Second-order Galerkin-Lagrange Method for the Navier-Stokes Equations

Mohamed Bensaada,[1] **Driss Esselaoui,**[1] **Pierre Saramito**[2]
[1]*Laboratoire SIANO, Département de Mathématiques et d'Informatique, Faculté des Sciences, Université Ibn Tofail, B.P. 133, 14000 Kénitra, Morocco*

[2]*LMC-IMAG, BP 53, 38041 Grenoble Cedex 9, France*

Received 12 July 2004; accepted 3 February 2005
Published online in Wiley InterScience (www.interscience.wiley.com).
DOI 10.1002/num.20080

This article introduces a new second-order characteristic mixed finite element approximation for Navier-Stokes equation. Optimal error estimates are proved in the framework of L^2-theory. © 2005 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 21: 000–000, 2005

Keywords: second-order characteristic method; mixed finite element method; Navier-Stokes equation; convergence

I. INTRODUCTION

The Lagrange-Galerkin method, has been proposed for the numerical treatment of convection-dominated diffusion equation (see [1–4]). It is based on combining a Galerkin finite element procedure with a special discretization of the material derivative along trajectories and has been shown to possess remarkable stability properties. Pironneau [5–8] have studied a characteristic finite element method for the Navier-Stokes equations. Other authors have used this method for the study and the simulation of viscoelastic fluid flows (see [9, 10]). In those works the first order characteristic finite element method is analyzed and used successfully. As for higher order methods, Pironneau et al. [8, 11] have mentioned a second-order Runge-Kutta (RK) approximation to the material derivative term, but they did not give a correct second-order approximation to the diffusion term. Hence, the final scheme is not of second order in time increment. For the convection-diffusion problems, Rui and Tabata [12] use also the second-order RK approximation to the material derivative term and point out that the scheme is second order in time increment.

Correspondence to: Driss Esselaoui, Laboratoire SIANO, Département de Mathématiques et d'Informatique, Faculté des Sciences, Université Ibn Tofail, B.P. 133, 14000 Kénitra, Morocco (e-mail: desselaoui@yahoo.fr)

The object of this article is to present optimal error estimates for a second-order Lagrange-Galerkin mixed finite element approximation of the Navier-Stokes equation. We also use the second-order RK method to approximate the material derivative term. Our point is to evaluate the convection term at the correct position to maintain a second-order accuracy in time increment Δt. Consequently we need an additional correction term of order Δt.

An outline of the article follows. In the next section a discretization of the Lagrangian material derivative along particle trajectories is introduced. The variational form of our second-order characteristic-mixed method, which we define in section 3. Section 4 is devoted to the convergence analysis of the Lagrange-Galerkin method, followed by the conclusion.

II. STATEMENT OF THE PROBLEM

Let Ω denote a bounded open subset of R^d ($d = 2, 3$) with a lipschitz continuous boundary $\partial\Omega$. In the following we shall use the classical Sobolev spaces $W^{s,p}(\Omega)$ and $W_0^{s,p}(\Omega)$, $s \geq 0$, $1 \leq p \leq \infty$, endowed with the norm $\|\cdot\|_{s,p}$ and the seminorm $|\cdot|_{s,p}$ [13]. For $p = 2$, $W^{s,2}(\Omega)$ will be denoted by $H^s(\Omega)$ and, if there is no ambiguity, we drop the subscript $p = 2$ in the corresponding norms and the seminorms. The dual space of $H_0^s(\Omega)$ will be denoted by $H^{-s}(\Omega)$. We denote by $C^{0,1}(\bar{\Omega})$ the space of lipschitz continuous functions on the closure of Ω. Let T be a positive real number. For a real Banach space Y: $L^p(Y)$, $H^s(Y)$, and $C^m(Y)$ will denote the spaces $L^p(0, T; Y)$, $H^s(0, T; Y)$, and $C^m([0, T]; Y)$, respectively. In the cylindrical domain $\Omega \times (0, T)$, we consider the Navier-Stokes equations:

$$(NS)\begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u - \nu\Delta u + \nabla p = f \\ \nabla \cdot u = 0, \end{cases}$$

with the boundary condition

$$u/\partial\Omega = 0,$$

and the initial condition

$$u(x, 0) = u_0(x).$$

In these equations, $u(x, t)$ is the velocity of the fluid, $p(x, t)$ is the pressure, ν is the viscosity, $f(x, t)$ is a density of body forces per unit mass, and u_0 is the initial velocity.

The following special function spaces are well suited to the Navier-Stokes equations: $V = \{v \in H_0^1(\Omega)^d \mid \nabla \cdot v = 0 \text{ in } \Omega\}$, $H = \{v \in L^2(\Omega)^d \mid \nabla \cdot v = 0 \text{ in } \Omega \text{ and } v \cdot \vec{n} = 0 \text{ on } \partial\Omega\}$, where $\vec{n}$ is the unit outward normal to $\partial\Omega$. $L_0^2(\Omega)$ will denote the subspace of functions of $L^2(\Omega)$ with zero mean on Ω.

For f and u_0 given, $f \in L^2(H^{-1}(\Omega)^d)$, $u_0 \in H$, the mixed finite element approximation of (NS) problem is based on the following saddle-point weak formulation: find $(u, p) \in [L^\infty(L^2(\Omega)^d) \cap L^2(H_0^1(\Omega)^d)] \times L^2(L_0^2(\Omega))$ such that

$$(WNS)\begin{cases} \dfrac{d}{dt}(u, v) + ((u \cdot \nabla)u, v) + \nu(\nabla u, \nabla v) - (\nabla \cdot v, p) = \langle f, v\rangle & \forall v \in H_0^1(\Omega)^d \\ (\nabla \cdot u, q) = 0 & \forall q \in L_0^2(\Omega), \end{cases}$$

where the inner products are to be interpreted to be in L^2 and $\langle , \rangle$ is the duality pairing between $H_0^1(\Omega)^d$ and $H^{-1}(\Omega)^d$. Hereafter we assume that Ω is regular in the sense that the mapping $-\nu\Delta u + \nabla p$ is an isomorphism from $[H^2(\Omega)^d \cap V] \times [H^1(\Omega) \cap L_0^2(\Omega)]$ onto $L^2(\Omega)^d$. For instance, Ω is regular if $\partial\Omega$ is of class C^2 or Ω is a convex plane polygonal domain (see [14]).

Theorem 2.1. *Assume that* $(f, u_0) \in L^2(L^2(\Omega)^d) \times V$, *then problem* (*WNS*) *has a unique solution* (u, p) *in* $[L^2(H^2(\Omega)^d) \cap H^1(L^2(\Omega)^d) \cap C(V)] \times L^2(H^1(\Omega))$.

For the proof, we refer for example to [15]. Under the assumptions of Theorem 2.1, $D_t u = \partial u/\partial t + (u \cdot \nabla)u$, the material derivative of u, belonging to $L^2(L^2(\Omega)^d)$, so that problem (*WNS*) can be rewritten as

$$(MNS)\begin{cases}(D_t u, v) + \nu(\nabla u, \nabla v) - (\nabla \cdot v, p) = \langle f, v\rangle & \forall v \in H_0^1(\Omega)^d,\\ (\nabla \cdot u, q) = 0 & \forall q \in L_0^2(\Omega).\end{cases}$$

The crucial aspect of the Lagrange-Galerkin method is the discretisation of the material derivative along trajectories.

A. Discretisation of the Material Derivative: Presentation

The lagrangian representation of the flow is based on the function $X : (x, t; s) \in \Omega \times (0, T)^2 \rightarrow X(x, t; s)$ is the position at time s of the particle of fluid, which is at point x at time $s = t$. Thus, $s \rightarrow X(x, t; s)$ is the parametric representation of the trajectories may be determined from the initial value problem

$$\frac{d}{ds} X(x, t; s) = u(X(x, t; s), s)$$

$$X(x, t; t) = x. \tag{2.1}$$

Lemma 2.2. *Assume that* $u \in C(C^{0,1}(\bar{\Omega})^d) \cap C(V)$. *If* $|t - s|$ *is sufficiently small, then* $x \rightarrow X(x, t; s)$ *is a quasi-isometric homeomorphism of* Ω *onto itself and its jacobian equals* 1 *a.e. on* Ω.

Proof. See [18]. ∎

In the next, we exploit the following second-order finite difference scheme:

$$\varphi'(t + \Delta t/2) = \frac{\varphi(t) - \varphi(t - \Delta t)}{\Delta t} + \mathcal{O}(\Delta t^2),$$

where φ belongs to C^3. In [12], Rui and Tabata adapted this idea to the linear diffusion-convection problem:

(*DC*): find ϕ defined in $\Omega \times]0, T[$ such that

$$\begin{cases}\dfrac{D\phi}{Dt}(x, t) - \Delta\phi(x, t) = f & (x, t) \in \Omega \times]0, T[\\ \phi(0) = \phi_0 & x \in \Omega,\end{cases}$$

where $D\phi/Dt = \partial\phi/\partial t + \mathbf{u}. \nabla\phi$ and the velocity field $\mathbf{u}$ of the convected derivative is given.

These authors introduced the scheme

$$\frac{\phi^{(n)}(x) - \phi^{(n-1)}(X(x, t_n; t_{n-1}))}{\Delta t} - \frac{(\Delta\phi^{(n)})(x) + (\Delta\phi^{(n-1)})(X(x, t_n; t_{n-1}))}{2} = f(x, t_{n-(1/2)})$$

$$\phi^{(0)}(x) = \phi_0(x),$$

where $t_n = n\Delta t$ and $\Delta t = T/M$.

