:

- let L be the *linear hull* of F
- then $F = L \cap K$

remark: we do not demand that L is a supporting hyperplane

in particular, if F is the minimal face of x, then x is in the interior of F

Faces of the PSD cone

let $K = \mathcal{S}_+^n$ be the cone of $n \times n$ PSD matrices

the faces of $\mathcal{S}_+^n$ are parameterized by the *linear subspaces* $V \subset \mathbb{R}^n$

$F_V = \{A \in \mathcal{S}^n \mid Ax = 0 \ \forall\, x \in V\}$

by an appropriate coordinate change the face F_V becomes the set of PSD matrices of the form $A = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$

the face F_V is the minimal face of A iff the upper left block is PD

Faces of the COP cone

the linear hull $\mathcal{L}^A$ of the minimal face of $A \in \mathcal{COP}^n$ has been described in [Dickinson, H. 2016] in terms of its *minimal zeros*

Definition

A vector $v \in \mathbb{R}^n_+ \setminus \{0\}$ is a *zero* of A if $v^T A v = 0$.
The *support* of v is the index set $\{i \mid v_i > 0\}$.
The zero v is *minimal* if there does not exist a zero with strictly smaller support.

let $\mathcal{V}^A_{\min}$ be the set of minimal zeros of A

$$\mathcal{L}^A = \{B \in \mathcal{S}^n \mid (Bv)_i = 0 \ \forall\ v \in \mathcal{V}^A_{\min},\ (Av)_i = 0\}$$

the minimal face of A is then $\mathcal{L}^A \cap \mathcal{COP}^n$

Structure of the set of faces of the PSD cone

the faces of $\mathcal{S}_+^n$ are organized in smooth manifolds:

- F_V analytically depends on the subspace V
- all subspaces V of the same dimension k form an analytic manifold
- all corresponding faces F_V are isomorphic

unions of interiors of faces corresponding to subspaces V of the same dimension $n - k$ form an analytic (even algebraic) manifold $\mathcal{F}_k$:

$$\mathcal{F}_k = \{A \in \mathcal{S}_+^n \mid \text{rk } A = k\}$$

$\mathcal{F}_k$ is an *open* subset of an algebraic variety $\mathcal{Z}_k$ determined by polynomial equations:
the determinant of every submanifold of A of size $k + 1$ vanishes

the cone $\mathcal{S}_+^n$ is a disjoint union of finitely many subsets $\mathcal{F}_k$

Analogous result for COP?

goal: carry over these results to the case of the COP cone

COP cone more complicated than PSD cone

summary:

- $\mathcal{COP}^n$ is the disjoint union of a *finite* number of sets $\mathcal{F}_\mathcal{E}$
- each $\mathcal{F}_\mathcal{E}$ is an *open* subset of an *algebraic* variety $\mathcal{Z}_\mathcal{E}$
- each $\mathcal{F}_\mathcal{E}$ is a disjoint union of interiors of faces F sharing some characteristics $\mathcal{E}$

Motivation

example: extreme rays of $\mathcal{COP}^5$

the exceptional extreme rays are in the orbit of either

$$H = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix}$$

or

$$T_\psi = \begin{pmatrix} 1 & -c_1 & c_{12} & c_{45} & -c_5 \\ -c_1 & 1 & -c_2 & c_{23} & c_{15} \\ c_{12} & -c_2 & 1 & -c_3 & c_{34} \\ c_{45} & c_{23} & -c_3 & 1 & -c_4 \\ -c_5 & c_{15} & c_{34} & -c_4 & 1 \end{pmatrix}$$

with $c_i = \cos\psi_i$, $c_{ij} = \cos(\psi_i + \psi_j)$, $\psi_i > 0$, $\sum_i \psi_i < \pi$

orbits of T_ψ form one manifold, orbit of H another one

they can be distinguished by their *minimal zero support set*

(supports of all minimal zeros)

Motivation

example: extreme rays of $\mathcal{COP}^6$

classification with respect to the minimal zero support set *not fine enough*:

Case 9.1

$$\begin{pmatrix} 1 & -1 & -\cos\phi_2 & \cos(\phi_1+\phi_2) & \cos(\phi_2+\phi_3) & \cos\phi_5 \\ -1 & 1 & \cos\phi_2 & \cos(\phi_4+\phi_5) & \cos(\phi_5-\phi_6) & -\cos\phi_5 \\ -\cos\phi_2 & \cos\phi_2 & 1 & -\cos\phi_1 & -\cos\phi_3 & \cos(\phi_1+\phi_4) \\ \cos(\phi_1+\phi_2) & \cos(\phi_4+\phi_5) & -\cos\phi_1 & 1 & \cos(\phi_4+\phi_6) & -\cos\phi_4 \\ \cos(\phi_2+\phi_3) & \cos(\phi_5-\phi_6) & -\cos\phi_3 & \cos(\phi_4+\phi_6) & 1 & -\cos\phi_6 \\ \cos\phi_5 & -\cos\phi_5 & \cos(\phi_1+\phi_4) & -\cos\phi_4 & -\cos\phi_6 & 1 \end{pmatrix},$$

