Internship Report

Continuous mathematical models of cell migration

Date: 03/07/2019

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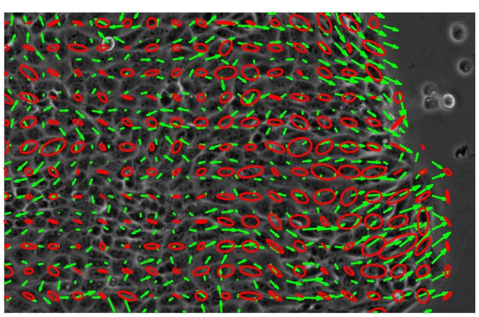
First of all, I would like to thank my supervisors, Prof. Pierre Saramito, Dr. Ibrahim Cheddadi, for their wise and precious advice. Thanks also to the students of my room, for their help and assistance to my work. Eventually, I would like to show my gratitude to the entire LJK team for their support and warm work culture.

Contents

1 Introduction

In this report, we have worked in Collective cell migration. The biological concept of our work in based on Refs. [2], [4]. According to the authors, cells connected by cell–cell adhesion can migrate collectively across several time scales and length scales in several biological processes like tissue formation during embryonic development, wound healing and some types of cancer invasion, all require the coordinated movement of cells in certain directions to specific positions (See Ref. [5]), where each single cell can grow, divide, die and migrate. We have considered cell-cell and cell-substrate interactions which conduct the cell migration. The polarized movements of collective cells depend on mechanical factors and external geometrical constraints (see Refs. [15], [16]). In case of collectively spreading and acquiring a free space, highly motile leader cells can appear ([17]) and locally guide small organized group of cells ([13]), albeit, it is not completely clear how the migration is sensitive to distance from the migrating front and how cells at far from the migrating front can maintain their polarity. It is known that cells move as collectives but the mechanisms remained controversial. There are several proposed theories describing the mechanisms ([30], [31]). Here, we have used the established fact that all cells within the tissue can move by an active force (say $\boldsymbol{f_{as}}$) which drives cells to go forward ([6]) and the active force is inversely proportional to the depth of collective cells ($\boldsymbol{f_{as}} \propto \frac{1}{h}$). By performed experiments on embryonic tissues ([23]), multicellular spheroids ([24], [25]), or cell monolayers with or without substrate ([26], [27]), tissues have been described as a viscoelastic liquid in nature. The flow of viscous fluid arround circular obstacle in two or three dimension was experimented by Stokes ([28]), mechanical behavior of viscoelastic or viscoplastic materials are described in Ref. [29]. The main purpose of the report is deriving a new simple model for a thin layer of viscoelastic fluid when the motion is essentially driven by viscosity and numerically investigate the prediction of that simple model. Mathematically, the problem is formulated as a system of partial differential equations. The set of equations is designed to describe the behavior of cell tissues, moving on a artificial substrate. The behavior of the cell tissues have been described as an active viscoelastic fluid. We need to obtain the reduced model because it is computationally much less costful and easy to solve numerically than the full Oldroyd model (See [22] for numerical simulations of full 3D model) and also the images are easier to analyse than three-dimensional (3D) ones, specially for extracting physical quantities like cell velocity, shape, deformation ([14]) etc. We can also easily investigate its predictions to show the capabilities and potentials of the model. In several works of mathematical literature, we can notice the reduced models of thin layers for viscoelastic flows. For example, take a look at Refs. [7] and [8] where reduced models are derived for the Oldroyd-B (OB) system of equations. Oldroyd models are widely-used differential model for viscoelastic fluids where viscosity plays mojor role for fluid flow. Various models for thin layers of non-Newtonian fluids have already been derived in the physics and applied mathematics literature. Based on the work of "Francois Bouchut and Sebastien Boyaval" (See Ref. [1]) in Upper Convected Maxwell (UCM) model, we have derived a thin layer form of the Oldroyd equations. There are some versions of reduced UCM models which can be cosidered close to our work, see Refs. [9] and [10] for instance. One can also find sketch of these works in Ref. [11]. But in these cases, the reduced models are obtained with another methodologies and with different perspectives and boundary conditions rather than applying asymptotic analysis for general fluid equations. There are also other existing thin layer models with different methodologies see for example, Refs. [12], [18], [19], [20], [21]. In our thin layer problem, the length (longitudinal characteristic length) L is extensively larger than the height (layer depth) h. So, the aspect ratio of height and lengh is intensively small ($\frac{h}{L} \approx \varepsilon \ll 1$). Here, the influence of each term is compared with ε. As the cells migrate over a flat surfacee and the only force is the active pulling force, So, there would be no gravity-driven free-surface thin-layer flows. The numerical resolution for viscous fluid is done in Rheolef C++ Library (See Ref. [3]).

In **Section 2**, we have introduced the 3-dimensional Oldroyd model for viscoelastic fluids with the equations for conservation of mass and conservation of momentum. In **Section 3**, we developed some of its properties in the mathematical setting that is adequate to take step towards creating a thin layer problem from the initially defined Oldroyd model. Firstly, we have expanded the primary 3D model to make our problem simple. We take the axes wise components and introducing new notations, we transformed the model in 2D form. We, then make the physical terms dimensionless. Next, the dimensionless model is expanded asymptotically. An asymptotic expansion is a special kind of an asymptotic expression, in which a function is approximated by partial sums of some series (may be convergent or divergent). According to the need of our problem, we take partial sums till second order of ε and ignore higher order terms because of very small ε. After that, we have used some clear mathematical hypothesis to reduce the system of equations and finally derived the closed system of equations. We also have discussed the general assumptions we made to reach the desirable closed system of equations. **Section 4** is devoted for doing the Variational formulation of our new reduced model and generating a discontinuous finite element space, where our continuous system of equations can be approximated discretely. In this section, we also provide the numerical solutions and explain the results. Finally, **Section 5** is engaged to the conclusion, where physical interpretation of situations, modeled by our system of equations and some idea about further works are discussed.



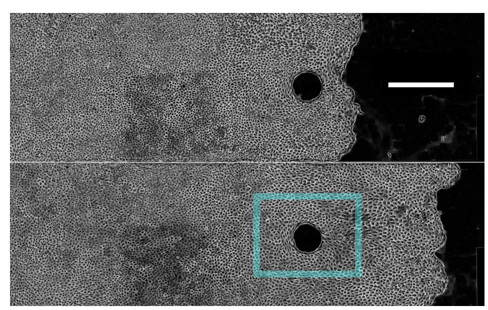


Figure 1: Figure in left shows Cell migration in 2-Dimension, without obstacle, See Ref. [5] and Figure in right shows Cell migration around an ostacle in 2- Dimension, See Ref. [2]

2 Problem formulation

Here, we have considered the Oldroyd model together with the equations for conservation of mass and conservation of momentum. It forms a very commonly used mathematical model to describe viscoelastic fluids. The viscoelastic fluid is incompressible in nature. In this model, we consider the x, y, z-components. So, $\boldsymbol{u}$, $\boldsymbol{\tau}$, $\boldsymbol{\sigma}$, are functions of t, x, y, z where $\boldsymbol{u}$ is the cell's velocity, $\boldsymbol{\tau}$ is the elastic stress tensor and $\boldsymbol{\sigma}$ is the total stress tensor. Depth (h) is the function of t, x, y.

2.1 Notations

Here we denote the velocity $\boldsymbol{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$ and the 'nabla' operator $\boldsymbol{\nabla} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$.

The elastic stress tensor $\boldsymbol{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \tau_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{pmatrix}$.

The gradient of velocity tensor can be written as:

$\boldsymbol{\nabla}\, \boldsymbol{u} = \begin{pmatrix} \partial_x u_x & \partial_y u_x & \partial_z u_x \\ \partial_x u_y & \partial_y u_y & \partial_z u_z \\ \partial_x u_z & \partial_y u_z & \partial_z u_z \end{pmatrix}$.