This scheme is not suitable for a variational formulation because of the presence of the term $(\Delta\phi^{(n-1)})(X(x, t_n; t_{n-1}))$. Thus, this term is rewritten. For all φ defined in Ω:

$$\frac{\partial}{\partial x_i}(\varphi \circ X(x, t_n; t_{n-1})) = \sum_{j=1}^{d} \frac{\partial X_j}{\partial x_i}(x, t_n; t_{n-1}) \frac{\partial \varphi}{\partial x_j}(X(x, t_n; t_{n-1})).$$

Introducing the jacobian matrix

$$J = \left(\frac{\partial X_i}{\partial x_j}\right)_{1\le i,j\le d},$$

this identity becomes

$$(\nabla\varphi)X(x, t_n; t_{n-1}) = J^{-T}(x, t_n; t_{n-1})\nabla(\varphi(X(x, t_n; t_{n-1}))).$$

From the first-order approximation of $X(x, t_n; t_{n-1})$:

$$X_1(x, t_n; t_{n-1}) = x - \Delta t u(x, t_{n-1}), \tag{2.2}$$

of the trajectory, these authors exhibit a second-order approximation of J^{-T}:

$$J^{-T}(x, t_n; t_{n-1}) = I + \Delta t(\nabla u)^T(x, t_{n-1}) + \mathcal{O}(\Delta t^2).$$

Thus,

$$(\nabla\varphi)(X(x, t_n; t_{n-1})) = (I + \Delta t(\nabla u)^T(x, t_{n-1}))\nabla(\varphi(X(x, t_n; t_{n-1}))) + \mathcal{O}(\Delta t^2).$$

On the other hand, for all ξ defined in Ω, we have the identity

$$\nabla \cdot (\nabla u \xi) = \nabla(\nabla \cdot u)\xi + (\nabla u)^T \nabla \xi,$$

recall that $\nabla \cdot u = 0$. Then choosing $\xi = \varphi(X(\cdot, t_n; t_{n-1}))$, we get

$$(\nabla\varphi)(X(x, t_n; t_{n-1})) = \nabla[\varphi(X(x, t_n; t_{n-1}))] + \Delta t \nabla \cdot [\nabla u(x, t_{n-1})\varphi(X(x, t_n; t_{n-1}))] + \mathcal{O}(\Delta t^2)$$

or by expanding at the component level:

$$\left(\frac{\partial \varphi}{\partial x_i}\right)(X(x,\, t_n;\, t_{n-1})) = \sum_{j=1}^{d} \frac{\partial}{\partial x_i}\left\{\left(\delta_{i,j} + \Delta t\,\frac{\partial u_i}{\partial x_j}\right)\varphi(X(x,\, t_n;\, t_{n-1}))\right\} + \mathcal{O}(\Delta t^2.$$

Finally, by choosing $\varphi = \nabla \phi^{(n-1)}$ and after summation over the indices:

$$(\Delta \phi^{(n-1)})(X(x,\, t_n;\, t_{n-1})) = \nabla \cdot [(I + \Delta t \nabla u(x,\, t_{n-1}))(\nabla \phi^{(n-1)})(X(x,\, t_n;\, t_{n-1}))] + \mathcal{O}(\Delta t^2). \tag{2.3}$$

Using a second-order approximation $X_2^{(n-1)}$ of $X(x,\, t_n;\, t_{n-1})$, Rui and Tabata [12] introduce the following second-order scheme:

$$\frac{\phi^{(n)} - \phi^{(n-1)} \circ X_2^{(n-1)}}{\Delta t} - \frac{1}{2}\,\nabla \cdot (\nabla \phi^{(n)} + (I + \Delta t \nabla \mathbf{u}^{(n-1)})\{(\nabla \phi^{(n-1)}) \circ X_1^{(n-1)}\}) = f(x,\, t_{n-(1/2)})$$

$$\phi^{(0)} = \phi_0,$$

where $\mathbf{u}^{(n-1)}$ denotes $u(\cdot,\, t_{n-1})$ and $X_1^{(n-1)}$ is a first-order approximation of $X(x,\, t_n;\, t_{n-1})$. In [12], subject to an initial condition $X(x,\, t_{n+1};\, t_{n+1}) = x$ we get approximate value of $X(x,\, t_{n+1};\, \cdot)$ at t_n by the Euler method and the second-order Runge-Kutta method, respectively,

$$X_1^n(x) = x - u^n(x)\Delta t$$

$$X_2^n(x) = x - u^{n+(1/2)}(x - u^n(x)\Delta t/2)\Delta t.$$

As application to the (*MNS*) problem, we take in the coming development $u_i = \phi$, $(i = 1,\, d)$. Concerning X_i^n $(i = 1,\, 2)$, we give the following useful result.

Lemma 2.3. *Under the condition $\Delta t \leq 1/\|u\|_{C^0(W^{1,\infty}(\Omega))}$, it holds that $X_1^n(\Omega) = X_2^n(\Omega) = \Omega$ and there exists a constant C independent of Δt such that for all $v \in (L^2(\Omega))^d$*

$$\|v \circ X_i^n\| \leq (1 + C\Delta t)\|v\|, \qquad i = 1,\, 2.$$

Proof. We only show a proof in the case $i = 1$. Let $d_1(x) = \text{distance}(x,\, \partial\Omega)$ for $x \in \Omega$. Since u vanishes on the boundary, it holds that

$$|X_1^n(x) - x| = \Delta t |u(x,\, t_n)| \leq d_1(x)\Delta t \|u\|_{C^0(W^{1,\infty}(\Omega))} < d_1\,(x),$$

which implies that $X_1^n(x) \in \Omega$. On the other hand, let J_1^n be the jacobian matrix ($\det J_1^n > 0$) on the transformation $y = X_1^n(x)$. We have,

$$\|v \circ X_1\|^2 = \int_\Omega v(X_1(x))^2\, dx = \int_\Omega v(y)^2 (\det J_1^n)^{-1}\, dy.$$

Since it holds that

$$\left|\det J_1^n - 1\right| \leq C\Delta t \|u\|_{C^0(W^{1,\infty}(\Omega))},$$

we complete the proof. ■

Remark. In [8] and [11] they proposed the scheme,

$$\frac{\phi^{(n+1)} - \phi^{(n)} \circ X_2^n}{\Delta t} + \frac{1}{2} \nabla \cdot [\nabla \phi^{(n+1)} + \nabla \phi^{(n)}] = \frac{1}{2} (f^{(n+1)} + f^{(n)})(x)$$

$$\phi(0) = \phi_0.$$

This scheme is better than the first-order characteristic scheme, but it is not a second-order scheme. In fact, although it holds that

$$\frac{\phi^{(n+1)} - \phi^{(n)} \circ X_2^n}{\Delta t} = \left(\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi\right)\left(\frac{x + X_2^n}{2}, t_{n+(1/2)}\right) + \mathcal{O}(\Delta t^2),$$

$$\frac{1}{2} (\nabla \phi^{(n+1)} + \nabla \phi^{(n)})(x) = \nabla \phi(x, t_{n+(1/2)}) + \mathcal{O}(\Delta t^2)$$

and

$$\frac{1}{2} (f^{(n+1)} + f^{(n)})(x) = f(x, t_{n+(1/2)}) + \mathcal{O}(\Delta t^2).$$

However, the distance of the point $(x + X_2^n)/2$ and x is $\mathcal{O}(\Delta t)$, which reduces the total accuracy to $\mathcal{O}(\Delta t)$.

III. FORMULATION OF THE FINITE ELEMENT PROCEDURE

In this section, the mathematical framework and approximation properties are summarized and we recall the most important fact for our purpose about the velocity/pressure finite element method.

A. Finite Element Space

Let $\Omega \subset R^d$ $(d = 2, 3)$ be a polygonal domain and let $\mathfrak{T}_h$ be a triangulation on Ω made on triangles K (in R^2) or tetrahedrals (in R^3). Thus, the computational domain is defined by

$$\bar{\Omega} = \cup K, \qquad K \in \mathfrak{T}_h.$$

We assume that $\mathfrak{T}_h$ uniformly regular, there exists (ν_0, ν_1) such that $\nu_0 h \leq h_K \leq \nu_1 \varrho_K$, where ϱ_K is the diameter of the greatest ball included in K and $h = \max_{K \in \mathfrak{T}_h} h_K$. Let assume that $0 < h \leq h_0$ $(h_0 \leq 1)$, and let W_h (resp. M_h) be a finite element space, $W_h \subset H_0^1(\Omega)$ (resp. $M_h \subset L_0^2(\Omega)$) associated with a partition $\mathfrak{T}_h$ of Ω and piecewise polynomial functions of some fixed degree l (resp. $l-1$). We define $\mathbf{W}_h = (W_h)^d$, $\mathbf{V}_h = \{v_h \in (H_0^1(\Omega))^d \mid (\nabla \cdot v_h, q_h) = 0, \forall q_h \in M_h\}$.

In the following, it will be assumed that the continuous and the discrete spaces are related by the following hypotheses.

$\mathbf{H_1}$. There exists a positive constant C_1 such that

$$\inf_{v_h \in \mathbf{W}_h} [\|v - v_h\|_{0,\Omega} + h\|v - v_h\|_{1,\Omega}] \leq C_1 h^{r+1} \|v_h\|_{r+1,\Omega}, \tag{3.1}$$

$\forall v \in (H_0^1(\Omega))^d \cap (H^{r+1}(\Omega))^d$ and $\forall r$, $1 \leq r \leq l$.

$\mathbf{H_2}$. There exists a positive constant C_2 such that

$$\inf_{q_h \in M_h} \|q - q_h\|_{0,\Omega} \leq C_2 h^r \|q\|_{r,\Omega}, \tag{3.2}$$

$\forall q \in H^r(\Omega) \cap L_0^2(\Omega)$ and all r, $0 \leq r \leq l$.