$\phi_i > 0, \phi_2+\phi_3 < \pi, \phi_2+\phi_3+\phi_5 < \pi+\phi_6, \phi_1+\phi_4+\phi_6 < \phi_3$, $\phi_2+\phi_3+\phi_6 < \pi+\phi_5$, excluding $\phi_2+\phi_3+\phi_6 = \phi_5$.

Case 9.2

$$\begin{pmatrix} 1 & -1 & -\cos\phi_2 & \cos(\phi_1+\phi_2) & \cos(\phi_2+\phi_3) & \cos\phi_5 \\ -1 & 1 & \cos\phi_2 & \cos(\phi_4+\phi_5) & -\cos(\phi_2+\phi_3) & -\cos\phi_5 \\ -\cos\phi_2 & \cos\phi_2 & 1 & -\cos\phi_1 & -\cos\phi_3 & \cos(\phi_1+\phi_4) \\ \cos(\phi_1+\phi_2) & \cos(\phi_4+\phi_5) & -\cos\phi_1 & 1 & \cos(\phi_4+\phi_6) & -\cos\phi_4 \\ \cos(\phi_2+\phi_3) & -\cos(\phi_2+\phi_3) & -\cos\phi_3 & \cos(\phi_4+\phi_6) & 1 & -\cos\phi_6 \\ \cos\phi_5 & -\cos\phi_5 & \cos(\phi_1+\phi_4) & -\cos\phi_4 & -\cos\phi_6 & 1 \end{pmatrix},$$

$\phi_i > 0, \phi_2+\phi_3 < \pi, \phi_2+\phi_3+\phi_5 < \pi+\phi_6, \phi_1+\phi_4+\phi_6 < \phi_3$, $\phi_2+\phi_3+\phi_6 > \pi+\phi_5$.

minimal zero support sets the same, but matrices have a different structure

Extended minimal zero support set

Definition

Let $A \in \mathcal{COP}^n$, $v \in \mathbb{R}^n_+ \setminus \{0\}$ a zero of A. Let $I_v = \{i \mid v_i > 0\}$, $J_v = \{j \mid (Av)_j = 0\}$.
We call (I_v, J_v) the *extended support set* of v.
The set $\mathcal{E}(A) = \{(I_v, J_v) \mid v \in \mathcal{V}^A_{\min}\}$ is the *extended minimal zero support set* of A.

- $\emptyset \neq I_v \subset J_v$, because $A_{I_v} \succeq 0$ and hence $(Av)_{I_v} = 0$
- $A_{J_v \times I_v} v_{I_v} = 0$ by definition
- hence submatrix $A_{J_v \times I_v}$ is rank-deficient $\to$ polynomial equations

classification of $A \in \mathcal{COP}^n$ according to the extended minimal zero support set is finer than by just the minimal zero support set

adapted to the description of the linear hull of the minimal face F_A

Main result

Theorem

Let $A \in \mathcal{COP}^n$, and let $\mathcal{E} = \{(I_\alpha, J_\alpha) \mid \alpha = 1, \dots, |\mathcal{E}|\}$ be the extended minimal zero support set of A. Set $\mathcal{Z}_\mathcal{E} = \{B \in \mathcal{COP}^n \mid \operatorname{rk} B_{J_\alpha \times I_\alpha} < |I_\alpha| \ \forall \ \alpha\}$.
Then there exists a neighbourhood U of A such that for all $A' \in U \cap \mathcal{Z}_\mathcal{E}$ we have $A' \in \mathcal{COP}^n$ and A' has the same extended minimal zero support set as A.

let $\mathcal{F}_\mathcal{E}$ be the set of all $A \in \mathcal{COP}^n$ such that $\mathcal{E}$ is the extended minimal zero support set of A

- $\mathcal{COP}^n$ is a disjoint union of finitely many $\mathcal{F}_\mathcal{E}$
- each $\mathcal{F}_\mathcal{E}$ is an open subset of the algebraic variety $\mathcal{Z}_\mathcal{E}$
- each $\mathcal{F}_\mathcal{E}$ is a disjoint union of interiors of faces of $\mathcal{COP}^n$