The rate of deformation tensor is denoted as:

$D(\boldsymbol{u}) = \frac{\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T}{2}$.

The divergence of a vector (velocity vector) is:

$\text{div}\, \boldsymbol{u} = \partial_x u_x + \partial_y u_y + \partial_z u_z$.

The divergence of a tensor (total stress) is:

$\textbf{div}\, \boldsymbol{\sigma} = \begin{pmatrix} \partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz} \\ \partial_x \sigma_{xy} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz} \\ \partial_x \sigma_{xz} + \partial_y \sigma_{yz} + \partial_z \sigma_{zz} \end{pmatrix}$.

2.2 Problem statement

The constitutive equations for viscoelastic materials are:

$\boldsymbol{\sigma} = -p\, I + 2\eta_0 D(\boldsymbol{u}) + \boldsymbol{\tau}$.

$\lambda \overset{\nabla}{\boldsymbol{\tau}} + \boldsymbol{\tau} - 2\, \eta_p\, D(\boldsymbol{u}) = 0$.

Where $\overset{\nabla}{\boldsymbol{\tau}}$ denotes the upper-convected derrivative of the tensor $\boldsymbol{\tau}$, such that:

$\overset{\nabla}{\boldsymbol{\tau}} = \partial_t\, \boldsymbol{\tau} + (\boldsymbol{u}.\boldsymbol{\nabla})\boldsymbol{\tau} - \boldsymbol{\nabla u}\, \boldsymbol{\tau} - \boldsymbol{\tau}\, \boldsymbol{\nabla u^T}$.

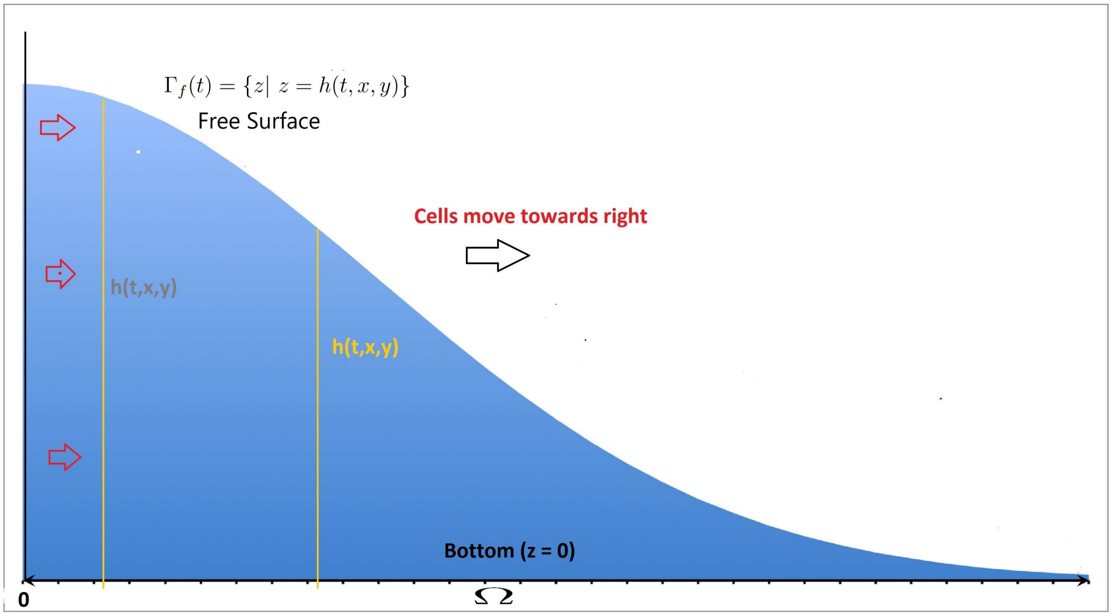
These constitutive equations are completed by the equations of mass conservation and conservation of momentum.

$\text{div}(\boldsymbol{u}) = 0$.

$\rho\Big(\partial_t \boldsymbol{u} + (\boldsymbol{u}.\boldsymbol{\nabla})\boldsymbol{u}\Big) - \textbf{div}\, \boldsymbol{\sigma} = 0$.

Here, the density ρ is considered constant (because of the imcompressible nature) and so, the mass conservation equation reduces to $\text{div}(\boldsymbol{u}) = 0$.

We have considered a free surface flow problem, where the flow domain is defined by:
$\Lambda(t) = \{(x, y, z) \in \mathbb{R}^3 |\ (x, y) \in \Omega,\ 0 \le z \le h(t, x, y)\}$.
where $h(t, x, y)$ is the height and Ω is the bottom surface of the flow.
The boundary $\partial\Lambda(t)$ splits in two components $\Gamma_f(t)$ and Ω.



$\Gamma_f(t) = \{z |\ (x,\ y) \in \Omega\ \&\ z = h(t, x, y)\}$ and Ω is in the bottom $z = 0$.

The kinematic condition at free surface can be described by $\frac{dF}{dt} = 0$ at $z = h(t, x, y)$, where $F(t, x, y, z) = h(t, x, y) - z$. Hence, The kinematic equation of the free surface is:
$\partial_t h + u_x\, \partial_x h + u_y\, \partial_y h - u_z = 0$ on $\Gamma_f(t)$ or at $z = h$.
The boundary conditions are:
$\boldsymbol{\sigma}.\boldsymbol{n} = 0$ on $\Gamma_f(t)$.
$\boldsymbol{\sigma_{nt}} + c_f\, u_t = \boldsymbol{f_a}$ and $\boldsymbol{u}.\boldsymbol{n} = 0$ at Ω.
where $c_f > 0$ is the friction coefficient and $\boldsymbol{n}$ is the outward unit normal vector. The outward unit normal vector on Γ_f is $\boldsymbol{n} = \begin{pmatrix} -\partial_x h \\ -\partial_y h \\ 1 \end{pmatrix}$ and the outward unit normal vector on Ω is $\boldsymbol{n} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$.

Here, $\boldsymbol{f_a}$ denotes the given active force, $\boldsymbol{u_t}$ denotes the tangential component of the velocity vector $\boldsymbol{u}$ and $\boldsymbol{\sigma_{nt}}$ denotes the tangential component of the vector $\boldsymbol{\sigma}\, \boldsymbol{n}$, represented as:
$\boldsymbol{u_t} = \boldsymbol{u} - (\boldsymbol{u}.\, \boldsymbol{n})\boldsymbol{n}$.
$\boldsymbol{\sigma_{nt}} = \boldsymbol{\sigma}\, \boldsymbol{n} - \big((\boldsymbol{\sigma}\, \boldsymbol{n}).\boldsymbol{n}\big)\boldsymbol{n}$.