$\mathbf{H_3}$ (Brezzi-Babuska condition). There exists a positive constant C_3 such that

$$\inf_{q_h \in M_h} \sup_{v_h \in \mathbf{W}_h} \frac{(\nabla \cdot v_h, q_h)}{\|v_h\|_{1,\Omega} \|q_h\|_{0,\Omega}} \geq C_3. \tag{3.3}$$

Example 1. For any integer l, let $P_l(K)$ denote the usual polynomial space on K. Consider the following choice of finite element spaces

$$\mathbf{W}_h = \{v_h \in (H_0^1(\Omega))^d \mid v_{h|K} \in P_2(K)^d \quad \forall K \in \mathfrak{T}_h\},$$

$$M_h = \{q_h \in L_0^2(\Omega) \cap C(\bar{\Omega}) \mid q_{h|K} \in P_1(K) \quad \forall K \in \mathfrak{T}_h\}.$$

These satisfy the assumptions $\mathbf{H_1}$ to $\mathbf{H_3}$ with $l = 2$ and are usually referred to as the Taylor-Hood finite element spaces.

Next, for $u \in L^\beta(V)$ and $p \in L^\beta(L_0^2(\Omega))$, $\beta \in [1, \infty]$, we define $(\tilde{u}_h, \tilde{p}_h) \in L^\beta(W_h) \times L^\beta(M_h)$ as the solution of the following problem:

find $(\tilde{u}_h, \tilde{p}_h) : [0, T] \rightarrow \mathbf{W_h} \times M_h$ such that, for $t \in [0, T]$,

$$\begin{aligned} (\nabla(u(t) - \tilde{u}_h(t)), \nabla v_h) - (\nabla \cdot v_h, p(t) - \tilde{p}_h(t)) &= 0 \qquad \forall\, v_h \in \mathbf{W_h} \\ (\nabla \cdot (u(t) - \tilde{u}_h(t)), q_h) &= 0 \qquad \forall\, q_h \in M_h. \end{aligned} \tag{3.4}$$

The technique described in [13, 17] coupled with a duality argument and the H^2-regularity of the Stokes operator in regular domains yields the following result.

Lemma 3.1. *Suppose that s is a real number, $s \geq 1$, and that hypotheses* $\mathbf{H_1}$ *to* $\mathbf{H_3}$ *hold. If*

$$u \in L^\beta(H^{s+1}(\Omega)^d \cap V) \qquad \textit{and} \quad p \in L^\beta(H^s(\Omega) \cap L^2_0(\Omega)), \quad \beta \in [1, \infty),$$

then

$$\frac{d\tilde{u}}{dt} = \widetilde{\left(\frac{du}{dt}\right)},$$

there exists a positive constant C_4 *such that*

$$\|(u - \tilde{u}_h)(t)\|_{L^\beta(L^2(\Omega)^d)} + h[\|(u - \tilde{u}_h)(t)\|_{L^\beta(H^1(\Omega)^d)} + \|(p - \tilde{p}_h)(t)\|_{L^\beta(H^1(\Omega))}] \leq C_4 h^{r+1}(\|u\|_{L^\beta(H^{r+1}(\Omega)^d)} + \|p\|_{L^\beta(H^r(\Omega))}),$$

and

$$\left\|\left(\frac{du}{dt}\right) - \widetilde{\left(\frac{du}{dt}\right)}\right\|_{L^\beta(L^2(\Omega)^d)} + h\left\|\left(\frac{du}{dt}\right) - \widetilde{\left(\frac{du}{dt}\right)}\right\|_{L^\beta(H^1(\Omega)^d)} \leq C_4 h^{r+1}(\|u\|_{L^\beta(H^{r+1}(\Omega)^d)}),$$

for all r, $1 \leq r \leq \min(l, s)$.

B. Construction of the Second-Order Lagrange-Galerkin Method

Let u_h^n and p_h^n denote the approximations of the velocity and the pressure at $t = t_n$. Then, we define $(u_h^{n+1}, p_h^{n+1}) \in X_h \times M_h$ as the solution of the following discrete problem:

$$(MNS)_h^{(n+1)} \begin{cases} (d_t u_h^{n+1}, v_h) + \dfrac{\nu}{2}(\nabla u_h^{n+1} + \nabla u_h^n \circ X_{h1}^n, \nabla v_h) + \dfrac{\nu \Delta t}{2}(J_h^n \nabla u_h^n \circ X_{h1}^n, \nabla v_h) \\ - (\nabla \cdot v_h, p_h^{n+1}) = \dfrac{1}{2}(f^{n+1} + f^n \circ X_{h1}^n, v_h) \qquad \forall v_h \in \mathbf{W}_h, \\ (\nabla \cdot u_h^{n+1}, q_h) = 0 \qquad \forall q_h \in M_h, \end{cases}$$

where J_h^n is a matrix defined by $(J_h^n)_{ij} = \partial u_{hi}^n / \partial x_j$ and

$$d_t u_h^{n+1} = \frac{u_h^{n+1} - u_h^n(X_{h2}^n)}{\Delta t},$$

and we take the following approximation $X_{h1}^n = x - \Delta t u_h^n$ and $X_{h2}^n = x - (\Delta t/2)(u_h^n(x) + u_h^n(X_{1h}^n))$ of $X_h(x, t_{n+1}, \cdot)$, solution of the initial value problem:

$$\frac{dX_h^n}{ds}(x, t_{n+1}; s) = u_h^n(X_h^n(x, t_{n+1}; s)), \qquad t_n \leq s < t_{n+1}$$

$$X_h^n(x, t_{n+1}; t_{n+1}) = x. \tag{3.5}$$

Remark. The first-order characteristic finite element scheme is

$$\begin{cases} \left(\dfrac{u_h^{n+1} - u_h^n \circ X_{1h}^n}{\Delta t}, v_h\right) + \nu(\nabla u_h^{n+1}, \nabla v_h) - (\nabla \cdot v, p_h^{n+1}) = (f_h^{n+1}, v_h) & \forall v_h \in \mathbf{W}_h, \\ (\nabla \cdot u_h^{n+1}, q_h) = 0 & \forall q_h \in M_h, \end{cases}$$

which was studied in [7]. This scheme is symmetric in the unknown vector function u_h^{n+1} and unconditionally stable, these properties are also maintained in our scheme $(MNS)_h^{(n+1)}$.

Because $(\mathbf{W}_h, M_h)$ satisfies the inf sup condition (3.3), the problem $(MNS)_h^{(n+1)}$ is also equivalent to the following:

$(MNSR)_h^{(n+1)}$: Find $u_h^{n+1} \in \mathbf{V}_h$ such that,

$$\langle \mathcal{A}_h^{n+(1/2)} u_h, v_h \rangle = \langle \mathcal{F}_h^{n+(1/2)}, v_h \rangle \qquad \forall\ v_h \in V_h, \tag{3.6}$$

where

$$\langle \mathcal{A}_h^{n+(1/2)} u_h, v_h \rangle = (d_t u_h^{n+1}, v_h) + \nu_2(\nabla u_h^{n+1} + \nabla u_h^n \circ X_{h1}^n, \nabla v_h) + \frac{\nu \Delta t}{2}(J_{1h}^n \nabla u_h^n \circ X_{h1}^n, \nabla v_h),$$

and

$$\langle \mathcal{F}_h^{n+(1/2)}, v_h \rangle = \frac{1}{2}(f^{n+1} + f^n \circ X_{h1}^n, v_h).$$

The initial approximate velocity u_h^0 will be chosen to be the $H_0^1(\Omega)^d$-projection of u_0 onto V_h. If $u_0 \in H^{s+1}(\Omega)^n \cap V$, $s \geq 1$, and the hypotheses $\mathbf{H_1}$ to $\mathbf{H_3}$ hold, we have from Lemma 3.1, that

$$\|u_0 - u_h^0\|_{0,\Omega} + h\|u_0 - u_h^0\|_{1,\Omega} \leq C_4 h^{r+1} \qquad \forall r : 1 \leq r \leq \min(l, s). \tag{3.7}$$

IV. CONVERGENCE OF THE LAGRANGE-GALERKIN METHOD

The object of this section is to derive optimal error estimates for the second-order Lagrange-Galerkin method. In the following C denote the generic positive constant independent of discretization parameters, which can take different values at different places.

A. Main Result

In addition to hypotheses $\mathbf{H_1}$ to $\mathbf{H_3}$, the following inverse properties will be assumed to hold (see [18]).

$\mathbf{H_4}$. Let r and p be reals with $1 \leq r, p \leq \infty$ and let $l \geq 0$ and $m \geq 0$ be integers such that $l \leq m$. Then there exists a constant $C = C(\nu_0, \nu_1, l, r, m, p)$ such that

$$\forall v_h \in W_h \cap W^{l,r}(\Omega) \cap W^{m,p}(\Omega), \qquad \|v_h\|_{m,p} \leq Ch^{l-m-d\max\{0,1/r-1/p\}}\|v_h\|_{l,r}, \tag{4.1}$$

and

$$\forall v_h \in W_h, \qquad \|v_h\|_{0,\infty} \leq Ch^{1-(d/2)}\left(\mathrm{Log}\,\frac{1}{h}\right)^{1-(1/d)} \|v_h\|_1. \tag{4.2}$$

In order to obtain optimal error estimates, we are led to make the following mesh restriction:

$$\Delta t = o(h^{d/2}) \qquad \text{as } h \to 0. \tag{4.3}$$

For this reason we define

$$\delta_d(h) = \sqrt{\Delta t h^{-d/2}}.$$

At this stage we study the approximation of particle trajectories. In the first we prove the following results.

