Some additional comments on the presentation of the rheological equation of state according to Weissenberg and Grossmann

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translation from German by
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1 Introduction

In a recently published study, Grossman [1] has reformulated Weissenberg's [2] concept concerning the representation of a rheological equation of state that is as universal as possible.

1. The set of permissible deformation and strain measures contains a subset that is preferable for representing the equation of state, insofar as these measures are particularly closely related to the stress tensor. Such a measure is given by the tensor[1] $\boldsymbol{W}$, which assigns a body-fixed vector[2] $d\boldsymbol{w}$ to a plane quantity $d\boldsymbol{f}$ characterizing a body-fixed flat element:

$$d\boldsymbol{w} = \boldsymbol{W}.d\boldsymbol{w}, \quad (1)$$

whereby a specific state (for example, the "natural state") is characterized by the equality[3] of $d\boldsymbol{w}$ and $d\boldsymbol{f}$ and thus by $\mathring{\boldsymbol{W}} = 1$.

2. According to Weissenberg's principle, the rheological equation of state of an elastic body can be represented as a quasi-linear[4] expression in the tensors $\mathbf{1}.\boldsymbol{W}$ and the stress tensor $\boldsymbol{\sigma}$

Gieskus has addressed questions relating to the same set of problems in various studies. Therefore, the following section will highlight the relationships between the results obtained by both authors and draw some further conclusions from them. First, it is deduced that the tensor $\boldsymbol{W}$ represents the tensor density associated with the spatially contravariant deformation measure. First, it is deduced that the tensor $\boldsymbol{W}$ represents the tensor density associated with the spatial contravariant deformation measure. Subsequently, it is shown that the state tensor $\boldsymbol{S}$, which characterizes the flow behavior of a large class of fluids (Weissenberg fluids), can be related to the tensor $\boldsymbol{W}$. Finally, with the help of this connection, Weissenberg's principle is transferred to elastoviscous fluids. The equation of state obtained in this way can be represented as a quasi-linear relationship in the contravariant kinematic tensors.

2 The tensor $\boldsymbol{W}$ in the carried coordinate system

The following result was obtained in the study [3]: if the rheological state equation is formulated using tensor components of a coordinate system, it is appropriate for a very general class of bodies (characterized by the existence of a scalar state function) to represent the stress tensor by a contravariant tensor density. Consequently, in the usual spatial (so-called Eulerian) representation, contravariant measures, e.g., the spatial Green strain measure, and contravariant convective derivatives are given preferential importance.

To examine the tensor $\boldsymbol{W}$ more closely, we represent it in a concomitant coordinate system. Such a system is determined – as in [3], Appendix I – by the covariant basis $\bar{\boldsymbol{e}}_\nu(t)$ fixed in the field, $\bar{\boldsymbol{e}}^\nu(t)$ denotes

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1 At Grossman, this is referred to as the *separation tensor* and represented by the symbol $\boldsymbol{S}$.

2 $d\boldsymbol{w}$ is accordingly referred to as the *separation vector* and represented by the symbol $\boldsymbol{s}$.

3 Since $d\boldsymbol{w}$ and $d\boldsymbol{f}$ have different dimensions, but $\boldsymbol{W}$ is supposed to be a dimensionless quantity, constant factor of the dimension of a reciprocal length would actually have to be added to the right-hand side of (1).

4 Here, "quasi-linear" means that the coefficients may be scalar functions of the tensor invariants.

the associated contravariant basis, and the metric is accordingly given by:

$$\gamma_{\nu\mu} = \bar{\boldsymbol{e}}_\nu.\bar{\boldsymbol{e}}_\mu,\ \gamma^{\nu\mu} = \bar{\boldsymbol{e}}^\nu.\bar{\boldsymbol{e}}^\mu,\ \gamma = |\gamma_{\nu\mu}| = [\bar{\boldsymbol{e}}_1\,\bar{\boldsymbol{e}}_2\,\bar{\boldsymbol{e}}_3]^2 \quad (2)$$