Let, $T > 0$ be the final time. Now, the final problem can be written as:

Find $\boldsymbol{\tau}$, $\boldsymbol{u}$, $\boldsymbol{h}$ such that:

$$\operatorname{div}(\boldsymbol{u}) = 0 \quad in \ \ ]0,T[\times\Lambda(t) \tag{1a}$$

$$\boldsymbol{\sigma} = -p\,I + 2\eta_0 D(\boldsymbol{u}) + \boldsymbol{\tau} \quad in \ \ ]0,T[\times\Lambda(t) \tag{1b}$$

$$\rho\Big(\partial_t\boldsymbol{u} + (\boldsymbol{u}.\boldsymbol{\nabla})\boldsymbol{u}\Big) - \mathbf{div}\,\boldsymbol{\sigma} = 0 \quad in \ \ ]0,T[\times\Lambda(t) \tag{1c}$$

$$\lambda\left(\partial_t\,\boldsymbol{\tau} + (\boldsymbol{u}.\boldsymbol{\nabla})\boldsymbol{\tau} - \boldsymbol{\nabla}\boldsymbol{u}\,\boldsymbol{\tau} - \boldsymbol{\tau}\,\boldsymbol{\nabla}\boldsymbol{u}^{\boldsymbol{T}}\right) + \boldsymbol{\tau} - 2\,\eta_p\,D(\boldsymbol{u}) = 0 \quad in \ \ ]0,T[\times\Lambda(t) \tag{1d}$$

$$\partial_t h + u_x\,\partial_x h + u_y\,\partial_y h - u_z = 0 \quad in \ \ ]0,T[\times\Gamma_f(t) \tag{1e}$$

$$\boldsymbol{\sigma}.\boldsymbol{n} = 0 \quad in \ \ ]0,T[\times\Gamma_f(t) \tag{1f}$$

$$\boldsymbol{u}.\boldsymbol{n} = 0 \quad in \ \ ]0,T[\times\Omega \tag{1g}$$

$$\sigma_{nt} + c_f\,u_t = \boldsymbol{f_a} \quad in \ \ ]0,T[\times\Omega \tag{1h}$$

Where, the entire problem is closed by the initial conditions for h, $\boldsymbol{u}$, $\boldsymbol{\tau}$:
$h(t=0) = h_0$, in Ω
$\boldsymbol{u}(t=0) = \boldsymbol{u_0}$ in $\Lambda(0)$
$\boldsymbol{\tau}(t=0) = \boldsymbol{\tau_0}$ in $\Lambda(0)$.

3 Derivation of a thin layer approximation

3.1 Expansion

We need to expand the primary equations and create more simplified forms for our computational ease. Initial equations are three dimensional. It is tough to handle them and without reducing the dimension it would be computationally costful to look for numerical solutions and also it would not be straightforwrd to include active forces. So, here our first step is expanding the initial equations and consider their components axes wise. By this way, we have generated simplified versions of the equations and then by introducing the planar and vertical components, we write the primary three dimensional equations in two dimensional forms. The detailed expansion is shown in Appendix (6.1).

3.2 Splitting planar and vertical components

We introduce the new notations:

- $\boldsymbol{u_s} = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$. Hence, $\boldsymbol{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} \boldsymbol{u_s} \\ u_z \end{pmatrix}$.
- $\boldsymbol{\nabla_s} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}$. Hence, $\boldsymbol{\nabla} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = \begin{pmatrix} \boldsymbol{\nabla_s} \\ \partial_z \end{pmatrix}$.
- $\operatorname{div}_s \boldsymbol{u_s} = \boldsymbol{\nabla_s}.\boldsymbol{u_s}$.

- The planar elastic stress tensor $\boldsymbol{\tau_s} = \begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{xy} & \tau_{yy} \end{pmatrix}$, $\boldsymbol{\tau_{sz}} = \begin{pmatrix} \tau_{xz} \\ \tau_{yz} \end{pmatrix}$.

  Hence, $\boldsymbol{\tau} = \begin{pmatrix} \boldsymbol{\tau_s} & \boldsymbol{\tau_{sz}} \\ \boldsymbol{\tau_{sz}}^T & \tau_{zz} \end{pmatrix}$.

- $\mathbf{div_s}\, \boldsymbol{\tau_s} = \boldsymbol{\nabla_s}.\boldsymbol{\tau_s}$.

- Similarly, total stress tensor $\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\sigma_s} & \boldsymbol{\sigma_{sz}} \\ \boldsymbol{\sigma_{sz}}^T & \sigma_{zz} \end{pmatrix}$.

- $\mathbf{div_s}\, \boldsymbol{\sigma_s} = \boldsymbol{\nabla_s}.\boldsymbol{\sigma_s}$.

We have introduced the new notations and accordingly change the one dimensional equations in two dimensional forms, by using the new notations. The detailed transformation is shown in the Appendix 6.2. Eventually, changing the forms of primary three dimensional equations, we get:

Equation for conservation of mass:

$$\mathrm{div_s}\, \boldsymbol{u_s} + \partial_z u_z = 0 \tag{2a}$$

Equations for conservation of momentum:

$$\rho\Big(\partial_t u_s + (\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z)\, \boldsymbol{u_s}\Big) - \mathbf{div_s}\, \boldsymbol{\sigma_s} - \partial_z \boldsymbol{\sigma_{sz}} = 0 \tag{2b}$$

$$\rho\Big(\partial_t u_z + (\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\, \partial_z) u_z\Big) - \mathrm{div_s}\, \boldsymbol{\sigma_{sz}}^T - \partial_z\, \sigma_{zz} = 0 \tag{2c}$$

Constitutive equations:

$$\boldsymbol{\sigma_s} = -pI + \boldsymbol{\tau_s} + 2\eta_0\, D_s(\boldsymbol{u_s}) \tag{2d}$$

$$\boldsymbol{\sigma_{sz}} = \boldsymbol{\tau_{sz}} + \eta_0\, (\partial_z \boldsymbol{u_s} + \boldsymbol{\nabla_s} u_z) \tag{2e}$$

$$\sigma_{zz} = -p + \tau_{zz} + 2\eta_0\, \partial_z u_z \tag{2f}$$

$$\lambda\Big[\partial_t \boldsymbol{\tau_s} + \big(\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z\big)\boldsymbol{\tau_s} - (\boldsymbol{\nabla_s}\boldsymbol{u_s})\boldsymbol{\tau_s} - \boldsymbol{\tau_s}(\boldsymbol{\nabla_s}\boldsymbol{u_s})^T - (\partial_z \boldsymbol{u_s})\boldsymbol{\tau_{sz}}^T - \boldsymbol{\tau_{sz}}(\partial_z \boldsymbol{u_s})^T\Big] + \boldsymbol{\tau_s} = 2\eta_p\, D_s(\boldsymbol{u_s}) \tag{2g}$$

$$\lambda\Big[\partial_t \boldsymbol{\tau_{sz}} + \big(\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z\big)\boldsymbol{\tau_{sz}} - (\boldsymbol{\nabla_s}\boldsymbol{u_s} + \partial_z u_z)\boldsymbol{\tau_{sz}} - (\partial_z \boldsymbol{u_s})\tau_{zz} - \boldsymbol{\tau_s}\boldsymbol{\nabla_s} u_z\Big] + \boldsymbol{\tau_{sz}} = \eta_p\, \big(\partial_z \boldsymbol{u_s} + \boldsymbol{\nabla_s} u_z\big) \tag{2h}$$

$$\lambda\Big[\partial_t \tau_{zz} + \big(\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z\big)\tau_{zz} - 2(\partial_z u_z)\tau_{zz} - (\boldsymbol{\nabla_s} u_z)^T.\boldsymbol{\tau_{sz}} - \boldsymbol{\tau_{sz}}^T.(\boldsymbol{\nabla_s} u_z)\Big] + \tau_{zz} = 2\eta_p\, \partial_z u_z \tag{2i}$$

Kinematic equation:

$$\partial_t h + (\boldsymbol{\nabla_s} h).\boldsymbol{u_s} - u_z = 0 \tag{2j}$$

Equations for boundary condition at free surface:

$$-\, \boldsymbol{\sigma_s}\, (\boldsymbol{\nabla_s} h) + \boldsymbol{\sigma_{sz}} = 0 \tag{2k}$$

$$-\, \boldsymbol{\sigma_{sz}}^T\, (\boldsymbol{\nabla_s} h) + \sigma_{zz} = 0 \tag{2l}$$

Equations for boundary conditions at bottom:

$$u_z = 0 \tag{2m}$$

$$\boldsymbol{\sigma_{sz}} - c_f \boldsymbol{u_s} = -\boldsymbol{f_{as}} \tag{2n}$$