Lemma 4.1. *Assume that* $u_0 \in H^2 \cap V \cap C^{0,1}(\bar{\Omega})$, $u \in L^\infty(H^2(\Omega)) \cap C(V) \cap C(C^{0,1}(\bar{\Omega})$ *and that the hypotheses* $\mathbf{H_1}$ *to* $\mathbf{H_4}$ *and* (4.3) *hold. Furthermore, let* $\sqrt{\Delta t}\|u(\cdot, t_n) - u_h^n\|_{1,\Omega} \leq C(h + \Delta t^2)$ *for all* $h \in (0, h_0]$, *C is a positive constant, independent of n and* $(h, \Delta t)$. *Then there exists a constant* $h_1 \in (0, h_0]$ *independent of n and* Δt, *such that*

$$\Delta t\|u_h^n\|_{1,\infty} < \delta_d(h), \qquad \forall h \in (0, h_1]. \tag{4.4}$$

Proof. Let us first show that $\Delta t\|u_h^n\|_{0,\infty} < \delta_d(h)/2$. We define

$$D_d(h) = h^{1-(d/2)}\left(\log\frac{1}{h}\right)^{1-(1/d)}.$$

From the hypothesis $\mathbf{H_4}$ we have by virtue of (4.3), for h sufficiently small,

$$\begin{aligned}\Delta t\|u_h^n\|_{0,\infty} &\leq C\Delta t D_d(h)\|u_h^n\|_1 \leq C\Delta t D_d(h)(\|u_h^n - u^n\|_1 + \|u^n\|_1)\\ &\leq C\sqrt{\Delta t}\, D_d(h)(h + \Delta t^2) + C\Delta t D_d(h)\|u^n\|_1.\end{aligned}$$

Hence, by virtue of (4.3) and for h sufficiently small we obtain that,

$$\Delta t\|u_h^n\|_{0,\infty} \leq C(\sqrt{\Delta t}\, hD_d(h) + \Delta t^2\sqrt{\Delta t}\, D_d(h)) + C\Delta t D_d(h)\|u\|_{C(C^{0,1}(\bar{\Omega})^d)} < \delta_d(h)/2.$$

The proof of the estimate $\Delta t\|\nabla u_h^n\|_{0,\infty} < \delta_d(h)/2$ is similar because from the hypothesis $\mathbf{H_4}$ we have

$$\begin{aligned}\Delta t\|\nabla u_h^n\|_{0,\infty} \leq Ch^{-d/2}(\Delta t^2 + h)\sqrt{\Delta t} + C\Delta t h^{-d/2}\|\nabla u^n\| &\leq C\delta_d(h)h^{-d/4}(h + \Delta t^2)\\ &\quad + C\delta_d(h)^2\|u\|_{C(C^{0,1}(\bar{\Omega})^d)} < \delta_d(h)/2.\end{aligned}$$

Hence, the proof of the lemma follows for $h_1 \leq h_0$. ∎

Next, we establish the discrete analogue of Lemma 2.2 and one result concerning $G_\theta(\cdot) = (1 - \theta)X_h^n(\cdot, t_{n+1}; t_n) + \theta X(\cdot, t_{n+1}; t_n)$.

Lemma 4.2. *Under the hypotheses of Lemma* 4.1, *there exists a constant* h_1 *in* $(0, h_0]$, *independent of* n *and* Δt, *such that* $x \to X_{hi}^n(x, t_{n+1}; t)$ $(i = 1, 2)$ *is a quasi-isometric homeomorphism of* Ω *onto itself, for all* h *in* $(0, h_1]$ *and all* $t \in [t_n, t_{n+1}]$, *and there exists a positive constant* C *independent of* $(\Delta t, h, n)$ *and* $h_2 \in (0, h_1]$ *such that for* $0 \leq h \leq h_2$ *the jacobian* j_{hi}^n $(i = 1, 2)$ *of the mapping* $x \to X_{hi}^n(x)$ *satisfies*

$$\frac{1}{2} \leq 1 - C\delta_d(h) \leq j_{h1}^n \leq 1 + C\delta_d(h) \leq \frac{3}{2}, \tag{4.5}$$

$$\frac{5}{8} \leq 1 - \frac{C}{2}\,\delta_d(h)(2 + C\delta_d(h)) \leq j_{h2}^n \leq 1 + \frac{C}{2}\,\delta_d(h)(2 + C\delta_d(h)) \leq \frac{13}{8}. \tag{4.6}$$

Proof. We give only the proof for the case $i = 1$.

Let $y = X_{h1}^n(x) = x - \Delta t u_h^n(x)$, we have $y \in \Omega$ because

$$\begin{aligned}
\text{distance}(y, \partial\Omega) &= \min_{z\in\partial\Omega}|x - \Delta t u_h^n(x) - z| \\
&= \min_{z\in\partial\Omega}\left|x - z - \Delta t\int_x^z \nabla u_h^n \frac{x-z}{|x-z|}\right| \\
&\geq \min_{z\in\partial\Omega}|x - z| \times |1 - \Delta t\|\nabla u_h^n\|_{0,\infty}| \\
&\geq (1 - \delta_d(h)) \times \text{distance}(x, \partial\Omega).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
|X_{h1}^n(x_1) - X_{h1}^n(x_2)| &= \left|x_1 - x_2 + \Delta t\int_{x_1}^{x_2} \nabla u_h^n \frac{x_1 - x_2}{|x_1 - x_2|}\right| \\
&\geq |x_2 - x_1||1 - C\Delta t\|\nabla u_h^n\|_{0,\infty}| \\
&\geq (1 - \delta_d(h))|x_1 - x_2|,
\end{aligned}$$

so for $x_1 \neq x_2$, $X_{h1}^n(x_1) \neq X_{h2}^n(x_2)$, and the injectivity propriety of X_{h1}^n follow. Surjectivity is proved by reductio ad absurdum. Let us first extend u_h^n outside $\bar{\Omega}$ to an open set Ω_1, $\Omega \subset \Omega_1$ such that distance$(R^d \backslash \Omega_1, \Omega) > 2d$, where $d = \sup_{h\in(0,h_1]}\delta_d(h)$. Suppose that X_{h1}^n is not surjective for some $h \in (0, h_1]$. Then there exists $y_0 \in \Omega$ such that $y_0 \neq X_{h1}^n$ for all $x \in \Omega$. Consider the closed ball $B(y_0, 2d)$ of radius $2d$ and define the mapping $\Phi : \Omega_1 \to \Omega_1$ by

$$\begin{cases} \Phi(x) = X_{h1}^n & x \in \Omega \\ \Phi(x) = x & x \in \Omega_1 \backslash \Omega. \end{cases}$$

As Φ is a continuous mapping, the set $\hat{B} = \{\Phi(x) \mid x \in B(y_0, 2d)\}$ is simply-connected. Using the X_{h1}^n-definition, we obtain by applying Lemma 4.1 that

$$|x - \Phi(x)| \leq \Delta t \|u_h^n\|_{0,\infty} < \delta_d \qquad \forall x \in \Omega_1.$$

Hence, $y_0 \in \text{int}(B)$, which is a contradiction.
For the jacobian born, we have from the definition of X_{h1}^n and using the Einstein notation,

$$(\nabla X_{h1}^n)_{ij} = \delta_{ij} - \Delta t \, \frac{\partial u_{hi}}{\partial x_j}, \qquad i, j = 1, 2.$$

Hence,

$$\begin{aligned} |(\nabla X_{h1}^n)_{ij}| &\geq \left| \delta_{ij} - \Delta t \, \frac{\partial u_{hi}}{\partial x_j} \right|, \\ &\geq |1 - \Delta t \|\nabla u_h^n\|_{0,\infty}|. \end{aligned}$$

Using Lemma 4.1 result there exists $0 < h_1 \leq h_0$ such that for $h \leq h_1$ we have

$$|(\nabla (X_{h1}^n)_{ij}| \geq 1 - C\delta_d(h) \qquad \text{if } i = j \ (i, j = 1, d)$$

and

$$|(\nabla X_{h1}^n)_{ij}| \geq C\delta_d(h) \qquad \text{if } i \neq j \ (i, j = 1, d),$$

where C is a constant independent of $(h, \Delta t)$. In the same way we prove that

$$|(\nabla X_{h1}^n)_{ij}| \leq 1 + C\delta_d(h) \qquad \text{if } i = j \ (i, j = 1, d)$$

and

$$|(\nabla X_{h1}^n)_{ij}| \leq C\delta_d(h) \qquad \text{if } i \neq j \ (i, j = 1, d).$$

Therefore, the jacobian and the $\delta_d(\cdot)$ definitions give the lower and the upper bounds of j_{h1}^n. ■

Concerning $G_\theta(\cdot) = (1 - \theta) X_h^n(\cdot, t_{n+1}; t_n) + \theta X(\cdot, t_{n+1}, t_n)$, we give the following result. For the sake of brevity we omit the proof, which is very similar to the last lemma.

Lemma 4.3. *Assume that the hypotheses of Lemma* 4.1 *hold and* $N = T/\Delta t \geq C(T, u)$. *Then there exists* $h_3 \in (0, h_2]$ *independent of n and* Δt *such that* $x \to G_\theta(x) = (1 - \theta) X_h^n(x, t_{n+1}; t_n) + \theta X(x, t_{n+1}, t_n)$ *is quasi-isometric homeomorphism of* Ω *onto itself with a jacobian* $\geq \frac{1}{2}$ *for all* $\theta \in [0, 1]$ *and all* $h \in (0, h_3]$.

Also we use the discrete Gronwall's lemma, which plays an important role in the following analysis.