Then the body-fixed vector $d\boldsymbol{w}$ is represented by

$$d\boldsymbol{w}(t) = d\psi^\nu \bar{\boldsymbol{e}}_\nu(t), \quad (3)$$

the plan size $d\boldsymbol{f}$ of the surface element fixed in the body by

$$d\boldsymbol{f}(t) = d\varphi_\mu(t)\,\bar{\boldsymbol{e}}^\mu(t) \quad (4)$$

and the tensor $\boldsymbol{W}$ linking both sizes is represented by

$$\boldsymbol{W}(t) = \psi^{\nu\mu}(t)\,\bar{\boldsymbol{e}}_\nu(t)\bar{\boldsymbol{e}}_\mu(t) \quad (5)$$

so that (1) in component notation reads as follows:

$$d\psi^\nu = \psi^{\nu\mu}(t)\,d\varphi_\mu(t). \quad (1a)$$

While $d\psi^\nu$ is a time-independent quantity, this is not the case for $d\varphi_\mu$ and consequently also for $\psi^{\nu\mu}$. If $d\eta^\rho$ and $d\zeta^\sigma$ are the components of the vectors spanning the plan quantity $d\boldsymbol{f}$, then

$$d\varphi_\mu = \sqrt{\gamma(t)}\varepsilon_{\mu\rho\sigma}d\eta^\rho d\zeta^\sigma, \quad (6)$$

where $\varepsilon_{\mu\rho\sigma}$ denotes the well-known permutation symbol, therefore, $\gamma^{-1/2}d\varphi_m u$ and, correspondingly, $\gamma^{1/2}\psi^{\nu\mu}$ are time-independent quantities.

If we now want to equate the quantities $d\boldsymbol{w}$ and $d\boldsymbol{f}$ at time $t = 0$, this means for the above time-independent quantities $d\psi^\nu$ and $\gamma^{-1/2}d\varphi_\mu$ the existence of the relation

$$d\psi^\nu = \mathring{\gamma}^{1/2}\mathring{\gamma}^{\nu\mu}(\gamma^{-1/2}d\varphi_\mu) \quad (7)$$

By comparison with (1a), we immediately obtain the convective components of $\boldsymbol{W}$:

$$\psi^{\nu\mu} = (\mathring{\gamma}/\gamma)^{1/2}\,\mathring{\gamma}^{\nu\mu}. \quad (8)$$

If, in addition to the entrained system, a space-fixed system is introduced by the time-independent basis $\boldsymbol{e}_i$, where for some body-fixed position vector

$$d\boldsymbol{x}(t) = d\xi^\nu\,\bar{\boldsymbol{e}}_\nu(t) = dx^i(t)\,\boldsymbol{e}_i \quad (9)$$

holds, then $\boldsymbol{W}$ can easily be expressed in terms of components of this system. It is, in fact[5]

$$\bar{\boldsymbol{e}}_\nu(t) = x^i_{,\nu}\,\boldsymbol{e}_i,\quad \gamma^{1/2} = \mathring{\gamma}^{1/2}\,|x^i_{,\nu}| \quad (10)$$

and therefore

$$\begin{aligned}\boldsymbol{W} &= \psi^{\nu\mu}\bar{\boldsymbol{e}}_\nu\bar{\boldsymbol{e}}_\mu = \psi^{\nu\mu}x^i_{,\nu}x^k_{,\mu}\,\boldsymbol{e}_i\,\boldsymbol{e}_k\\ &= \mathring{\gamma}^{\nu\mu}\,\frac{x^i_{,\nu}x^k_{,\mu}}{|x^m_{,\rho}|}\,\boldsymbol{e}_i\,\boldsymbol{e}_k. \quad (11)\end{aligned}$$

If one identifies the carried basis at time $t = 0$ with the space-fixed basis, i.e. sets

$$d\xi^L = d\mathring{x}^L \qquad \mathring{\gamma}^{LM} = g^{LM} \quad (12)$$

then

$$\begin{aligned}\boldsymbol{W} &= g^{LM}\,\frac{x^i_{,L}\,x^k_{,M}}{|x^m_{,N}|}\,\boldsymbol{e}_i\,\boldsymbol{e}_k\\ &= \frac{g^{LM}\,x^i_{,L}\,x^k_{,M}}{|g^{LM}\,x^m_{,L}\,x^n_{,M}|^{1/2}}\,\frac{\boldsymbol{e}_i\,\boldsymbol{e}_k}{[\bar{\boldsymbol{e}}_1\,\bar{\boldsymbol{e}}_2\,\bar{\boldsymbol{e}}_3]}. \quad (13)\end{aligned}$$

The expression on the right in the numerator means precisely the spatial contravariant deformation tensor[6] derived in [3]

$$\hat{\boldsymbol{c}} = g^{LM}\,x^i_{,L}\,x^k_{,M}\,\boldsymbol{e}_i\,\boldsymbol{e}_k \quad (14)$$

and the denominator expression, the associated volume dilation, so that

$$\boldsymbol{W} = \frac{\hat{\boldsymbol{c}}}{|\hat{\boldsymbol{c}}|^{1/2}} = \frac{\hat{\boldsymbol{c}}}{III^{1/2}_{\hat{\boldsymbol{c}}}}, \quad (16)$$

[5] Indices preceded by a comma, as usual, signify differentiation with respect to spatial coordinates. Specifically, these are convective coordinates ξ^ν when Greek indices are used, and the initial coordinates $\mathring{x}^L$ when large Latin indices are used.