Where, the entire problem is closed by the initial conditions for h, $\boldsymbol{u_s}$, $\boldsymbol{\tau_s}$, τ_{zz}:
$h(t=0) = h_0$, in Ω
$\boldsymbol{u_s}(t=0) = \boldsymbol{u_{s\,0}}$ in $\Lambda(0)$
$\boldsymbol{\tau_s}(t=0) = \boldsymbol{\tau_{s\,0}}$ in $\Lambda(0)$.
$\tau_{zz}(t=0) = \tau_{zz\,0}$ in $\Lambda(0)$

3.3 Dimension analysis

Dimensionless numbers are necessary to reduce the number of parameters that describe a system. In this way, we can reduce the amount of experimental data required to make correlations of physical phenomena to scalable systems. The most common dimensionless number in fluid dynamics are the Reynolds number (Re) and the Weissenberg number (We). The Reynolds number is a dimensionless number used to predict the flow patterns in different fluid flow situations. It is used in several situations where the fluid is in a relative motion to surface. Low reynolds number means that the fluid flow tend to be dominated by laminar flow which is our case of thin layer problem whereas, high Reynolds number creates turbulence in fluid's velocity and direction of flow. The Weissenberg number is also a dimensionless number used in the study of viscoelastic flows. It compares the elastic forces to the viscous forces. More specifically, the Weissenberg number is the ratio of elastic forces and viscous forces. We have denoted the characteristic dimensions of length by H, L, the characteristic dimensions of velocity by U, V, the characteristic dimensions of tensor or pressure by Σ.

3.3.1 Choices of characteristic quantities

As we have a thin layer problem, we can say that the ratio between height and length is very small.
So, $\varepsilon = \frac{H}{L} \Rightarrow H = \varepsilon L$.
$x = L\tilde{x}$, $y = L\tilde{y}$, $z = H\tilde{z}$, $h = H\,\tilde{h}$.
$\boldsymbol{\nabla_s} = \frac{1}{L}\boldsymbol{\tilde{\nabla}_s}$, $\mathrm{div}_s = \frac{1}{L}\tilde{\mathrm{div}}_s$, $\partial_z = \frac{1}{H}\partial_{\tilde{z}}$.
$\boldsymbol{u_s} = U\boldsymbol{\tilde{u}_s}$, $u_z = V\tilde{u}_z$ and $V = \varepsilon U$.
$\boldsymbol{\tau} = \Sigma\boldsymbol{\tilde{\tau}}$, $\boldsymbol{\sigma} = \Sigma\boldsymbol{\tilde{\sigma}}$, $p = \Sigma p$, $\boldsymbol{f_a} = \Sigma\boldsymbol{f_a}$.
Where $\Sigma = \frac{(\eta_p+\eta_0)\,U}{L} = \frac{\eta U}{L}$, $[\eta = \eta_p + \eta_0]$.
The characteristic time is: $t = \frac{L}{U}\tilde{t}$.
$u_z \ll \boldsymbol{u_s}$.
All the terms of $\boldsymbol{\tilde{X}}$ forms are dimensionless. In further sections, we know the terms are dimensionless but we ignore "tilde" for reducing writing complexity. Our objective is to change the differential equations in a manner such that all the equations become dimensionless. So, we will replace the physical terms by newly assumed terms. The idea is simple, any physically meaningful equation must have same dimensions on the left and right sides.
Reynolds number: $Re = \frac{\rho U L}{\eta}$.
Weissenberg number $We = \frac{\lambda U}{L}$.

3.3.2 Forming dimensionless equations

We are generationg the dimentionless terms from previous equations of the problem.

3.3.2.1 Conservation of mass

For the equation of mass conservation:
$\frac{1}{L}\mathrm{div_s}(U\boldsymbol{u_s}) + \frac{\partial(Vu_z)}{\partial(Hz)} = 0. \Rightarrow \frac{U}{L}\mathrm{div_s}\,\boldsymbol{u_s} + \frac{\partial(\varepsilon U u_z)}{\partial(\varepsilon L z)} = 0.$
$\Rightarrow \frac{U}{L}\mathrm{div_s}\,\boldsymbol{u_s} + \frac{\partial(Uu_z)}{\partial(Lz)} = 0. \Rightarrow \frac{U}{L}\Big(\mathrm{div_s}\,\boldsymbol{u_s} + \frac{\partial(u_z)}{\partial(z)}\Big) = 0.$
$\Rightarrow \mathrm{div_s}\,\boldsymbol{u_s} + \partial_z u_z = 0.$

3.3.2.2 Conservation of momentum

We consider the projections on the planar and vertical axis, the equations for conservation of momentum are:

1. Planer component:

$$\rho\Big(\frac{\partial(U\boldsymbol{u_s})}{\partial(\frac{L}{U}t)} + (U\,\boldsymbol{u_s}.\frac{1}{L}\boldsymbol{\nabla_s})\,U\boldsymbol{u_s} + Vu_z\frac{\partial(U\boldsymbol{u_s})}{\partial(Hz)}\Big) - \frac{\Sigma\mathbf{div_s}\,\boldsymbol{\sigma_s}}{L} + \frac{\Sigma\boldsymbol{\sigma_{sz}}}{\partial(Hz)} = 0.$$

$$\Rightarrow \rho\frac{U^2}{L}\Big(\frac{\partial\boldsymbol{u_s}}{\partial t} + (\boldsymbol{u_s}.\boldsymbol{\nabla_s})\,\boldsymbol{u_s} + u_z\frac{\partial(\boldsymbol{u_s})}{\partial(z)}\Big) - \frac{\Sigma}{L}\big(\mathbf{div_s}\,\boldsymbol{\sigma_s}\big) - \frac{\Sigma}{\varepsilon L}\Big(\frac{\partial\boldsymbol{\sigma_{sz}}}{\partial z}\Big) = 0.$$

Multiplying both sides by $\frac{\varepsilon L}{\Sigma}$, we get:

$$\rho\frac{U^2\varepsilon}{\Sigma}\Big(\frac{\partial\boldsymbol{u_s}}{\partial t} + (\boldsymbol{u_s}.\boldsymbol{\nabla_s})\,\boldsymbol{u_s} + u_z\frac{\partial(\boldsymbol{u_s})}{\partial z}\Big) - \varepsilon\big(\mathbf{div_s}\,\boldsymbol{\sigma_s}\big) - \frac{\partial\boldsymbol{\sigma_{sz}}}{\partial z} = 0.$$

Now $\rho\frac{U^2}{\Sigma} = \rho\frac{U^2}{\frac{\eta U}{L}} = \frac{\rho U L}{\eta} = Re$.
Hence,

$$\varepsilon\,Re\Big[\partial_t\boldsymbol{u_s} + (\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z)\boldsymbol{u_s}\Big] - \varepsilon\mathbf{div_s}\,\boldsymbol{\sigma_s} - \partial_z\boldsymbol{\sigma_{sz}} = 0.$$

2. Vertical component:

$$\rho\Big(\frac{\partial\varepsilon U u_z}{\partial(\frac{L}{U}t)} + (U\boldsymbol{u_s}.\frac{1}{L}\boldsymbol{\nabla_s})\varepsilon U u_z + \varepsilon U u_z\frac{\partial(\varepsilon U u_z)}{\partial(\varepsilon L z)}\Big) - \frac{\Sigma}{L}\mathrm{div_s}\,\boldsymbol{\sigma_{sz}}^T - \frac{\Sigma\partial\sigma_{zz}}{\partial(\varepsilon L z)} = 0.$$

$$\Rightarrow \rho\frac{U^2\varepsilon}{L}\Big(\partial_t u_z + (\boldsymbol{u_s}.\boldsymbol{\nabla_s})u_z + u_z\partial_z\,u_z\Big) - \frac{\Sigma}{L}\mathrm{div_s}\,\boldsymbol{\sigma_{sz}}^T - \frac{\Sigma}{\varepsilon L}\partial_z\sigma_{zz} = 0.$$