Lemma 4.4. (*Discrete Gronwall's Lemma*). *Let* Δt, H, *and* a_n, b_n, c_n, γ_n (*for integers* $n \geq 0$), *be nonnegative numbers such that*

$$a_l + \Delta t \sum_{n=0}^{l} b_n \leq \Delta t \sum_{n=0}^{l} \gamma_n a_n + \Delta t \sum_{n=0}^{l} c_n + H, \qquad \text{for } l \geq 0.$$

Suppose that $\Delta t \gamma_n < 1$, *for all* n, *and set* $\sigma_n = (1 - \Delta t \gamma_n)^{-1}$. *Then*

$$a_l + \Delta t \sum_{n=0}^{l} b_n \leq \exp\left(\Delta t \sum_{n=0}^{l} \sigma_n \gamma_n\right)\left[\Delta t \sum_{n=0}^{l} c_n + H\right] \qquad \text{for } l \geq 0.$$

B. Error Bound: $L^\infty(L^2(\Omega))$ and $L^2(H^1(\Omega))$ Estimates

For a Banach space Y normed by $\|\cdot\|_Y$, we define for $1 \leq p < \infty$,

$$l^p(0, T; Y) = \left\{ v : \{t_1, \ldots, t_M\} \mapsto Y \mid \|v_h\|_{l^p(0,T;Y)} = \left[\Delta t \sum_{i=1}^{M} \|v(t_i)\|_Y^p\right]^{1/p} < \infty \right\}$$

and

$$l^\infty(0, T; Y) = \{v : \{t_1, \ldots, t_M\} \mapsto Y \mid \|v_h\|_{l^\infty(0,T;Y)} = \max_{1 \leq i \leq M} \|v(t_i)\|_Y < \infty\}.$$

In this section we suppose that

$$f \in \bigcap_{i=0,2} C^i(H^{2-i}) \cap C^0(H) \qquad \text{and} \qquad u_0 \in H^{r+1}(\Omega)^N \cap C^{0,1}(\bar{\Omega})^N \cap \mathbf{V}. \tag{4.7}$$

The corresponding solution (u, p) of problem (MNS) will be assumed to satisfy some regularity hypotheses. Also we assume that

$$\Delta t = O(h^{d/2} \varepsilon(h)) \qquad \text{where } \varepsilon(h) \mapsto 0 \text{ as } h \mapsto 0. \tag{4.8}$$

With these assumptions, we have the following result.

Theorem 4.5. *Assume that the hypotheses* $\mathbf{H_1}$ *to* $\mathbf{H_4}$ *and* (4.3) *to* (4.8) *hold. Then there exists a positive constant* C_0, *independent of* $(h, \Delta t)$ *and* $h_4 \in (0, h_0]$ *such that if problem* (MNS) *admits a solution* (u, p) *with*

$$u \in \bigcap_{i=0,3} C^i(H^{3-i}) \cap L^\infty(H^{s+1}(\Omega)^d) \cap C(C^{0,1}(\bar{\Omega})^d) \cap C(\mathbf{V}),$$

$$du/dt \in L^2(H^{s+1}(\Omega)^d) \cap L^2(H), \qquad D_t \in L^2(H)$$

$$\textit{and} \qquad p \in L^\infty(H^s) \cap L^\infty(L_0^2(\Omega)), \tag{4.9}$$

then the solution $(u_h^i, p_h^i)_{1\le i\le M}$ *of* $(MNS)_h^i$ $(1 \le i \le M)$ *satisfies*

$$\|u_h - u\|_{l^\infty(L^2(\Omega))^d)} + \|u_h - u\|_{l^2(H^1(\Omega))^d)} \le C_0(h^r + \Delta t^2), \tag{4.10}$$

where $r : 1 \le r \le \min(l, s)$ *and* $\forall h \in (0, h_4]$.

Proof. For notational simplicity in this proof, we note $\delta_d(h)$ by δ_d. According to (3.7), we have

$$\|u(\cdot, t_0) - u_h^0\|_{0,\Omega} \le C_4 h^{r+1} \le C_0(h^{r+1} + \Delta t^2),$$
$$\|u(\cdot, t_0) - u_h^0\|_{1,\Omega} \le C_4 h^r \le C_0(h^r + \Delta t^2),$$

where C_0 denotes an arbitrary constant $\ge C_4$. Let us suppose that m is an integer, $0 \le m \le M - 1$, and that we have already shown the estimates

$$\max_{0\le i\le k}\|u(\cdot, t_i) - u_h^i\|_{0,\Omega} + \left(\sum_{i=1}^{i=k} \Delta t\|u(\cdot, t_k) - u_h^k\|_{1,\Omega}^2\right)^{1/2} \le C_0(h^r + \Delta t^2).$$

We shall prove that (4.11) hold for $k = m + 1$ and by induction, this will complete the proof.

Let us introduce the functions $e_h^i = u_h^i - \tilde{u}_h^i$ and $\eta_h^i = u^i - \tilde{u}_h^i$, $1 \le i \le k + 1$, where $u^i = u(\cdot, t_i)$ and $\tilde{u}_h^i = \tilde{u}(\cdot, t_i)$ is defined by (3.4). Formulation $(WMNR)_h^{(n+1)}$ and the exact consistency equation yields that

$$\begin{aligned}(d_t e_h^{k+1}, v_h) &+ \frac{\nu}{2}(\nabla e_h^{k+1} + \nabla e_h^k \circ X_{h1}^k, \nabla v_h) + \frac{\nu\Delta t}{2}(J_h^k \Delta e_h^k \circ X_{h1}^k, \nabla v_h) = (D_t u(Y^k, t_{k+(1/2)}) \\ &- \nu\Delta u(Y^k, t_{k+(1/2)}), v_h) - \frac{\nu\Delta t}{2}(J_h^k \nabla \tilde{u}_h^k \circ X_{h1}^k, \nabla v_h) - (d_t \tilde{u}_h^{k+1}, v_h) - \frac{\nu}{2}(\nabla \tilde{u}_h^{k+1} \\ &+ \nabla \tilde{u}_h^k \circ X_{h1}^k, \nabla v_h) + \left(\frac{1}{2}(f^{k+1} + f^k \circ X_{h1}^k) - f(Y^k, t_{k+(1/2)}), v_h\right) \\ &- (p(Y^k, t_{k+(1/2)}), \nabla \cdot v_h), \qquad \forall v_h \in \mathbf{V}_h,\end{aligned} \tag{4.12}$$

where

$$Y^k(x) = \frac{x + X_1^k(x)}{2} = x - \frac{\Delta t}{2} u^k(x).$$

Corresponding to $\mathscr{A}_h^{k+(1/2)}u$ and $\mathscr{F}_h^{k+(1/2)}$, we define linear forms $\mathscr{A}^{k+(1/2)}u$ and $\mathscr{F}^{k+(1/2)}$ on $(H_0^1(\Omega))^d$ for $k = 0, \ldots, m$ by

$$\langle \mathscr{A}^{k+(1/2)}u, v\rangle = (D_t u(Y^k, t_{k+(1/2)}) - \nu\Delta u(Y^k, t_{k+(1/2)}), v)$$
$$\text{and} \qquad \langle \mathscr{F}^{k+(1/2)}, v\rangle = (f(Y^k, t_{k+(1/2)}), v) + (p(Y^k, t_{k+(1/2)}), \nabla \cdot v).$$

In view of Equation (4.12) we can write that

$$\langle \mathcal{A}_h^{k+(1/2)} e_h, e_h^{k+1} \rangle = \langle (\mathcal{F}_h^{k+(1/2)} - \mathcal{F}^{k+(1/2)}), v_h \rangle + \langle (\mathcal{A}^{k+(1/2)} - \mathcal{A}_h^{k+(1/2)}) u, v_h \rangle + \langle \mathcal{A}_h^{k+(1/2)} \eta_h, v_h \rangle, \qquad \forall v_h \in \mathbf{V}_h. \quad (4.13)$$

The proof of Theorem 4.5 is established using the following three steps:

1. Prove the $\mathcal{A}_h^{k+(1/2)}$-propriety and $\|X_{hi}^k - X_i^k\|$ $(i = 1, 2)$ estimate,
2. bound each term of the second member of (4.13) and
3. conclusion of the proof by using Lemma 4.4.

Step 1. Preliminary results. We prepare the corresponding following lemmas.

Lemma 4.6. (*The* $\mathcal{A}_h^{k+(1/2)}$*-propriety*). *Under the hypotheses of Lemma* 4.1, *there exists a constant* $C_5 = C(u, \nu_0, \nu_1)$ *independent of k and* $(\Delta t, h)$, *such that*

$$\begin{aligned}(\mathcal{A}_h^{k+(1/2)} e_h, e_h^{k+1}) \geq \frac{1}{2\Delta t}[\|e_h^{k+1}\|^2 - \|e_h^k\|^2] + \frac{1}{2\Delta t}\|e_h^{k+1} - e_h^k \circ X_2^n\|^2 + \frac{\nu}{4}(1 - \delta_d)\|\nabla e_h^{k+1}\|^2 \\ + \frac{\nu}{4}\|\nabla e_h^{k+1} + \nabla e_h^k \circ X_{h1}^k\|^2 - C_5\left[\|e_h^k\|^2 + \left[h^{-d/2}\left\|\frac{X_2^k - X_{2h}^k}{\Delta t}\right\| + \frac{3\nu}{8}(1 - \delta_d)\right]\|\nabla e_h^k\|^2 \right. \\ \left. + h^{-d/2}\left\|\frac{X_2^k - X_{2h}^k}{\Delta t}\right\| \|e_h^{k+1}\|^2\right].\end{aligned}$$

Proof. From the definition of $\mathcal{A}_h^{k+(1/2)}$ and $\mathbf{H_4}$ hypothesis we have

$$\begin{aligned}(d_t e_h^{k+1}, e_h^{k+1}) = \left(\frac{e_h^{k+1} - e_h^k \circ X_2^k}{\Delta t}, e_h^{k+1}\right) + \left(\frac{e_h^k \circ X_2^k - e_h^k \circ X_{2h}^k}{\Delta t}, e_h^{k+1}\right) \geq \frac{1}{2\Delta t}[\|e_h^{k+1}\|^2 \\ - \|e_h^k\|^2] - \frac{C(u)}{2}\|e_h^k\|^2 + \frac{1}{2\Delta t}\|e_h^{k+1} - e_h^k \circ X_2^n\|^2 - C(\nu_0, \nu_1) h^{-d/2}\left\|\frac{X_2^k - X_{2h}^k}{\Delta t}\right\| \|\nabla e_h^k\| \, \|e_h^{k+1}\|.\end{aligned}$$

