Mechanics fixed in the body in a plastically deformable state.

by

R. von Mises

with 4 figures in text.

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Translation from German by
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The mechanics of continua, which is based on the general concept of stress developed by Cauchy, has so far been applied almost exclusively to fluid and solid elastic bodies. For the field of plastic or permanent deformation of solid bodies, Saint-Venant[1] has outlined a theory that does not, however, provide the necessary number of equations to determine the motion. Other occasional attempts in this direction have also been unsuccessful[2].

The following lines lead to a complete approach to equations of motion for plastically deformable bodies – within the framework of Cauchy's mechanics and based on certain empirical facts characterizing the field of application.

1 Notations

The stress state in a coordinate system that is perpendicular to a point on the body is given by the three normal stresses σ_x, σ_y, σ_z and the tangential stresses τ_x, τ_x τ_z. In the scheme

$$\begin{matrix} \sigma_x & \tau_z & \tau_y \\ \tau_z & \sigma_y & \tau_x \\ \tau_y & \tau_x & \sigma_z \end{matrix} \tag{1}$$

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[1] Comptes Rendus Paris, t. 70, 72, 74. Journ. de math. 1871, p. 473.

[2] Haar und v. Kármán, Göttinger Nachr. 1909, leiten Bewegungsgleichungen aus einem neuen Variationsprinzip ab, dessen Verhältnis zur übrige Mechanik noch nicht aufgeklärt ist.

the quantities in the first row represent the components of the stress vector $\bar{\boldsymbol{\sigma}}_x$ for a surface element whose outer normal is in the direction of the positive x-axis, and so on. The vector structure represented by (1), which is transformed in a known manner using equation

$$\bar{\boldsymbol{\sigma}}_{x'} = \bar{\boldsymbol{\sigma}}_x \cos(x, x') + \bar{\boldsymbol{\sigma}}_y \cos(y, x') + \bar{\boldsymbol{\sigma}}_y \cos(z, x') \tag{2}$$

is also referred to as the stress dyad $\bar{\bar{\boldsymbol{\sigma}}}$ for short.

Analogous concept formation leads to the deformation dyad $\bar{\bar{\boldsymbol{\varepsilon}}}$ and to the deformation velocity dyad $\bar{\bar{\boldsymbol{\lambda}}}$. If ξ, η, ζ denote the infinitely small, elastic displacements of a point, then the strains and angular changes are:

$$\begin{aligned} &\varepsilon_x = \frac{\partial \xi}{\partial x}, \quad \varepsilon_y = \frac{\partial \eta}{\partial y}, \quad \varepsilon_z = \frac{\partial \zeta}{\partial z}, \\ &\gamma_x = \frac{1}{2}\left(\frac{\partial \eta}{\partial z} + \frac{\partial \zeta}{\partial y}\right), \quad \gamma_y = \frac{1}{2}\left(\frac{\partial \zeta}{\partial x} + \frac{\partial \xi}{\partial z}\right), \quad \gamma_z = \frac{1}{2}\left(\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}\right) \end{aligned} \tag{3}$$

and the dyad $\bar{\bar{\boldsymbol{\varepsilon}}}$ has the scheme:

$$\begin{matrix} \varepsilon_x & \gamma_z & \gamma_y \\ \gamma_z & \varepsilon_y & \gamma_x \\ \gamma_y & \gamma_x & \varepsilon_z \end{matrix} \tag{4}$$

If, instead of ξ, η, ζ, we take the components u, v, w of the velocity vector, we obtain the strain and shear velocities:

$$\begin{aligned} &\lambda_x = \frac{\partial u}{\partial x}, \quad \lambda_y = \frac{\partial v}{\partial y}, \quad \lambda_z = \frac{\partial w}{\partial z}, \\ &\nu_x = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \quad \nu_y = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \quad \nu_z = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \end{aligned} \tag{5}$$

and the scheme for the dyad $\bar{\bar{\boldsymbol{\lambda}}}$:

$$\begin{matrix} \lambda_x & \nu_z & \nu_y \\ \nu_z & \lambda_y & \nu_x \\ \nu_y & \nu_x & \lambda_z \end{matrix} \tag{6}$$

For each dyad, there is at least one coordinate system for which the scheme is reduced to the elements of the main diagonal, e.g., for (1) to the form:

$$\begin{matrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{matrix} \tag{7}$$