Multiplying both sides by $\frac{L}{\Sigma}$, we get:

$$\rho\frac{U^2\varepsilon}{\Sigma}\Big(\partial_t u_z + (\boldsymbol{u_s}.\boldsymbol{\nabla_s})u_z + u_z\partial_z\,u_z\Big) - \mathrm{div_s}\,\boldsymbol{\sigma_{sz}}^T - \frac{1}{\varepsilon}\partial_z\sigma_{zz} = 0.$$

As we know that $\frac{U^2\rho}{\Sigma} = Re$.

$$\Rightarrow \varepsilon^2 Re\Big[\partial_t u_z + (\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z)\,u_z\Big] - \varepsilon\mathrm{div_s}\,\boldsymbol{\sigma_{sz}}^T - \partial_z\sigma_{zz} = 0.$$

3.3.2.3 Constitutive equations:

Considering the projection for total stress tensor $\boldsymbol{\sigma}$, we get:

1. Planar component:
   $\Sigma\boldsymbol{\sigma_s} = -\Sigma pI + \Sigma\boldsymbol{\tau_s} + 2\eta_0\,\frac{U}{L}D_s(\boldsymbol{u_s})$.
   $\Rightarrow \boldsymbol{\sigma_s} = -pI + \boldsymbol{\tau_s} + 2\frac{\eta_0}{\eta}\,D_s(\boldsymbol{u_s})$.
   (Because $\Sigma = \frac{\eta U}{L} \Rightarrow \frac{U}{L} = \frac{\Sigma}{\eta}$).
   So, $\boldsymbol{\sigma_s} = -pI + \boldsymbol{\tau_s} + 2(1-\beta)\,D_s(\boldsymbol{u_s})$.
   (As $\beta = \frac{\eta_p}{\eta_p+\eta_0}$, So, $1-\beta = \frac{\eta_0}{\eta_p+\eta_0} = \frac{\eta_0}{\eta}$.)

2. Shear component:
   $\Sigma\boldsymbol{\sigma_{sz}} = \Sigma\boldsymbol{\tau_{sz}} + \eta_0(\frac{U}{\varepsilon L}\,\partial_z\boldsymbol{u_s} + \frac{\varepsilon U}{L}\boldsymbol{\nabla_s}u_z)$.
   $\Rightarrow \boldsymbol{\sigma_{sz}} = \boldsymbol{\tau_{sz}} + (1-\beta)(\frac{1}{\varepsilon}\,\partial_z\boldsymbol{u_s} + \varepsilon\boldsymbol{\nabla_s}u_z)$.
   $\Rightarrow \varepsilon\,\boldsymbol{\sigma_{sz}} = \varepsilon\,\boldsymbol{\tau_{sz}} + (1-\beta)(\partial_z\boldsymbol{u_s} + \varepsilon^2\boldsymbol{\nabla_s}u_z)$.

3. Vertical component:
   $\Sigma\sigma_{zz} = -\Sigma p + \Sigma\tau_{zz} + 2\eta_0\frac{U}{L}\,\partial_z u_z$
   $\Rightarrow \sigma_{zz} = -p + \tau_{zz} + 2(1-\beta)\,\partial_z u_z$.

Considering the projections for elastic tensor $\boldsymbol{\tau}$, we get:

1. Planar component:

$$\lambda\Big[\frac{\partial\Sigma\boldsymbol{\tau_s}}{\partial(\frac{L}{U}t)}+(U\boldsymbol{u_s}.\frac{\boldsymbol{\nabla_s}}{L}\,\Sigma\boldsymbol{\tau_s}+\varepsilon Uu_z\frac{\partial\Sigma\boldsymbol{\tau_s}}{\partial(\epsilon Lz)}-\frac{\boldsymbol{\nabla_s}U\boldsymbol{u_s}}{L}\,\Sigma\boldsymbol{\tau_s}-\Sigma\boldsymbol{\tau_s}(\frac{\boldsymbol{\nabla_s}\,U\boldsymbol{u_s}}{L})^T-\frac{\partial_z U\boldsymbol{u_s}}{\varepsilon L}\,\Sigma\boldsymbol{\tau_{sz}}^T$$
$$-\Sigma\boldsymbol{\tau_{sz}}\,\frac{(\partial_z U\boldsymbol{u_s})^T}{\varepsilon L}\Big] + \Sigma\boldsymbol{\tau_s} = \frac{2\eta_p\,U}{L}\,D_s(\boldsymbol{u_s}).$$

$$\Rightarrow \lambda\frac{U\Sigma}{L}\Big[\partial_t\boldsymbol{\tau_s}+\boldsymbol{u_s}.\boldsymbol{\nabla_s}\,\boldsymbol{\tau_s}+u_z\partial_z\boldsymbol{\tau_s}-\boldsymbol{\nabla_s}\boldsymbol{u_s}\,\boldsymbol{\tau_s}-\boldsymbol{\tau_s}\,\boldsymbol{\nabla_s}\boldsymbol{u_s}^T\Big]-\lambda\frac{U\Sigma}{\varepsilon L}\Big[\partial_z\boldsymbol{u_s}\,\boldsymbol{\tau_{sz}}^T+\boldsymbol{\tau_{sz}}\,(\partial_z\boldsymbol{u_s})^T\Big]$$
$$+\Sigma\boldsymbol{\tau_s} = \frac{2\eta_p\,U}{L}\,D_s(\boldsymbol{u_s}).$$

As $\Sigma = \frac{\eta U}{L}$, So, $\frac{\eta_p U}{L} = \frac{\Sigma\eta_p}{\eta}$ and Σ is constant term so we can cancel it.

$$\Rightarrow \lambda\frac{U}{L}\Big[\partial_t\boldsymbol{\tau_s}+\boldsymbol{u_s}.\boldsymbol{\nabla_s}\,\boldsymbol{\tau_s}+u_z\partial_z\boldsymbol{\tau_s}-\boldsymbol{\nabla_s}\boldsymbol{u_s}\,\boldsymbol{\tau_s}-\boldsymbol{\tau_s}\,\boldsymbol{\nabla_s}\boldsymbol{u_s}^T\Big]-\lambda\frac{U}{\varepsilon L}\Big[\partial_z\boldsymbol{u_s}\,\boldsymbol{\tau_{sz}}^T+\boldsymbol{\tau_{sz}}\,(\partial_z\boldsymbol{u_s})^T\Big]$$
$$+\boldsymbol{\tau_s} = \frac{2\eta_p}{\eta}\,D_s(\boldsymbol{u_s}).$$

We know that the Weissenberg number $We = \frac{\lambda U}{L}$ and $\frac{\eta_p}{\eta} = \beta$. Hence:

$$\varepsilon\,We\,\Big[\partial_t\boldsymbol{\tau_s}+(\boldsymbol{u_s}.\boldsymbol{\nabla_s}+u_z\partial_z)\boldsymbol{\tau_s}-(\boldsymbol{\nabla_s}\boldsymbol{u_s})\,\boldsymbol{\tau_s}-\boldsymbol{\tau_s}\,(\boldsymbol{\nabla_s}\boldsymbol{u_s}^T)\Big]-We\,\Big[(\partial_z\boldsymbol{u_s})\,\boldsymbol{\tau_{sz}}^T+\boldsymbol{\tau_{sz}}\,(\partial_z\boldsymbol{u_s})^T\Big]$$
$$+\varepsilon\boldsymbol{\tau_s} - 2\varepsilon\beta\,D_s(\boldsymbol{u_s}) = 0.$$