[6] From this, the *spatial* Green strain measure

$$\hat{\boldsymbol{e}} = \frac{1}{2}\,(\hat{\boldsymbol{c}} - \boldsymbol{1}) \quad (14a)$$

is derived, whereas the *material* Green strain measure is represented by the expression

$$\boldsymbol{E} = \frac{1}{2}\,(\boldsymbol{C} - \boldsymbol{1}) \quad (15a)$$

in which

$$\boldsymbol{C} = g_{ij}\,x^i_{,L}\,x^k_{,M}\,\boldsymbol{e}^L\,\boldsymbol{e}^M \quad (15)$$

denotes the material covariant deformation tensor.

as claimed, represents the tensor density associated with the spatial contravariant deformation tensor[7].

This also shows that the result formulated by Grossman under point 1 essentially agrees with the result by Giesekus cited above. However, the latter investigation contains a further addition, for it derives why it is appropriate to refer the stresses to body-fixed surfaces[8], whereas Grossman assumes this without further explanation.

3 The tensor $\boldsymbol{W}$ and the state tensor $\boldsymbol{S}$ of a Weissenberg liquid

Grossman applied Weissenberg's principle essentially to elastic bodies, whereas his considerations cannot be readily transferred to liquids, except for Newtonian and linear elastoviscious liquids. However, Giesekus has now shown in a further investigation [4] that a large class of liquids can be described to a good approximation by an equation of state of the form

$$\boldsymbol{\sigma} = -\frac{\mathfrak{D}\boldsymbol{S}}{\mathfrak{D}t} = -\frac{D\boldsymbol{S}}{Dt} + \boldsymbol{S}.\nabla\boldsymbol{v} + \boldsymbol{v}\nabla.\boldsymbol{S} - (\nabla.\boldsymbol{v})\boldsymbol{S} \tag{17}$$

where $\boldsymbol{\sigma}$ is an additional stress due to suspended (or loose) particles, $\mathfrak{D}\boldsymbol{S}/\mathfrak{D}t$ is the contravariant convective derivative of a symmetric tensor density $\boldsymbol{S}$, $D\boldsymbol{S}/Dt$ is its material derivative, and $\nabla\boldsymbol{v}$ is the gradient of the velocity field[9]. For such fluids, - just as Weissenberg had demanded - in the direction of the shear rate, the normal components of the stress disappear in steady layer flows[10], which is why Giesekus referred to such liquids as Weissenberg liquids.

[7]This result is also contained in Grossman's investigation. However, it is not so clear there that $\boldsymbol{W}$ is a spatial ("Eulerian") measure, but by no means a material ("Lagrangian") measure.

[8]This is just another way of expressing the statement that the stress tensor in the entrained system is to be considered a contravariant tensor density.

[9]In this equation, the anisotropic component of the disturbance of the suspension medium flow by the introduced particles is neglected.

[10]See also Giesekus [5] and [6].

The tensor $\boldsymbol{S}$ is, in general, an isotrope function of the kinematic tensors. For particles, which consist of cone-shaped resistive bodies, which are rigidly or elastically bound to one another in groups, this quantity can be represented in the form

$$\boldsymbol{S} = \sum_i \frac{n_i}{2B_i} \langle \boldsymbol{r}_i \boldsymbol{r}_i \rangle \tag{18}$$

where n_i is the number of particles of the i-th type[11]in the unit volume, B_i is their mobility in the suspension medium, $\boldsymbol{r}_i$ their position vector (from an arbitrary point at rest in the body), and the brackets express the average over all orientations.