Here, the *main stresses* σ_1, σ_2, σ_3, the roots of the secular equation, are also determined by the following three conditions:

$$
\begin{aligned}
&\sigma_1 + \sigma_2 + \sigma_3 = \sigma_x + \sigma_x + \sigma_x \\
&\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_z - \left(\tau_x^2 + \tau_y^2 + \tau_z^2\right) \\
&\sigma_1\sigma_2\sigma_3 = \begin{vmatrix} \sigma_x & \tau_z & \tau_y \\ \tau_z & \sigma_y & \tau_x \\ \tau_y & \tau_x & \sigma_z \end{vmatrix}
\end{aligned} \tag{8}
$$

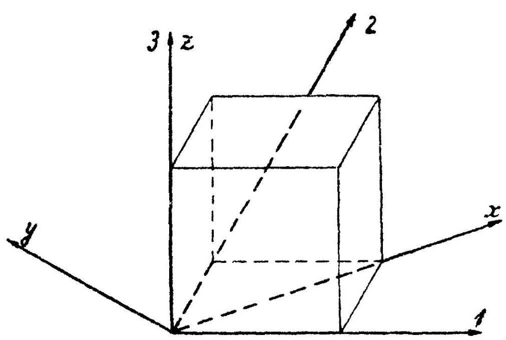

Figure 1:

If a coordinate system is placed such that the third principal axis coincides with the z-axis, while the x- and y-axes bisect the angles of the first two principal axes (Fig. 1), the following diagram results according to (2):

$$
\begin{matrix} \dfrac{\sigma_1 + \sigma_2}{2} & \dfrac{\sigma_2 - \sigma_1}{2} & 0 \\ \dfrac{\sigma_2 - \sigma_1}{2} & \dfrac{\sigma_1 + \sigma_2}{2} & 0 \\ 0 & 0 & \sigma_3 \end{matrix} \tag{9}
$$

At the same time, it can be seen that the τ values occurring here are extrema of the tangential stress, i.e., the three variables

$$
\tau_1 = \frac{\sigma_3 - \sigma_2}{2}, \quad \tau_2 = \frac{\sigma_1 - \sigma_3}{2}, \quad \tau_3 = \frac{\sigma_2 - \sigma_1}{2} \tag{10}
$$

always include the absolute largest and smallest tangential stresses. We call the τ_1, τ_2, τ_3, whose sum is zero, the principal tangential stresses.

The simplest of all stress dyads is that of the ideal fluid $-\bar{\bar{\boldsymbol{p}}}$. It has the following structure in every coordinate system

$$
\begin{matrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{matrix} \tag{11}
$$

If a stress state of the form (11) is subtracted from the stresses represented by (1), the tangential stresses remain unchanged and the following equation is obtained:

$$\begin{matrix} \sigma'_x & \tau_z & \tau_y \\ \tau_z & \sigma'_y & \tau_x \\ \tau_y & \tau_x & \sigma'_z \end{matrix} \tag{12}$$

with

$$\sigma'_x = \sigma_x + p, \quad \sigma'_y = \sigma_y + p, \quad \sigma'_z = \sigma_z + p \tag{13}$$

The dyad (12) has the same principal directions as (1), and the principal values σ'_1, σ'_2, σ'_3 are the principal values of (1) reduced by $-p$. It follows from (10) that the principal tangential stresses for (12) and (1) are identical. All these relationships also apply, of course, to the deformation dyad $\bar{\bar{\varepsilon}}$ or to $\bar{\bar{\boldsymbol{\lambda}}}$. We give another formula below that will be used later, which is derived from a combination of (10) and (8). It is:

$$\begin{aligned} \tau_1^2 + \tau_2^2 + \tau_3^2 &= \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - \frac{1}{2}\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right) \\ &= \frac{1}{2}\left(\sigma_1 + \sigma_2 + \sigma_3\right)^2 - \frac{3}{2}\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right) \\ &= \frac{1}{2}\left(\sigma_x + \sigma_y + \sigma_z\right)^2 - \frac{3}{2}\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x\right) + \frac{3}{2}\left(\tau_x^2 + \tau_y^2 + \tau_z^2\right) \\ &= \frac{1}{2}\left(\frac{\sigma_z - \sigma_y}{2}\right)^2 + \frac{1}{2}\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \frac{1}{2}\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \frac{3}{2}\left(\tau_x^2 + \tau_y^2 + \tau_z^2\right) \end{aligned} \tag{14}$$