2. Shear component:

$$\lambda\Big[\frac{U}{L}\partial_t\Sigma\boldsymbol{\tau_{sz}}+\big(\frac{U}{L}\,\boldsymbol{u_s}.\boldsymbol{\nabla_s}+\frac{\varepsilon U}{\varepsilon L}\,u_z\partial_z\big)\Sigma\boldsymbol{\tau_{sz}}-(\frac{U}{L}\,\boldsymbol{\nabla_s}\boldsymbol{u_s}+\frac{U}{L}\partial_z u_z)\Sigma\boldsymbol{\tau_{sz}}-\frac{U\Sigma}{\varepsilon L}(\partial_z\boldsymbol{u_s})\tau_{zz}-\frac{\Sigma\varepsilon U}{L}\,\boldsymbol{\tau_s}\boldsymbol{\nabla_s}u_z\Big]$$
$$+\Sigma\boldsymbol{\tau_{sz}}=\eta_p\,\big(\frac{U}{\varepsilon L}\,\partial_z\boldsymbol{u_s}+\frac{\varepsilon U}{L}\,\boldsymbol{\nabla_s}u_z\big).$$

$$\frac{\lambda U\Sigma}{L}\Big[\partial_t\boldsymbol{\tau_{sz}}+\big(\boldsymbol{u_s}.\boldsymbol{\nabla_s}+u_z\partial_z\big)\boldsymbol{\tau_{sz}}-(\boldsymbol{\nabla_s}\boldsymbol{u_s}+\partial_z u_z)\boldsymbol{\tau_{sz}}-\frac{1}{\varepsilon}(\partial_z\boldsymbol{u_s})\tau_{zz}-\varepsilon\,\boldsymbol{\tau_s}\boldsymbol{\nabla_s}u_z\Big]+\Sigma\boldsymbol{\tau_{sz}}$$
$$=\frac{\eta_p\,U}{L}\big(\frac{1}{\varepsilon}\,\partial_z\boldsymbol{u_s}+\varepsilon\,\boldsymbol{\nabla_s}u_z\big).$$

Doing similar operation like before, we get:

$$\frac{\lambda U\Sigma}{L}\Big[\partial_t\boldsymbol{\tau_{sz}}+\big(\boldsymbol{u_s}.\boldsymbol{\nabla_s}+u_z\partial_z\big)\boldsymbol{\tau_{sz}}-(\boldsymbol{\nabla_s}\boldsymbol{u_s}+\partial_z u_z)\boldsymbol{\tau_{sz}}-\frac{1}{\varepsilon}(\partial_z\boldsymbol{u_s})\tau_{zz}-\varepsilon\,\boldsymbol{\tau_s}\boldsymbol{\nabla_s}u_z\Big]+\Sigma\boldsymbol{\tau_{sz}}$$
$$=\frac{\Sigma\eta_p}{\eta}\big(\frac{1}{\varepsilon}\,\partial_z\boldsymbol{u_s}+\varepsilon\,\boldsymbol{\nabla_s}u_z\big).$$

$$\Rightarrow We\Big[\partial_t\boldsymbol{\tau_{sz}}+\big(\boldsymbol{u_s}.\boldsymbol{\nabla_s}+u_z\partial_z\big)\boldsymbol{\tau_{sz}}-(\boldsymbol{\nabla_s}\boldsymbol{u_s}+\partial_z u_z)\boldsymbol{\tau_{sz}}-\frac{1}{\varepsilon}(\partial_z\boldsymbol{u_s})\tau_{zz}-\varepsilon\,\boldsymbol{\tau_s}\boldsymbol{\nabla_s}u_z\Big]+\boldsymbol{\tau_{sz}}$$
$$=\beta\big(\frac{1}{\varepsilon}\,\partial_z\boldsymbol{u_s}+\varepsilon\,\boldsymbol{\nabla_s}u_z\big).$$

$$\Rightarrow \varepsilon We\Big[\partial_t\boldsymbol{\tau_{sz}}+\big(\boldsymbol{u_s}.\boldsymbol{\nabla_s}+u_z\partial_z\big)\boldsymbol{\tau_{sz}}-(\boldsymbol{\nabla_s}\boldsymbol{u_s}+\partial_z u_z)\boldsymbol{\tau_{sz}}\Big]-We\,(\partial_z\boldsymbol{u_s})\tau_{zz}-\varepsilon^2 We\,\boldsymbol{\tau_s}\boldsymbol{\nabla_s}u_z+\varepsilon\,\boldsymbol{\tau_{sz}}$$
$$-\beta\big(\partial_z\boldsymbol{u_s}+\varepsilon^2\,\boldsymbol{\nabla_s}u_z\big)=0.$$

3. Vertical component:

$$\lambda\Big[\frac{\Sigma U}{L}\partial_t\tau_{zz}+\big(\frac{U}{L}\boldsymbol{u_s}.\boldsymbol{\nabla_s}+\frac{\varepsilon U}{\varepsilon L}u_z\partial_z\big)\Sigma\tau_{zz}-2\frac{\varepsilon U}{\varepsilon L}(\partial_z u_z)\Sigma\tau_{zz}-\frac{\varepsilon U\Sigma}{L}(\boldsymbol{\nabla_s}u_z)^T.\boldsymbol{\tau_{sz}}-\frac{\Sigma\varepsilon U}{L}\boldsymbol{\tau_{sz}}^T.(\boldsymbol{\nabla_s}u_z)\Big]$$
$$+\Sigma\tau_{zz}=2\frac{\eta_p\,\varepsilon U}{\varepsilon L}\,\partial_z u_z.$$

$$\Rightarrow \lambda\frac{\Sigma U}{L}\Big[\partial_t\tau_{zz}+\big(\boldsymbol{u_s}.\boldsymbol{\nabla_s}+u_z\partial_z\big)\tau_{zz}-2(\partial_z u_z)\tau_{zz}-\varepsilon(\boldsymbol{\nabla_s}u_z)^T.\boldsymbol{\tau_{sz}}-\varepsilon\boldsymbol{\tau_{sz}}^T.(\boldsymbol{\nabla_s}u_z)\Big]+\Sigma\tau_{zz}$$
$$=2\frac{\Sigma\,\eta_p}{\eta}\,\partial_z u_z.$$

$$\Rightarrow We\Big[\partial_t\tau_{zz}+\big(\boldsymbol{u_s}.\boldsymbol{\nabla_s}+u_z\partial_z\big)\tau_{zz}-2(\partial_z u_z)\tau_{zz}\Big]-\varepsilon We\Big[(\boldsymbol{\nabla_s}u_z)^T.\boldsymbol{\tau_{sz}}+\boldsymbol{\tau_{sz}}^T.(\boldsymbol{\nabla_s}u_z)\Big]+\tau_{zz}$$
$$-2\beta\,\partial_z u_z=0.$$

3.3.2.4 Kinematic condition

The kinematic condition at free surface becomes:
$\frac{U\varepsilon L}{L}\partial_t h + \frac{\varepsilon LU}{L}(\boldsymbol{\nabla_s} h).\boldsymbol{u_s} - \varepsilon U u_z = 0.$
$\Rightarrow \partial_t h + (\boldsymbol{\nabla_s} h).\boldsymbol{u_s} - u_z = 0.$

3.3.2.5 Boundary conditions

- Boundary conditions at free surface are:

  1. $-\frac{\Sigma\varepsilon L}{L}\boldsymbol{\sigma_s}\,(\boldsymbol{\nabla_s} h) + \Sigma\boldsymbol{\sigma_{sz}} = 0.$
     $\Rightarrow -\varepsilon\boldsymbol{\sigma_s}\,(\boldsymbol{\nabla_s} h) + \boldsymbol{\sigma_{sz}} = 0.$
  2. $-\frac{\Sigma\varepsilon L}{L}\boldsymbol{\sigma_{sz}}^T\,(\boldsymbol{\nabla_s} h) + \Sigma\sigma_{zz} = 0.$
     $\Rightarrow -\varepsilon\boldsymbol{\sigma_{sz}}^T\,(\boldsymbol{\nabla_s} h) + \sigma_{zz} = 0.$