If we now assume that the rest state corresponds to a uniquely determined isotropic tensor

$$\mathring{\boldsymbol{S}} = \sum_i \frac{n_i}{2B_i} \langle \mathring{\boldsymbol{r}}_i \mathring{\boldsymbol{r}}_i \rangle = \sum_i \mathring{\eta}_i \mathbf{1} \tag{19}$$

then we can at least formally assign the vectors $\boldsymbol{r}_i$ to a tensor

$$\boldsymbol{A}_i = (x^j_{,L}\, \boldsymbol{e}_j\, \boldsymbol{e}^L)_i \tag{20}$$

such that

$$\boldsymbol{r}_i = \boldsymbol{A}_i.\mathring{\boldsymbol{r}}_i = \mathring{\boldsymbol{r}}_i.\tilde{\boldsymbol{A}}_i \tag{21}$$

holds[12]. Then, however

$$\langle \boldsymbol{r}_i \boldsymbol{r}_i \rangle = \boldsymbol{A}_i.\langle \mathring{\boldsymbol{r}}_i \mathring{\boldsymbol{r}}_i \rangle.\tilde{\boldsymbol{A}}_i = \langle \mathring{\boldsymbol{r}}_i \mathring{\boldsymbol{r}}_i \rangle.\boldsymbol{A}_i.\tilde{\boldsymbol{A}}_i. \tag{22}$$

The commutation between $\langle \mathring{\boldsymbol{r}}_i \mathring{\boldsymbol{r}}_i \rangle$ and $\boldsymbol{A}_i$ is therefore permissible because $\langle \mathring{\boldsymbol{r}}_i \mathring{\boldsymbol{r}}_i \rangle$ represents an isotropic tensor. Furthermore

$$n_i = \frac{\mathring{n}_i}{|\boldsymbol{A}_i|} \tag{23}$$

and thus we can represent (18) in the form:

$$\boldsymbol{S} = \sum_i \frac{\mathring{n}_i}{2B_i} \langle \mathring{\boldsymbol{r}}_i \mathring{\boldsymbol{r}}_i \rangle . \frac{\boldsymbol{A}_i.\tilde{\boldsymbol{A}}_i}{|\boldsymbol{A}_i|} = \sum_i \mathring{\eta}_i \frac{\boldsymbol{A}_i.\tilde{\boldsymbol{A}}_i}{|\boldsymbol{A}_i|} \tag{24}$$

However, the tensor expression on the right represents, as can easily be verified by substituting in (20)

[11]Note that i does not represent a tensor index and therefore the Einstein convention does not apply here.

[12]

$$.\tilde{\boldsymbol{A}}_i = (x^k_{,M}\, \boldsymbol{e}^M\, \boldsymbol{e}_k)_i \tag{20a}$$

means the tensor adjoined to $\boldsymbol{A}_i$.

and (20a) and comparing with (13), precisely the tensor $\boldsymbol{W}_i$ belonging to the mapping tensor $\boldsymbol{A}_i$. We have thus assigned a continuum to each particle type by (21) such that the change in the orientation and distance statistics of the position vectors $\boldsymbol{r}_i$ can be understood as a deformation of this continuum, and it has now turned out that $\boldsymbol{W}_i$ represents precisely the deformation measure by which the state tensor $\boldsymbol{S}$ can be expressed in the simplest way. Let us perform an averaging using the quantities $\mathring{\eta}_i$, which intuitively represent the initial viscosities corresponding to the undeformed continua:

$$\overline{\boldsymbol{W}} = \frac{1}{\mathring{\eta}} \sum_i \mathring{\eta}_i \boldsymbol{W}_i, \qquad \mathring{\eta} = \sum_i \mathring{\eta}_i \tag{25}$$

then (24) takes the even clearer form

$$\boldsymbol{S} = \sum_i \mathring{\eta}_i \boldsymbol{W}_i = \mathring{\eta} \overline{\boldsymbol{W}} \tag{24a}$$

We can clearly see from this how closely the equation of state (17) corresponds to the Weissenberg-Grossmann basic concept.

eine quasi-1ineare Funktion des Spannungstensors ist

4 Weissenberg liquids, whose state tensor is a quasi-linear function of the stress tensor

The result of the previous section now enables us to apply Weissenberg's principle to the elastoviscous fluid, more precisely: to the Weissenberg fluid, in the sense of (17), by linking the tensor $\boldsymbol{W}_i$ to the associated additional stresses $\boldsymbol{\sigma}$ via a quasi-linear relationship[13]:

$$\boldsymbol{\sigma}_i = \alpha_i \boldsymbol{I} + \beta_i \boldsymbol{W}_i \tag{26}$$