Using Lemma 4.1 result, we estimate other terms of $\langle \mathcal{A}_h^{k+(1/2)}, \cdot \rangle$ in the following way:

$$\begin{aligned}\frac{\nu}{2}(\nabla e_h^{k+1} + e_h^k \circ X_{h1}^k, \nabla e_h^{k+1}) &\geq \frac{\nu}{4}[\|\nabla e_h^{k+1}\|^2 - (1 + C\delta_d)\|\nabla e_h^k\|^2 + \|\nabla e_h^{k+1} + \nabla e_h^k \circ X_{h1}^k\|^2] \\ &\geq \frac{\nu}{4}\left[\|\nabla e_h^{k+1}\|^2 - \frac{3}{2}\|\nabla e_h^k\|^2\right] + \frac{\nu}{4}\|\nabla e_h^{k+1} + \nabla e_h^k \circ X_{h1}^k\|^2,\end{aligned}$$

and

$$\begin{aligned}\frac{\nu}{2}\Delta t(J_h^n \nabla e_h^k \circ X_{1h}^n, \nabla e_h^{k+1}) &\leq \frac{\nu\delta_d}{4}[(1 + C\delta_d)\|\nabla e_h^k\|^2 + \|\nabla e_h^{k+1}\|^2] \\ &\leq \frac{\nu\delta_d}{4}\left[\|\nabla e_h^{k+1}\|^2 + \frac{3}{2}\|\nabla e_h^k\|^2\right].\end{aligned}$$

Therefore,

$$(\mathcal{A}_h^{k+(1/2)}e_h,\, e_h^{k+1}) \le \frac{1}{2\Delta t}[\|e_h^{k+1}\|^2 - \|e_h^k\|^2] + \frac{1}{2\Delta t}\|e_h^{k+1} - e_h^k \circ X_2^n\|^2 + \frac{\nu}{4}(1-\delta_d)\|\nabla e_h^{k+1}\|^2$$
$$+ \frac{\nu}{4}\|\nabla e_h^{k+1} + \nabla e_h^k \circ X_{h1}^k\|^2 - \Bigg[C(u)\|e_h^k\|^2 + C(\nu_0,\, \nu_1)h^{-d/2}\Bigg(\left\|\frac{X_2^k - X_{2h}^k}{\Delta t}\right\| \|\nabla e_h^k\|^2$$
$$+ h^{-d/2}\left\|\frac{X_2^k - X_{2h}^k}{\Delta t}\right\| \|e_h^{k+1}\|^2\Bigg) + \frac{3\nu}{8}(1-\delta_d)\|\nabla e_h^k\|^2\Bigg],$$

which completes the proof of the lemma result. ∎

Lemma 4.7. *Under the hypotheses of Lemma* 4.2 *and Lemma* 4.3, *there exists a constant C independent of* $(h, \Delta t)$ *such that*

$$\|X_2^k - X_{2h}^k\| \le C\Delta t[\Delta t^2 + \|u_h^k - u^k\|]$$

and

$$\|X_1^k - X_{1h}^k\| \le \Delta t\|u^k - u_h^k\|.$$

Proof. The definition of X_2^k and X_{2h}^k give that

$$X_2^k - X_{2h}^k = \Delta t\left[u\left(x - \frac{\Delta t}{2}u^k(x),\, t_{k+(1/2)}\right) - \frac{1}{2}(u_h^k(x) + u_h^k(X_{1h}^k))\right].$$

Because

$$Y^k(x) = \frac{x + X_1^k}{2} = x - \frac{\Delta t}{2}u^k(x)$$

and using the Taylor's formula we have

$$\left\|u(Y^k,\, t_{k+(1/2)}) - \frac{u^k + u^k \circ X_1^k}{2}\right\| \le C(u)\Delta t^2.$$

From the result of Lemma 4.3 and the regularity assumption (4.9), we can see that

$$\|u^k(X_1^k) - u^k(X_{1h}^k)\| \le \|X_1^k - X_{1h}^k\|\, \|\nabla u(G_\theta(x))\|_{0,<} \le C(u)\|X_1^k - X_{1h}^k\|.$$

On the other hand, Lemma 4.2 results imply that

$$\|(u^k - u_h^k) \circ X_{1h}^k\| \le (1 + C\delta_d)\|u^k - u_h^k\|.$$

If we note $C = \max[C(u), 1 + C\delta_d])$, then, by the definition of $X_2^k - X_{2h}^k$, δ_d and the following result:

$$\|X_1^k - X_{1h}^k\| \leq \Delta t\|u^k - u_h^k\|,$$

we complete the proof of the lemma. ∎

Step 2. The (4.13)-second member estimates. This step is divided into three results object of the following lemmas.

Lemma 4.8. (*Perturbation error*). *There exists a constant* $C_6 = C(f)$ *and* $C_7 = C(p, \Omega)$ *independent of* $(h, \Delta t)$ *such that for all* $v_h \in V_h$ *it holds that*

$$|\langle(\mathcal{F}_h^{k+(1/2)} - \mathcal{F}^{k+(1/2)}), v_h\rangle| \leq C_6\Delta t^2\|v_h\| + [C_6\|X_1^k - X_{1h}^k\| + C_7 h^r]\|\nabla v_h\|.$$

Proof. In the first we have

$$\left\langle f(Y^k, t_{k+(1/2)}) - \frac{1}{2}\left(f^{k+1} + f^k \circ X_{1h}^k, v_h\right\rangle = \left\langle f(Y^k, t_{k+(1/2)}) - \frac{1}{2}\left(f^{k+1} + f^k \circ X_1^k, v_h\right\rangle + \frac{1}{2}\langle f^k \circ X_1^k - f^k \circ X_{1h}^k, v_h\rangle.$$

The relation $(Y^k, t_{k+(1/2)}) = \frac{1}{2}[(X_1^k(x), t_k) + (x, t_{k+1})]$ leads if we use the Taylor's formula that

$$\left\langle f(Y^k, t_{k+(1/2)}) - \frac{1}{2}\left(f^{k+1} + f^k \circ X_1^k, v_h\right\rangle \leq C\Delta t^2 \max_{i=0,2}\|f\|_{C^i(H^{2-i}(\Omega))}\|v_h\|.$$

On the other hand, we have

$$\langle f^k \circ X_1^k - f^k \circ X_{1h}^k, v_h\rangle \leq \|f^k \circ X_1^k - f^k \circ X_{1h}^k\|_{0,1}\|v_h\|_{0,\infty}$$
$$\leq \|X_1^k - X_{1h}^k\| \left\| \int_0^1 \frac{\partial f}{\partial \mu}(G_\theta(\cdot))\, d\theta \right\| \|v_h\|_{0,\infty},$$

where $\mu = [G_1(x) - G_0(x)]/[|G_1(x) - G_0(x)|]$. Hence, the $\mathbf{H_4}$ hypothesis and the regularity assumption of the continuous solution yield

$$\langle f^k \circ X_1^k - f^k \circ X_{1h}^k, v_h\rangle \leq CD_d(h)\|X_1^k - X_{1h}^k\| \, \|f\|_{C^0(H^1(\Omega))}\|\nabla v_h\|.$$

Combining the last inequality, we prove the first part of the lemma result with

$$C_6 = \max(C \max_{i=0,2}\|f\|_{C^i(H^{2-i}(\Omega))}, C\|f\|_{C^0(H^1(\Omega))}).$$

Now, remarking that $(\tilde{p}_h(\cdot), v_h) \in M_h \times \mathbf{V}_h$ and using the result of Lemma 2.3, we have

$$\left|(p(Y_1, t^{n+(1/2)}), \nabla \cdot v_h)\right| \leq C\|(p - \tilde{p}_h)(\cdot, t^{n+(1/2)})\| \, \|\nabla \cdot v_h\|.$$

Then from the $\mathbf{H_2}$ hypothesis we obtain the second part of the lemma result with $C_7 = C_4\|p\|_{L^\infty(H^r(\Omega))}$. ■

Lemma 4.9. (*Truncation error*). *It holds that, there exists a constant* $C_{13} = C(u)$ (*respectively,* $C_{14} = C(u)$) *is a constant independent of* $(h, \Delta t)$ *such that, for all* $v \in V_h$ *we have*

$$\left|\langle(\mathcal{A}^{k+(1/2)} - \mathcal{A}_h^{k+(1/2)})u, v_h\rangle\right| \leq C_{13}\left[\Delta t^2 + \left\|\frac{X_2^k - X_{2h}^k}{\Delta t}\right\|\right] \|v_h\| \\ + C_{14}\nu[\|X_1^k - X_{1h}^k\| + (\|\nabla(u^k - u_h^k)\| + \delta_d\|X_{1h}^k - X_1^k\|)] \, \|\nabla v_h\|.$$

Proof. Using Taylor's formula and the regularity assumption (4.9), we have

$$\frac{u(x, t_{k+1}) - u(X_2^k, t_k)}{\Delta t} = D_t u(Y^k, t_{k+(1/2)}) + R^k,$$

where $\|R^k\| \leq C_8\Delta t^2$ and $C_8 = C \max_{i=0,3}\|u\|_{C^i(H^{3-i}(\Omega))}$.

Similarly,

$$\left\|\Delta u^{k+(1/2)} \circ Y^k - \frac{1}{2}(\Delta u^{k+1} + \Delta u^k \circ X_1^k)\right\| \leq C_9\Delta t^2. \tag{4.14}$$

$C_9 = C \max_{i=0,2}|\Delta u|_{C^i(H^{2-i}(\Omega))}$. On the other hand, from the Sobolev's imbedding theorems [18] and Lemma 4.3 we can see

$$\left(\frac{u^k \circ X_2^k - u^k \circ X_{2h}^k}{\Delta t}, v_h\right) \leq \left\|\frac{X_2^k - X_{2h}^k}{\Delta t}\right\| \|v_h\| \left\|\int_0^1 \nabla u^k(G_\theta(\cdot))\, d\theta\right\|_{0,\infty} \\ \leq C_{10}\left\|\frac{X_2^k - X_{2h}^k}{\Delta t}\right\| \|v_h\|,$$

where $C_{10} = C\|\nabla u\|_{C^0(H^2(\Omega))}$.