- Boundary conditions at bottom are:

  1. $\varepsilon U u_z = 0 \Rightarrow u_z = 0.$
  2. $\Sigma\boldsymbol{\sigma_{sz}} - c_f U\boldsymbol{u_s} = -\Sigma\boldsymbol{f_{as}}.$
     $\Rightarrow \boldsymbol{\sigma_{sz}} - \frac{\varepsilon\hat{c}_f U}{\Sigma}\boldsymbol{u_s} = -\boldsymbol{f_{as}}$ (where $c_f = \varepsilon\hat{c}_f$, discussion about the assumption is made in 3.5.3).
     $\Rightarrow \boldsymbol{\sigma_{sz}} - \epsilon\alpha\boldsymbol{u_s} = -\boldsymbol{f_{as}}$ (where $\alpha = \frac{\hat{c}_f U}{\Sigma}$, is a dimensionless term).
     $\Rightarrow \boldsymbol{\sigma_{sz}} - \varepsilon\alpha\boldsymbol{u_s} = -\boldsymbol{f_{as}}.$

3.3.3 Final equations

Eventually, here we have written down the final set of dimensionless equations:
Conservation of mass:

$$\mathrm{div_s}\,\boldsymbol{u_s} + \partial_z u_z = 0 \tag{3a}$$

Conservation of momentum:

$$\varepsilon\,Re\Big[\partial_t\boldsymbol{u_s} + (\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z)\boldsymbol{u_s}\Big] - \varepsilon\,\mathbf{div_s}\boldsymbol{\sigma_s} - \partial_z\boldsymbol{\sigma_{sz}} = 0 \tag{3b}$$

$$\varepsilon^2 Re\Big[\partial_t u_z + (\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z)\,u_z\Big] - \varepsilon\,\mathrm{div_s}\,\boldsymbol{\sigma_{sz}}^T - \partial_z\sigma_{zz} = 0 \tag{3c}$$

Constitutive equations:

$$\boldsymbol{\sigma_s} = -p\,I + \boldsymbol{\tau_s} + 2(1-\beta)\,D_s(\boldsymbol{u_s}) \tag{3d}$$

$$\sigma_{zz} = -p + \tau_{zz} + 2(1-\beta)\,\partial_z u_z \tag{3e}$$

$$\varepsilon\,\boldsymbol{\sigma_{sz}} = \varepsilon\,\boldsymbol{\tau_{sz}} + (1-\beta)(\partial_z\boldsymbol{u_s} + \varepsilon^2\boldsymbol{\nabla_s} u_z) \tag{3f}$$

$$\varepsilon\,We\left[\partial_t\boldsymbol{\tau_s} + (\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z)\boldsymbol{\tau_s} - (\boldsymbol{\nabla_s}\boldsymbol{u_s})\,\boldsymbol{\tau_s} - \boldsymbol{\tau_s}\,(\boldsymbol{\nabla_s}\boldsymbol{u_s}^T)\right] - We\left[(\partial_z\boldsymbol{u_s})\,\boldsymbol{\tau_{sz}}^T + \boldsymbol{\tau_{sz}}\,(\partial_z\boldsymbol{u_s})^T\right]$$
$$+\varepsilon\boldsymbol{\tau_s} - 2\varepsilon\beta\,D_s(\boldsymbol{u_s}) = 0 \tag{3g}$$

$$\varepsilon We\Big[\partial_t\boldsymbol{\tau_{sz}} + \big(\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z\big)\boldsymbol{\tau_{sz}} - (\boldsymbol{\nabla_s}\boldsymbol{u_s} + \partial_z u_z)\boldsymbol{\tau_{sz}}\Big] - We\,(\partial_z\boldsymbol{u_s})\tau_{zz} - \varepsilon^2 We\,\boldsymbol{\tau_s}\boldsymbol{\nabla_s} u_z + \varepsilon\,\boldsymbol{\tau_{sz}}$$

$$-\beta\big(\partial_z \boldsymbol{u_s} + \varepsilon^2\, \boldsymbol{\nabla_s} u_z\big) = 0 \tag{3h}$$

$$We\Big[\partial_t \tau_{zz} + \big(\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z\big)\tau_{zz} - 2(\partial_z u_z)\tau_{zz}\Big] - \varepsilon We\Big[(\boldsymbol{\nabla_s} u_z)^T.\boldsymbol{\tau_{sz}} + \boldsymbol{\tau_{sz}}^T.(\boldsymbol{\nabla_s} u_z)\Big] + \tau_{zz}$$
$$-2\beta\, \partial_z u_z = 0 \tag{3i}$$

Kinematic condition at z=h:

$$\partial_t h + (\boldsymbol{\nabla_s} h).\boldsymbol{u_s} - u_z = 0 \tag{3j}$$

Boundary condition at $z = h$:

$$-\,\varepsilon \boldsymbol{\sigma_s}\,(\boldsymbol{\nabla_s} h) + \boldsymbol{\sigma_{sz}} = 0 \tag{3k}$$

$$-\,\varepsilon \boldsymbol{\sigma_{sz}}^T.(\boldsymbol{\nabla_s} h) + \sigma_{zz} = 0 \tag{3l}$$

Boundary condition at $z = 0$:

$$u_z = 0 \tag{3m}$$

$$\boldsymbol{\sigma_{sz}} - \varepsilon\alpha \boldsymbol{u_s} = -\boldsymbol{f_{as}} \tag{3n}$$

Where, the entire problem is closed by the initial conditions for h, $\boldsymbol{u_s}$, $\boldsymbol{\tau_s}$, τ_{zz}:
$h(t = 0) = h_0$, in Ω
$\boldsymbol{u_s}(t = 0) = \boldsymbol{u_{s\,0}}$ in $\Lambda(0)$
$\boldsymbol{\tau_s}(t = 0) = \boldsymbol{\tau_{s\,0}}$ in $\Lambda(0)$.
$\tau_{zz}(t = 0) = \tau_{zz\,0}$ in $\Lambda(0)$

3.4 Asymptotic expansion

Asymptotic expansion or asymptotic series is a series of functions. It has the property that truncating the series after a finite number of terms provides an approximation to a given function as the argument of the function tends towards a particular or more often an infinite point. Here, in the series expansion we have considered terms till $O(\varepsilon^2)$, terms with $\geq O(\varepsilon^3)$ are ignored because they are very small, so negligible. So, we expand $\boldsymbol{u}$, $\boldsymbol{\tau}$, $\boldsymbol{\sigma}$, h, p upto $O(\varepsilon^2)$.

$\boldsymbol{u_s} = \boldsymbol{u_s}^{(0)} + \varepsilon\, \boldsymbol{u_s}^{(1)} + \varepsilon^2\, \boldsymbol{u_s}^{(2)}$.
$u_z = u_z^{(0)} + \varepsilon\, u_z^{(1)} + \varepsilon^2\, u_z^{(2)}$.
$\boldsymbol{\tau_s} = \boldsymbol{\tau_s}^{(0)} + \varepsilon\, \boldsymbol{\tau_s}^{(1)} + \varepsilon^2\, \boldsymbol{\tau_s}^{(2)}$.
$\boldsymbol{\tau_{sz}} = \boldsymbol{\tau_{sz}}^{(0)} + \varepsilon\, \boldsymbol{\tau_{sz}}^{(1)} + \varepsilon^2\, \boldsymbol{\tau_{sz}}^{(2)}$.
$\tau_{zz} = \tau_{zz}^{(0)} + \varepsilon\, \tau_{zz}^{(1)} + \varepsilon^2\, \tau_{zz}^{(2)}$.
$h = h^{(0)} + \varepsilon\, h^{(1)} + \varepsilon^2\, h^{(2)}$.
$\boldsymbol{\sigma_s} = \boldsymbol{\sigma_s}^{(0)} + \varepsilon\, \boldsymbol{\sigma_s}^{(1)} + \varepsilon^2\, \boldsymbol{\sigma_s}^{(2)}$.
$\boldsymbol{\sigma_{sz}} = \boldsymbol{\sigma_{sz}}^{(0)} + \varepsilon\, \boldsymbol{\sigma_{sz}}^{(1)} + \varepsilon^2\, \boldsymbol{\sigma_{sz}}^{(2)}$.
$\sigma_{zz} = \sigma_{zz}^{(0)} + \varepsilon\, \sigma_{zz}^{(1)} + \varepsilon^2\, \sigma_{zz}^{(2)}$
$p = p^{(0)} + \varepsilon\, p^{(1)} + \varepsilon^2\, p^{(2)}$.