so that α_i and β_i are scalar functions of the invariants of $\boldsymbol{\sigma}_i$ and $\boldsymbol{W}_i$, respectively. Solving this relationship for $\boldsymbol{W}_i$, substituting it into (24a) and then into (17), we obtain the equation of state in a form where only stress and strain rate quantities appear. However, we do not want to carry this out explicitly for the general case here, but rather restrict ourselves to the simplest special case where there is only one particle type and (26) is also strictly linear. Because of the additional condition that $\mathring{\boldsymbol{W}} = \mathbf{1}$ and $\mathring{\boldsymbol{\sigma}} = 0$ must hold, one of the two constants is also used, so that (26) can be written in the form

$$\boldsymbol{\sigma} = \mu(\boldsymbol{W} - \boldsymbol{I}) \tag{26a}$$

Thus

$$\boldsymbol{S} = \mathring{\eta} \boldsymbol{W} = \frac{\mathring{\eta}}{\mu} \boldsymbol{\sigma} + \mathring{\eta} \mathbf{1} \tag{27}$$

Considering that

$$-\frac{\mathfrak{D}\mathbf{1}}{\mathfrak{D}t} = \nabla \boldsymbol{v} + \boldsymbol{v} \nabla = 2\boldsymbol{\epsilon}^{(1)} \tag{28}$$

represents twice the deformation rate, performing the convective differentiation and substituting it into (17) yields the equation of state

$$\boldsymbol{\sigma} + \frac{\mathring{\eta}}{\mu} \frac{\mathfrak{D}\boldsymbol{\sigma}}{\mathfrak{D}t} = 2\mathring{\eta} \boldsymbol{\epsilon}^{(1)} \tag{29}$$

i.e., the equation of a Maxwell-Oldroyd-B fluid with the relaxation time $\tau = \mathring{\eta}/\mu$. With the help of the relation defining the contravariant kinematic tensors

$$-\frac{1}{2} \frac{\mathfrak{D}^n \mathbf{1}}{\mathfrak{D}t^n} = \boldsymbol{\epsilon}^{(n)} \tag{30}$$

relation (29) can be solved for $\boldsymbol{\sigma}$, yielding the series expansion

$$\boldsymbol{\sigma} = 2\mathring{\eta} \sum_{n=1}^{\infty} (-\tau)^{n-1} \boldsymbol{\epsilon}^{(n)} \tag{29a}$$

In the case that i different continua exist, the equation of state can be found simply by summing equations of the type above, i.e., a linear expansion in terms of the kinematic tensors with constant coefficients, where, however, these are no longer linked to each other by a simple relationship as in (29a). If the stress-strain relationship is finally quasi-linear

[13] These are the *same* stresses that, on the one hand, cause the relative motion of the particles in the suspension medium and, in connection with this, the anisotropic distribution of the position vectors, and on the other hand, also appear as a consequence of this anisotropic distribution.

according to (26), then an equation of state of the form

$$\boldsymbol{\sigma} = \Gamma^{(0)}\mathbf{1} + \Gamma^{(1)}\boldsymbol{\epsilon}^{(1)} + \Gamma^{(2)}\boldsymbol{\epsilon}^{(2)} + \ldots \tag{31}$$

follows, in which the $\Gamma^{(n)}$ represent scalar functions of the invariants of the kinematic tensors. This most general equation of state of an elasto-viscous liquid, which can be generated using Weissenberg's principle, is therefore a quasi-linear relationship in the contravariant kinematic tensors.

References

[1] P. U. A. Grossman. Weissenberg's rheological equation of state. *Kolloid-Z.*, 174:97–109, 1961.

[2] K. Weissenberg. Specification of rheological phenomena by means of a rheogoniometer. *Proc. Int. Congr. Rheol.*, 3:36, 1949.

[3] H Giesekus. Der Spannungstensor des visko-elastischen Körpers. *Rheol. Acta*, 1(4):395–404, 1961.

[4] H. Giesekus. Elasto-viskose Flüssigkeiten, für die in stationären Schichtströmungen sämtliche Normalspannungskomponenten verschieden groß sind. *Rheol. Acta*, 2(1):50–62, 1962.

[5] H. Giesekus. Einige Bemerkungen zum Fließverhalten elasto-viskoser Flüssigkeiten in stationären Schichtströmungen. *Rheol. Acta*, 1(4):404–413, 1961.

[6] H. Giesekus. Einige ergänzende Bemerkungen zur Darstellung der rheologischen Zustandsgleichung nach Weissenberg und Grossman. *Z. Angew. Math. Mech.*, 42(6):259–262, 1962.