2 Fundamental principles based on experience

We now present the empirical facts that are taken into account in the equations of motion below. We do not claim to provide an axiomatic structure, i.e., we refrain from using even a minimum of assumptions.

a) All solid bodies behave like elastic bodies under sufficiently small stresses: there is a one-to-one correspondence between stress and strain. This sentence distinguishes solid bodies from viscous ones. "*Solid*" refers, for example, to malleable wax, which yields even under slight external pressure, as well as iron, which only reaches its elasticity limit under very high pressure. In contrast, pitch or similar substances are not plastically deformable at normal temperatures, but are liquid. We will discuss the significance and form of the elasticity limit further below. The relationship between stress and strain dyads $\bar{\bar{\boldsymbol{\sigma}}}$ and $\bar{\bar{\varepsilon}}$ is assumed to be linear in mathematical elasticity theory:

$$\bar{\bar{\boldsymbol{\sigma}}} = L(\bar{\bar{\varepsilon}}) \tag{15}$$

The most general linear relationship, in which no direction in space is preferred, is that the two dyads have the same principal directions and the principal values are related as follows:

$$\sigma_1 = \alpha\varepsilon_1 + \beta(\varepsilon_1 + \varepsilon_2 + \varepsilon_3), \quad \sigma_2 = \alpha\varepsilon_2 + \beta(\varepsilon_1 + \varepsilon_2 + \varepsilon_3),$$
$$\sigma_3 = \alpha\varepsilon_3 + \beta(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \tag{16}$$

Here, α and β are the elastic constants. In a known manner, (16) can be transformed so that relationships between the components related to arbitrary axes are created.

b) Once the elasticity limit has been reached, the solid body behaves essentially like a viscous, almost incompressible fluid. The behavior of the fluid referred to here is characterized by the fact that it is not the state of deformation, as in the case of an elastic body, but the deformation process that causes stresses. However, one cannot simply assume the stress dyad $\bar{\bar{\boldsymbol{\sigma}}}$ as a function of the deformation velocity dyad $\bar{\bar{\boldsymbol{\lambda}}}$, but must note that a volume element under external pressure that is the same on all sides does not experience a finite deformation velocity. The resulting change in volume always remains, as observations consistently show, of the same order of magnitude as the elastic displacements. It follows, therefore, that in the mechanics of viscous fluids, a part $-\bar{\bar{\boldsymbol{p}}}$ must be subtracted from the stress dyad $\bar{\bar{\boldsymbol{\sigma}}}$ which corresponds to an equal pressure on all sides. The remainder $\bar{\bar{\boldsymbol{\sigma}}}'$, see (1) in section 1, can then be written as a linear function:

$$\bar{\bar{\boldsymbol{\sigma}}}' = L(\bar{\bar{\boldsymbol{\lambda}}}) \tag{17}$$

If the same symmetry as above is observed, the analogous result is (16):

$$\sigma_1' = k\lambda_1 + k'(\lambda_1 + \lambda_2 + \lambda_3), \ldots \tag{18}$$

But the expression in parentheses measures precisely the divergence or the change in volume, which was just mentioned as being negligible compared to λ_1. This gives us:

$$\sigma_1' = k\lambda_1, \quad \sigma_2' = k\lambda_2, \quad \sigma_3' = k\lambda_3 \tag{19}$$

These equations simply state that $\bar{\bar{\boldsymbol{\sigma}}}'$ results from $\bar{\bar{\boldsymbol{\lambda}}}$ when each component of $\bar{\bar{\boldsymbol{\lambda}}}$ is multiplied by k, i.e.:

$$\sigma_x' = \sigma_x + p = k\lambda_x, \quad \sigma_y' = \sigma_y + p = k\lambda_y, \quad \sigma_z' = \sigma_z + p = k\lambda_z,$$
$$\tau_x' = k\nu_x, \quad \tau_y' = k\nu_y, \quad \tau_z' = k\nu_z \tag{20}$$

These are exactly the same equations that Navier-Stokes' theory of viscous fluids leads to. A significant difference only becomes apparent when we examine the meaning of the variable k more closely. This leads to the following crucial empirical theorem.

c) If, while maintaining all other conditions, the absolute value of the velocities at which a movement takes place is changed, the work required to achieve a certain change in shape does not change in the case of plastically deformable bodies. We derive this statement from all the observational data that has been gathered to date in the field of permanent changes in shape, particularly in technology. For the most part, technology uses formulas for the amount of work required that disregard the influence of speed from the outset. Where this influence was particularly observed, it proved to be minor[3]. The constancy asserted in