Now Equation (2.3) yields that

$$(\Delta u(\cdot, t_k) \circ X_1^k, v_h) = -(\nabla u(\cdot, t_k) \circ X_1^k, \nabla v_h) - \Delta t(J_1^n \nabla u^n \circ X_1^n, \nabla v_h) \\ + \sum_{i=1}^{N} (R_i^k, v_h), \qquad \forall v_h \in \mathbf{V_h}, \tag{4.15}$$

where $\|R_i\| \leq C\Delta t^2\|u\|_{C^0(H^2(\Omega))}$. On the other hand,

$$(\nabla u^k \circ X_{1h}^k - \nabla u^k \circ X_1^k, \nabla v_h) \le \|\nabla v_h\| \, \|\nabla u^k \circ X_{1h}^k - \nabla u^k \circ X_1^k\|$$
$$\le \|\nabla v_h\| \, \|X_1^k - X_{1h}^k\| \left\| \int_0^1 \frac{\partial(\nabla u^k)}{\partial \mu} (G_\theta(\cdot)) \, d\theta \right\|_{0,\infty}.$$

Hence, using the regularity assumption on the continuous solution, there exists a constant $C_{11} = C\|u\|_{C^0(H^3(\Omega))}$ such that

$$(\nabla u^k \circ X_{1h}^k - \nabla u^k \circ X_1^k, \nabla v_h) \le C_{11}\|X_1^k - X_{1h}^k\| \, \|\nabla v_h\|.$$

In the end, we rewrite the last term in the $\langle(\mathcal{A}_h^{k+(1/2)} - \mathcal{A}^{k+(1/2)})u, \cdot\rangle$ expression as the following:

$$(J_{1h}^k \nabla u^k \circ X_{1h}^k - J_1^k \nabla u^k \circ X_1^k, \nabla v_h) = (J_{1h}^k \nabla u^k \circ (X_{1h}^k - X_1^k), \nabla v_h) + ((J_{1h}^k - J_1^k)\nabla u^k \circ X_1^k, \nabla v_h).$$

By definition of J_{1h}^k (respectively, J_1^k), result of Lemma 4.2 and the regularity assumption (4.9) we have

$$((J_{1h}^k - J_1^k)\nabla u^k \circ X_1^k, \nabla v_h) \le \|\nabla u^k\|_{0,\infty} \|\nabla(u_h^k - u^k)\| \, \|\nabla v_h\|.$$

Then from the injection imbedding result (see [18]), there exists a constant C independent of $(h, \Delta t)$ such that

$$((J_{1h}^k - J_1^k)\nabla u^k \circ X_1^k, \nabla v_h) \le C\|u^k\|_3 \|\nabla(u_h^k - u^k)\| \, \|\nabla v_h\|.$$

Similarly we prove that

$$(J_{1h}^k \nabla u^k \circ (X_{1h}^k - X_1^k), \nabla v_h) \le \|J_{1h}^k\|_{0,\infty} \|X_{1h}^k - X_1^k\| \int_0^1 \frac{\partial(\nabla u^k)}{\partial \mu} (G_\theta(\cdot)) \, d\theta\|_{0,\infty} \|\nabla v_h\|$$
$$\le C\|u^k\|_3 \|\nabla v_h\| \, \|X_{1h}^k - X_1^k\|.$$

Therefore, from Lemma 4.2 we have

$$\frac{\nu \Delta t}{2} (J_{1h}^k \nabla u^k \circ X_{1h}^k - J_1^k \nabla u^k \circ X_1^k, \nabla v_h) \le C_{12}\nu[\|\nabla(u^k - u_h^k)\| + \delta_d \|X_{1h}^k - X_1^k\|] \, \|\nabla v_h\|.$$

Hence, the lemma result follow with $C_{13} = \max(C_9, C_{10})$ and $C_{14} = \max(C_{11}, C_{12})$. ■

Lemma 4.10. (*Interpolation error*). *It holds that there exists a constant* C_{15} (*respectively,* $C_{16} = C(u)$ *and* $C_{17} = C(u)$) *such that*

$$\langle \mathcal{A}_h^{k+(1/2)} \eta_h, v_h \rangle \leq \frac{\nu}{2} [(\nabla \eta_h^{k+1}, \nabla e_h^{k+1}) - (\nabla \eta_h^k, \nabla e_h^k)] + C_{16} \left[\frac{h^r}{\sqrt{\Delta t}} \left(\left\| \frac{\partial u}{\partial t} \right\|_{L^2(t_k,t_{k+1};L^2(\Omega))} + \|u_h\|_{L^2(t_k,t_{k+1};H^1(\Omega))} \right) + h^{r-(d/2)} \left\| \frac{X_2^k - X_{h2}^k}{\Delta t} \right\| \|u\|_{C^0(H^1(\Omega))} \right] \|e_h^{k+1}\| + \nu C_{16} h^r \|e_h^{k+1} + e_h^k \circ X_{h1}^k\| + C_{17} \nu h^r \|\nabla e_h^{k+1}\|.$$

Proof. From the definition of $\langle \mathcal{A}_h^{k+(1/2)} \eta_h, \cdot \rangle$ we have

$$\langle \mathcal{A}_h^{k+(1/2)} \eta_h, v_h \rangle = (d_t \eta_h^{k+1}, v_h) + \frac{\nu}{2} (\nabla \eta_h^{k+1} + \nabla \eta_h^k \circ X_{h1}^k, \nabla v_h) + \frac{\nu \Delta t}{2} (J_{h1}^k \nabla \eta_h^k \circ X_{h1}^k, \nabla v_h).$$

Since it holds that

$$\left(\frac{\eta_h^{k+1} - \eta_h^k \circ X_2^k}{\Delta t}, v_h \right) \leq \frac{C}{\sqrt{\Delta t}} \left\{ \left\| \frac{\partial \eta_h}{\partial t} \right\|_{L^2(t_k,t_{k+1};L^2(\Omega))} + \|\eta_h\|_{L^2(t_k,t_{k+1};H^1(\Omega))} \right\} \times \|v_h\|,$$

also using Lemma 4.3 result, it holds that

$$\left(\frac{\eta_h^k \circ X_2^h - \eta_h^k \circ X_{h2}^k}{\Delta t}, v_h \right) = \frac{1}{\Delta t} \left(\int_0^1 (X_2^k - X_{h2}^k) \cdot \nabla \eta_h^k (G_\theta(\cdot)) \, d\theta, v_h \right) \leq 2 \left\| \frac{X_2^k - X_{h2}^k}{\Delta t} \right\| \|\eta_h^k\| \|v_h\|_{0,\infty}.$$

Then it follows from the $\mathbf{H_4}$ hypothesis that

$$(d_t \eta_h^{k+1}, v_h) \leq C_{15} \left[\frac{h^r}{\sqrt{\Delta t}} \left(\left\| \frac{\partial u}{\partial t} \right\|_{L^2(t_k,t_{k+1};L^2(\Omega))} + \|u_h\|_{L^2(t_k,t_{k+1};H^1(\Omega))} \right) + h^{r-(d/2)} \left\| \frac{X_2^k - X_{h2}^k}{\Delta t} \right\| \|u\|_{C^0(H^1(\Omega))} \right] \|v_h\|,$$

where $C_{15} = C_4 C$.

To estimate the second term in the $\langle \mathcal{A}_h^{k+(1/2)} \eta_h, \cdot \rangle$ expression, we first divide it into three parts:

$$(\nabla(\eta_h^{k+1}) + \nabla(\eta_h^k) \circ X_{1h}^k, \nabla e_h^{k+1}) = [(\nabla \eta_h^{k+1}, \nabla e_h^{k+1}) - (\nabla \eta_h^k, \nabla e_h^k)] + (\nabla \eta_h^k \circ X_{1h}^k, \nabla e_h^{k+1} + \nabla e_h^k \circ X_{1h}^k) + [(\nabla \eta_h^k, \nabla e_h^k) - (\nabla \eta_h^k \circ X_{1h}^k, \nabla e_h^k \circ X_{1h}^k)].$$

We remark that

$$(\nabla \eta_h^k, \nabla e_h^k) - (\nabla \eta_h^k \circ X_{1h}^k, \nabla e_h^k \circ X_{1h}^k)$$

$$= \int_\Omega \nabla \eta_h^k \nabla e_h^k \, dx - \int_{X_{1h}^k(\Omega)} \nabla \eta_h^k \nabla e_h^k |(j_{h1}^k)^{-1} \, dx$$

$$= \int_\Omega \nabla \eta_h^k \nabla e_h^k (1 - (j_{h1}^k)^{-1}) \, dx.$$

Then using Lemma 4.2 result, we have

$$(\nabla \eta_h^k, \nabla e_h^k) - (\nabla \eta_h^k \circ X_{1h}^k, \nabla e_h^k \circ X_{1h}^k) \leq C\delta_d \|\nabla \eta_h^k\| \, \|\nabla e_h^k\|.$$

For the same motive, we have

$$(\nabla \eta_h^k \circ X_{1h}^k, \nabla e_h^{k+1} + \nabla e_h^k \circ X_{1h}^k \leq (1 + C\delta_d) \|\nabla \eta_h^k\| \, \|\nabla e_h^{k+1} + \nabla e_h^k \circ X_{1h}^k\|.$$