Limits of the analogy with PSD cone

not all nice properties from the decomposition of the PSD cone carry over

- the faces in irreducible components of $\mathcal{F}_{\mathcal{E}}$ have a generic dimension, but there may be faces with *higher* dimension among them
- irreducible components of $\mathcal{F}_{\mathcal{E}}$ may not be manifolds (M. Manainen et al 2022)

depending on whether the generic dimension is > 1 or $= 1$:

$\mathcal{F}_{\mathcal{E}}$ either does not contain extremal rays at all, or the generic matrix in $\mathcal{F}_{\mathcal{E}}$ is extremal

Example

consider the family of matrices in $\mathcal{COP}^6$

$$\begin{pmatrix} 1 & -\cos\phi_1 & \cos(\phi_1+\phi_2) & \cos\phi_4 & -1 & \cos\phi_1 \\ -\cos\phi_1 & 1 & -\cos\phi_2 & \cos(\phi_2+\phi_3) & \cos\phi_1 & -1 \\ \cos(\phi_1+\phi_2) & -\cos\phi_2 & 1 & -\cos\phi_3 & \cos(\phi_3+\phi_4) & \cos\phi_2 \\ \cos\phi_4 & \cos(\phi_2+\phi_3) & -\cos\phi_3 & 1 & -\cos\phi_4 & \cos(\phi_4+\phi_5) \\ -1 & \cos\phi_1 & \cos(\phi_3+\phi_4) & -\cos\phi_4 & 1 & -\cos\phi_5 \\ \cos\phi_1 & -1 & \cos\phi_2 & \cos(\phi_4+\phi_5) & -\cos\phi_5 & 1 \end{pmatrix}$$

where $\phi_i > 0$, $i = 1, \dots, 5$, $\phi_1 < \phi_5$, $\phi_2 + \phi_3 + \phi_4 + \phi_5 < \pi$

- if $\phi_1 + \phi_5 \neq \pi$, then dim $F_A = 1$
- if $\phi_1 + \phi_5 = \pi$, then dim $F_A = 2$

but the extended minimal zero support set is the same

Conditions on the extended minimal zero support set

Given a dimension n, which sets $\mathcal{E}$ are the extended minimal zero support sets of some (extremal) matrix in $\mathcal{COP}^n$?

some conditions on $\mathcal{E}$ can be given:

- $\emptyset \neq I_\alpha \subset J_\alpha$, $I_\alpha \not\subset I_\beta$
- $\mathcal{E} = \emptyset$ corresponds to the interior of $\mathcal{COP}^n$
- for $I \subset \{1, \dots, n\}$ arbitrary, $\mathcal{E} = \{(I, I)\}$ corresponds to generic boundary points of $\mathcal{COP}^n$
- $\mathcal{E} = \{(\{k\}, \{1, \dots, n\}) \mid k = 1, \dots, n\}$ corresponds to $A = 0$
- collection of admissible sets $\mathcal{E}$ is invariant under the action of the permutation group S_n
- $I_\alpha \subset J_\beta$ iff $I_\beta \subset J_\alpha$

Boundaries

for every $\mathcal{E}$ with $\mathcal{F}_{\mathcal{E}} \neq \emptyset$, the boundary $\partial \mathcal{F}_{\mathcal{E}} \subset \mathcal{Z}_{\mathcal{E}}$ consists of matrices A belonging to different $\mathcal{F}_{\mathcal{E}'}$ with $\mathcal{E}' \neq \mathcal{E}$

Lemma

Let $\mathcal{E} = \{(I_\alpha, J_\alpha) \mid \alpha = 1, \ldots, |\mathcal{E}|\}$,
$\mathcal{E}' = \{(I_{\alpha'}, J_{\alpha'}) \mid \alpha' = 1, \ldots, |\mathcal{E}'|\}$ *be related as above.*
Then for every α, *there exists* α' *such that* $I_{\alpha'} \subset I_\alpha$, $J_\alpha \subset J_{\alpha'}$.
Moreover, $\mathcal{Z}_{\mathcal{E}'} \subset \mathcal{Z}_{\mathcal{E}}$.

open question: does the inclusion $\mathcal{F}_{\mathcal{E}'} \subset \mathcal{F}_{\mathcal{E}}$ hold?

Application

given a (extremal) matrix $A \in \mathcal{COP}^n$, we may construct families of other (extremal) matrices $A' \in \mathcal{COP}^n$ by considering a small enough neighbourhood of A in $\mathcal{Z}_{\mathcal{E}(A)}$

technically: solve the polynomial system on A' which defines $\mathcal{Z}_{\mathcal{E}(A)}$ consider solutions A' which are close enough to A

close enough means absence of additional minimal zeros and additional indices $j : (A'v)_j = 0$, where v is a minimal zero of A

Thank you!