[3] For references, see also bibliography, cf. my encyclopedia article IV 10, no. 5, p. 187.

sentence c) must be understood in a similar way to, for example, the constancy of friction coefficients with respect to changing normal pressure in the sliding friction of solid bodies. At least, accepting c) specifies an ideal case that allows for a certain theory and provides a useful approximation for the actual behavior of the bodies. The work per unit volume to be expended per second is generally given by:

$$\sigma_x'\lambda_x + \sigma_y'\lambda_y + \sigma_z'\lambda_z + 2\tau_x\nu_x + 2\tau_y\nu_y + 2\tau_z\nu_z = k\left(\lambda_x^2 + \lambda_y^2 + \lambda_z^2 + 2\nu_x^2 + 2\nu_y^2 + 2\nu_z^2\right) \tag{21}$$

If all speeds are multiplied by a factor c, this expression changes proportionally to kc^2. At the same time, however, the duration of the deformation process is shortened in a ratio of $1/c$, so the total work becomes proportional to kc. Therefore, the proportionality factor k introduced in (20) must be inversely proportional to the speed. Or, in other words: the voltage dyad $\bar{\bar{\boldsymbol{\sigma}}}'$ remains the same if all components of $\bar{\bar{\boldsymbol{\lambda}}}$ are changed in the same ratio. It follows from the latter formulation that the stresses in a plastically deformable body must vary in a range of lower diversity than, for example, in an elastic body. It is clear that this range can be nothing other than the elastic limit. This means that our statement c) can also be formulated as follows:

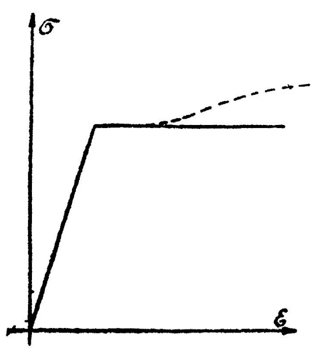

Figure 2:

c') In the case of plastic deformation, the stress always remains at the elastic limit. This sentence implies the requirement that the elasticity limit must be independent of an additive component of the formula (11), see below. One can verify c') directly through observation. In the one-dimensional case of tensile stress on a rod, the stress-strain diagram according to c') should have the shape shown in Fig. 2 First, an inclined straight line for the elastic state, then a horizontal line corresponding to the speed-independent limit stress in the plastic region. Observation shows that in iron, steel, and similar materials, a horizontal section connects to the inclined straight line, but this soon transitions into a slightly rising line. This is attributed to a process strongly influenced by heat and related to the crystalline nature of the body, which is called "*hardening*". Our theory does not take this hardening into account. However, it must be borne in mind that the actual scope of application of plasticity theory lies in the field of positive p. It has not yet been sufficiently clarified whether such hardening also occurs when iron and other materials are subjected to pressure. In any case, it does not seem unlikely that, in the case of easily malleable bodies such as wax and other similar materials, "*hardening*" plays a very limited role. We now turn to a final list concerning the nature of the elasticity limit:

d) In a coordinate system where the principal tangential stresses are coordinates, the elasticity limit appears as a closed curve in the plane