Hence, from Lemma 3.1, we obtain

$$\frac{\nu}{2} (\nabla \eta_h^{k+1} + \nabla \eta_h^k \circ X_{1h}^k, \nabla e_h^{k+1}) \leq \frac{\nu}{2} [(\nabla \eta_h^{k+1}, \nabla e_h^{k+1}) - (\nabla \eta_h^k, \nabla e_h^k)]$$
$$+ \nu C_4 (1 + C\delta_d) h^r \|u\|_{C^0(H^1(\Omega))} [\|\nabla e_h^{k+1} + \nabla e_h^k \circ X_{1h}^k\| + \|\nabla e_h^k\|].$$

We use a similar device and Lemma 4.1 to bound the last term in the $\langle \mathscr{A}_h^{k+(1/2)} \eta_h, \cdot \rangle$ expression:

$$\frac{\nu \Delta t}{2} (J_{h1}^k \nabla \eta_h^k \circ X_{h1}^k, \nabla v_h) \leq \frac{\nu C_4}{2} (1 + C\delta_d) \delta_d h^r \|u\|_{C^0(H^1(\Omega))} \|\nabla v_h\|,$$

which completes the proof of the lemma for $C_{16} = C_4(1 + C\delta_d)\|u\|_{C^0(H^1(\Omega))})$ and $C_{17} = \max[C_{16}, (C_4/2)(1 + C\delta_d)\|u\|_{C^0(H^1(\Omega))}]$. ■

Step 3. Conclusion of the proof. We estimate $\|e_h\|_{l^\infty(0,T;L^2(\Omega))}$ and $\|e_h\|_{l^2(0,T;H^1(\Omega))}$. Substituting the results of Lemma 4.6 to Lemma 4.9 into (4.13), we obtain

$$\frac{1}{2\Delta t} [\|e_h^{k+1}\|^2 - \|e_h^k\|^2] + \frac{1}{2\Delta t} \|e_h^{k+1} - e_h^k \circ X_2^n\|^2 + \frac{\nu}{4} (1 - \delta_d) \|\nabla e_h^{k+1}\|^2 + \frac{\nu}{4} \|\nabla e_h^{k+1}$$
$$+ \nabla e_h^k \circ X_{h1}^k\|^2 \leq C_5 \left[\|e_h^k\|^2 + \left[Ch^{-d/2}(\Delta t^2 + \|u_h^k - u^k\|) + \frac{3\nu}{8}(1 - \delta_d) \right] \|\nabla e_h^k\|^2 + Ch^{-d/2}(\Delta t^2 \right.$$
$$\left. + \|u_h^k - u^k\|) \|e_h^{k+1}\|^2 \right] + C_{16} \left[\frac{h^r}{\sqrt{\Delta t}} \left(\left\| \frac{\partial u}{\partial t} \right\|_{L^2(t_k, t_{k+1}; L^2(\Omega))} + \|u_h\|_{L^2(t_k, t_{k+1}; H^1(\Omega))} \right) + Ch^{r-(d/2)}(\Delta t^2 \right.$$
$$\left. + \|u_h^k - u^k\|) \|u\|_{C^0(H^1(\Omega))} \right] \|e_h^{k+1}\| + C_6 \Delta t^2 \|e_h^{k+1}\| + [C_6 \Delta t \|u^k - u_h^k\| + C_7 h^r] \|\nabla e_h^{k+1}\| + C_{13}[\Delta t^2$$

$$+ C(\Delta t^2 + \|u_h^k - u^k\|)]\, \|e_h^{k+1}\| + C_{14}\nu[\Delta t\|u^k - u_h^k\| + (\|\nabla(u^k - u_h^k)\| + \delta_d\Delta t\|u^k - u_h^k\|)]\, \|\nabla e_h^{k+1}\| + \frac{\nu}{2}[(\nabla\eta_h^{k+1}, \nabla e_h^{k+1}) - (\nabla\eta_h^k, \nabla e_h^k)] + \nu C_{16}h^r\|e_h^{k+1} + e_h^n \circ X_{h1}^k\| + C_{17}\nu h^r\|\nabla e_h^{k+1}\|.$$

Summing up the above equation from $k = 0$ to $k = m$ and using the induction hypotheses:

$$\sum_{k=0}^{k=m} \Delta t\|\nabla(u^k - u_h^k)\|^2 \le C_0^2(\Delta t^2 + h^r)^2,\ \max_{0\le k\le m}\|u_h^k - u^k\| \le C_0(\Delta t^2 + h^r)$$

$$\text{and } \sum_{k=0}^{k=m} \Delta t\|\nabla e_h^k\|^2 \le C(\Delta t^2 + h^r)^2,$$

we get that there exists a constant $C_{18} = C(u)$, $C_{19} = C(u)$, and $h_4 \le h_0$ independent of $(h, \Delta t)$ such that for $h \in (0, h_4]$ we have

$$\frac{1}{2}\|e_h^{m+1}\|^2 + \frac{\nu}{4}\left(\frac{1}{2} - \delta_d\right)\sum_{k=1}^{k=m+1} \Delta t\|\nabla e_h^k\|^2 \le \left[C_{18} + \frac{3\nu}{8}(1 - \delta_d)CC_5\right](\Delta t^2 + h^r)^2 + C_{19}\sum_{k=0}^{m+1} \Delta t\|e_h^k\|^2 + \frac{\Delta t\nu}{2}[(\nabla\eta_h^{m+1}, \nabla e_h^{m+1}) - (\nabla\eta_h^0, \nabla e_h^0)]. \quad (4.16)$$

Using the definition of e_h^k, $0 \le k \le m + 1$, the estimates

$$\frac{\Delta t\nu}{2}(\nabla\eta_h^{m+1}, \nabla e_h^{m+1}) \le \frac{\Delta t\nu}{16}\|\nabla e_h^{m+1}\|^2 + C\nu\Delta t h^{2r}\|u^m\|_{r+1}^2,$$

and the Gronwall's lemma, we get the result of the theorem by virtue of the result of Lemma 3.1. ■

Remark. In the two-dimensional case, subject the mesh restriction $\Delta t = o(h)$ (respectively, $\Delta t = o(h^{1/2})$), in [19] the author proves (respectively, [16]) that

$$\|u - u_h\|_{l^2(H^1(\Omega))} \le C_0(\Delta t + h^r).$$

Remark. For the Hood-Taylor mixed finite element method (see Example 1) Theorem 4.5 yields the optimal error estimates

$$\|u_h - u\|_{l^\infty(L^2(\Omega))^d)} + \|u_h - u\|_{l^2(H^1(\Omega))^d)} \le C_0(h^2 + \Delta t^2).$$

V. CONCLUSION

We have presented a new characteristic-mixed finite element scheme for Navier-Stokes equation, which is of second order in time increment. We have established optimal error estimates.

In the future this result will be generalized to the Oldroyd derivative in the tensorial transport problem and to transient viscoelastic fluid flows.

The authors thank referees for remarks. This work was supported partially by the CNRST-CNRS Program STIC01/03 and by the CNRST-GRICES Portugal project.

References

1. T. Arbogast and M. F. Wheeler, A characteristics-mixed finite element method for advection-dominated transport problems, SIAM J Numer Anal 32(2) (1995), 404–424.
2. M. Bause and P. Knabner, Uniform error analysis for Lagrange-Galerkin approximation of convection-dominared problems, SIAM J Numer Anal 39(26) (2002), 1954–1984.
3. P. Chen, Characteristic mixed discontinuous finite element methods for advection-dominated diffusion problems, Comput Methods Appl Mech Engrg 191 (2002), 2509–2538.
4. M. R. Kaazempur-Mofrad, P. D. Minev, and MC. R. Ethier, A characteristic/finte element algorithm for time-dependent 3-D advection-dominated transport using unstructured grids, Comput Methods Appl Mech Engrg 192 (2003), 1281–1298.
5. K. Boukir, Y. Maday, and B. Métivet, A high order characteristics method for the incompressible Navier-Stokes equations, Comput Methods Appl Mech Eng 116 (1994), 211–218.
6. K. Boukir, Y. Maday, B. Métivet, and Razafindrakoto, A high order characteristics/finite element method for the incompressible Navier-Stokes equations, Int J Num Meth Fluids 25 (1997), 1421–1454.
7. O. Pironneau, On the transport-diffusion algorithm and its application to the Navier-Stokes equations, Numer Math 38 (1982), 309–332.
8. O. Pironneau, Méthodes des éléments finis pour les fluides, Masson, Paris, 1988.
9. M. Fortin and D. Esselaoui, A finite element procedure for viscoelastic flows, Int J Num Meth Fluids 7 (1987), 1035–1052.
10. A. Machmoum and D. Esselaoui, Finite element approximation of viscoelastic fluid flow using characteristics method, Comp Meth Appl Mech Engrg 190 (2001), 5603–5618.
11. O. Pironneau, J. Liou, and T. Tezduyar, Characteristic-Galerkin and characteristic/least-squares space-time formulations for the advection-diffusion equations with time dependent domain, Comput Methods Appl Mech Engrg 100 (1992), 117–141.
12. H. Rui and M. Tabata, A second order characteristic finite element scheme for convetion-diffusion problems, Numer Math (2001), 18.
13. R. A. Adams, Sobolev space, Academic Press, New York, 1975.
14. P. Grisvard, Singularité des solutions du problème de Stokes dans un polygone, Publication de l'université de Nice, 1978.
15. R. Temam, Navier-Stokes equations. Theory and Numerical analysis, North-Holland, Amsterdam, 1977.
16. E. Sûli, Convegence and nonlinear stability of the Lagrange-Galerkin method for the Navier-Stokes equations, Numer Math 53 (1988), 459–483.
17. P. A. Raviart and J. M. Thomas, Introduction à l'analyse numérique des E.D.P., Masson, Paris, 1983.
18. P. G. Ciarlet, The finite element method for elliptic problems, North-Holland, Amsterdam, 1978.
19. E. Sûli, Lagrange-Galerkin mixed finite element approximation of the Navier-Stokes equations. K. W. Morton, M. J. Baines, editors, Numerical methods for fluid dynamics, Oxford University Press, 1985, pp 439–448.