$$\tau_1 + \tau_2 + \tau_3 = 0 \tag{22}$$

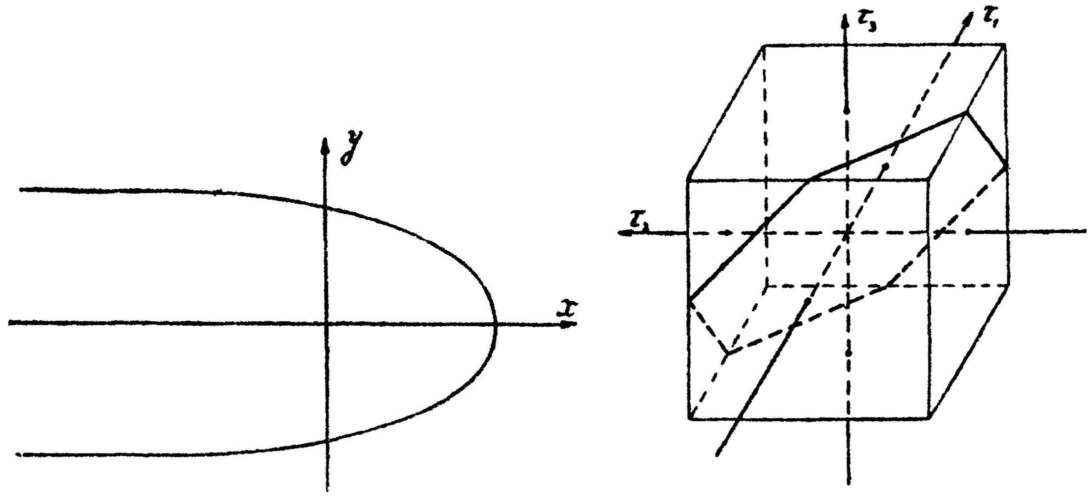

Figure 3:

that includes the zero point. It is well known that O. Mohr was responsible for the first detailed investigations into elasticity and breaking points[4]. According to Mohr, only the largest and smallest of the three principal stresses, let's say σ_1 and σ_2, are relevant here. In a coordinate system

$$x = \frac{\sigma_1 + \sigma_2}{2}, \quad y = \frac{\sigma_1 - \sigma_2}{2} = -\tau_3 \tag{23}$$

the fracture boundary, taking into account not only Mohr's considerations but also the new experiments by Kärmän[5]looks something like Fig. 3.left. The big difference between the behavior at positive x (tension) and negative x (compression) stems from the fact that there is tearing when tension is applied on all sides, but no crushing when pressure is applied equally on all sides. It is not very likely that the analogous contrast also exists for the limit of elastic behavior. Since we are primarily concerned with conditions at high mean pressure, it is permissible to regard the horizontal asymptotes in Fig. 3.left. as the essential limitation. This view, which is widely held in other contexts, leads to the elasticity limit:

$$|\tau_1| \leqslant K, \quad |\tau_2| \leqslant K, \quad |\tau_3| \leqslant K \tag{24}$$

The cube (24) is cut by the plane (22) into a regular hexagon (Fig. 3.right), so that our condition d) is fulfilled. However, we want to modify Mohr's approach in another direction. From the hexagon (22) and (24), only the corner points have been determined by the experiments so far, i.e., states in which one of the τ is zero and the absolute value of the other two is the same. The straight-line connection is based on the assumption that the

[4]O. Mohr, Abhandl. a. d. Gebiete der techn. Mechanik, Berlin 1906, p. 197.
[5]Zeitschr. d. Vereines deutscher Ingenieure 1911, p. 1749.

average principal stress and the smaller principal tangential stresses are irrelevant. This assumption does not seem so plausible that one should not attempt to replace the hexagon with a simpler structure, namely the circumscribed circle. Instead of the cube (24), we then have the sphere:

$$\tau_1^2 + \tau_2^2 + \tau_3^2 = 2K^2 \tag{25}$$

In any case, (25) allows for a much simpler analytical treatment, without the difference from (24) being greater than the margin allowed by the attempts made so far.

3 Equations of motion

We denote the specific mass of the body by ρ, and the components of the specific volume force (gravity, etc.) by χ_x, χ_y, χ_z. Then the equations of motion are as follows:

$$\text{(I)} \qquad \begin{aligned} \rho \frac{du}{dt} &= \chi_x - \frac{\partial p}{\partial x} + \frac{\partial \sigma_x'}{\partial x} + \frac{\partial \tau_z}{\partial y} + \frac{\partial \tau_y}{\partial z} \\ \rho \frac{dv}{dt} &= \chi_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_z}{\partial x} + \frac{\partial \sigma_y'}{\partial y} + \frac{\partial \tau_x}{\partial z} \\ \rho \frac{dw}{dt} &= \chi_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_y}{\partial x} + \frac{\partial \tau_x}{\partial y} + \frac{\partial \sigma_z'}{\partial z} \end{aligned}$$

The six stress components $\sigma_x' \dots \tau_z$ are expressed after (20) and (5) by the three velocity quantities u, v, w as follows:

$$\text{(II)} \qquad \begin{aligned} &\sigma_x' = k\frac{\partial u}{\partial x}, \quad \sigma_y' = k\frac{\partial v}{\partial y}, \quad \sigma_z' = k\frac{\partial w}{\partial z}, \\ &\tau_x = \frac{k}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \quad \tau_y = \frac{k}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \quad \tau_z = \frac{k}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \end{aligned}$$

As in hydromechanics, the continuity equation is used to eliminate p:

$$\text{(III)} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

This assumes incompressibility in accordance with statement b) and the associated comments, but our theory could also easily handle the more general case. Approaches (I) to (III) are entirely consistent with those for viscous fluids. However, in those cases, the variable k is the given viscosity coefficient, whereas in our case it is a reaction variable that can only be calculated based on knowledge of the motion itself. This is supported by the statement that the stress remains at the elastic limit during plastic deformation. If we assume the limit to be a circle in the form (25) and insert the value (14) into it, we obtain:

$$\left(\sigma_x' + \sigma_y' + \sigma_z'\right)^2 - 3\left(\sigma_x'\sigma_y' + +\sigma_y'\sigma_z' + +\sigma_z'\sigma_x'\right) + 3\left(\tau_x^2 + \tau_y^2 + \tau_z^2\right) = 4K^2 \tag{26}$$

This is because the last form of expression (14) implies that σ can also be replaced by σ'. However, if we add the first three of (II) and take (III) into account, we find that:

$$\sigma'_x + \sigma'_y + \sigma'_z = 0 \tag{27}$$

so that (26) reduces to:

$$\frac{4K^2}{3} = \tau_x^2 + \tau_y^2 + \tau_z^2 - \left(\sigma'_x\sigma'_y + +\sigma'_y\sigma'_z + +\sigma'_z\sigma'_x\right) \tag{IV}$$

If we substitute the values from (II) here, we obtain the equation we are looking for for k. Equations (I) to (IV) constitute the complete system of equations of motion for plastically deformable bodies. The following boundary condition must also be added here: the specification of the velocity components u, v, w for each surface point. It can be replaced by specifying the surface tension across the entire surface or part of it. In the case of plane motion, our approach reduces to Saint-Venant's. This is partly because, in the plane case, the difference between the elasticity limit according to (24) (hexagon) and according to (26) (circle) disappears. Because now there are only two main tangential stresses τ_1 and τ_2 with

$$\tau_1 + \tau_2 = 0 \tag{28}$$

so that $\tau_1^2 + \tau_2^2 \leqslant 2K^2$ says the same thing as $|\tau_1| \leqslant K$, $|\tau_2| \leqslant K$. Equations (I) to (IV) can be written very simply using vector symbols. If $\bar{\boldsymbol{v}}$ denotes the velocity vector and $\bar{\boldsymbol{\chi}}$ the vector of specific force, then we have:

$$\rho\frac{d\bar{\boldsymbol{v}}}{dt} = \bar{\boldsymbol{\chi}} - \mathbf{grad}\, p + \nabla\bar{\bar{\boldsymbol{\sigma}}}' \tag{I'}$$

$$\bar{\bar{\boldsymbol{\sigma}}}' = k\bar{\bar{\boldsymbol{\lambda}}} \tag{II'}$$

$$\operatorname{div}\bar{\boldsymbol{v}} = 0 \tag{III'}$$

$$-\left(\bar{\bar{\boldsymbol{\sigma}}}'\right)_2 = \frac{4K^2}{3} \tag{IV'}$$

Here, the symbol ∇ in (I') denotes the differentiation to be performed on the dyad, which is determined by (I). The index 2 in (IV') indicates that the second of the orthogonal invariants written in section 1 Eq. (8) is to be taken from the dyad $\bar{\bar{\boldsymbol{\sigma}}}'$. From (I') to (IV'), $\bar{\bar{\boldsymbol{\sigma}}}'$. can also be easily eliminated, yielding:

$$\rho\frac{d\bar{\boldsymbol{v}}}{dt} = \bar{\boldsymbol{\chi}} - \mathbf{grad}\, p + \nabla\left(k\bar{\bar{\boldsymbol{\lambda}}}\right) \tag{a}$$

$$\operatorname{div}\bar{\boldsymbol{v}} = 0 \tag{b}$$

$$k^2 = -\frac{4K^2}{3\left(\bar{\bar{\boldsymbol{\lambda}}}\right)_2} \tag{c}$$

If we multiply (I') scalarly by $\bar{\boldsymbol{v}}$ and integrate over the volume, we find, after appropriate transformation, that the dissipation function is represented by (21), which proves the consistency of the approach with our assumption c) in section 2.

Strasbourg, October 4, 1913.