3.4.1 Expansion in finite series form

3.4.1.1 Conservation of mass

$\mathrm{div_s}\,\boldsymbol{u_s} + \partial_z u_z = 0.$
$\Rightarrow \mathrm{div_s}\,\boldsymbol{u_s}^{(0)} + \partial_z u_z^{(0)} + \varepsilon\big(\mathrm{div_s}\,\boldsymbol{u_s}^{(1)} + \partial_z u_z^{(1)}\big) + \varepsilon^2\big(\mathrm{div_s}\,\boldsymbol{u_s}^{(2)} + \partial_z u_z^{(2)}\big) = 0.$
Equating the coefficient of ε^i in LHS and RHS, we get:
$\mathrm{div_s}\,\boldsymbol{u_s}^{(i)} + \partial_z u_z^{(i)} = 0, \quad i = 0, 1, 2...$

3.4.1.2 Conservation of momentum

- Planar component: $\varepsilon\, Re\Big[\partial_t \boldsymbol{u_s} + (\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z)\boldsymbol{u_s}\Big] - \varepsilon\mathbf{div_s}\boldsymbol{\sigma_s} - \partial_z\boldsymbol{\sigma_{sz}} = 0.$
  $\Rightarrow \varepsilon\, Re\Big[\partial_t\boldsymbol{u_s}^{(0)} + (\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s} + u_z^{(0)}\partial_z)\boldsymbol{u_s}^{(0)} + \varepsilon\,\Big(\partial_t\boldsymbol{u_s}^{(1)} + (\boldsymbol{u_s}^{(1)}.\boldsymbol{\nabla_s} + u_z^{(1)}\partial_z)\boldsymbol{u_s}^{(1)}\Big)\Big]$
  $-\varepsilon\,(\mathbf{div_s}\boldsymbol{\sigma_s}^{(0)} + \varepsilon\,\mathbf{div_s}\boldsymbol{\sigma_s}^{(1)}) - (\partial_z\boldsymbol{\sigma_{sz}}^{(0)} + \varepsilon\,\partial_z\boldsymbol{\sigma_{sz}}^{(1)} + \varepsilon^2\,\partial_z\boldsymbol{\sigma_{sz}}^{(2)}) = 0.$
  $$\Rightarrow \begin{cases} \partial_z\boldsymbol{\sigma_{sz}}^{(0)} = 0. \\ Re\Big[\partial_t\boldsymbol{u_s}^{(0)} + (\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s} + u_z^{(0)}\partial_z)\boldsymbol{u_s}^{(0)}\Big] - \mathbf{div_s}\,\boldsymbol{\sigma_s}^{(0)} - \partial_z\boldsymbol{\sigma_{sz}}^{(1)} = 0. \end{cases}$$

- Vertical component: $\varepsilon^2 Re\Big[\partial_t u_z + (\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z)\,u_z\Big] - \varepsilon\mathrm{div_s}\,\boldsymbol{\sigma_{sz}}^T - \partial_z\sigma_{zz} = 0.$
  $\Rightarrow \varepsilon^2 Re\Big[\partial_t u_z^{(0)} + (\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s} + u_z^{(0)}\partial_z)\,u_z^{(0)} + \varepsilon\,(\partial_t u_z^{(1)} + (\boldsymbol{u_s}^{(1)}.\boldsymbol{\nabla_s} + u_z^{(1)}\partial_z)\,u_z^{(1)})\Big]$
  $-\varepsilon(\mathrm{div_s}\,\boldsymbol{\sigma_{sz}}^{(0)\,T} + \varepsilon\,\mathrm{div_s}\,\boldsymbol{\sigma_{sz}}^{(1)\,T}) - (\partial_z\sigma_{zz}^{(0)} + \varepsilon\,\partial_z\sigma_{zz}^{(1)}) = 0.$
  $$\Rightarrow \begin{cases} \partial_z\sigma_{zz}^{(0)} = 0. \\ \mathrm{div_s}\,(\boldsymbol{\sigma_{sz}}^{(0)})^T + \partial_z\sigma_{zz}^{(1)} = 0. \end{cases}$$

3.4.1.3 Constitutive equations

Equations for total stress tensor $\boldsymbol{\sigma}$ are:

- $\boldsymbol{\sigma_s} = -pI + \boldsymbol{\tau_s} + 2(1-\beta)\,D_s(\boldsymbol{u_s}).$
  $\Rightarrow \boldsymbol{\sigma_s}^{(0)} + \varepsilon\boldsymbol{\sigma_s}^{(1)} = -(p^{(0)}I + \varepsilon p^{(1)}I) + (\boldsymbol{\tau_s}^{(0)} + \varepsilon\boldsymbol{\tau_s}^{(1)}) + 2(1-\beta)\,\big[D_s(\tilde{\boldsymbol{u}}_{\boldsymbol{s}}^{(0)})$
  $+\varepsilon D_s(\boldsymbol{u_s}^{(1)})\big].$
  $\Rightarrow \boldsymbol{\sigma_s}^{(0)} = -p^{(0)}I + \boldsymbol{\tau_s}^{(0)} + 2(1-\beta)\,D_s(\boldsymbol{u_s}^{(0)}).$

- $\sigma_{zz} = -p + \tau_{zz} + 2(1-\beta)\,\partial_z u_z$
  $\Rightarrow \sigma_{zz}^{(0)} = -p^{(0)} + \tau_{zz}^{(0)} + 2(1-\beta)\,\partial_z u_z^{(0)}.$

- $\varepsilon\,\boldsymbol{\sigma_{sz}} = \varepsilon\,\boldsymbol{\tau_{sz}} + (1-\beta)(\partial_z\boldsymbol{u_s} + \varepsilon^2\boldsymbol{\nabla_s}u_z).$
  $\Rightarrow \varepsilon\,\Big[\boldsymbol{\sigma_{sz}}^{(0)} + \varepsilon\boldsymbol{\sigma_{sz}}^{(1)}\Big] = \epsilon\,\Big[\boldsymbol{\tau_{sz}}^{(0)} + \epsilon\boldsymbol{\tau_{sz}}^{(1)}\Big] + (1-\beta)\Big[(\partial_z\boldsymbol{u_s}^{(0)} + \varepsilon\partial_z\boldsymbol{u_s}^{(1)}) + \varepsilon^2\boldsymbol{\nabla_s}u_z^{(0)}\Big].$
  $$\Rightarrow \begin{cases} (1-\beta)\partial_z\boldsymbol{u_s}^{(0)} = 0. \\ \boldsymbol{\sigma_{sz}}^{(0)} = \boldsymbol{\tau_{sz}}^{(0)} + (1-\beta)\partial_z\boldsymbol{u_s}^{(1)}. \end{cases}$$

Equations for the elasticity tensor $\boldsymbol{\tau}$ are:

- $$\varepsilon\, We\left[\partial_t \boldsymbol{\tau_s} + (\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z)\boldsymbol{\tau_s} - (\boldsymbol{\nabla_s u_s})\,\boldsymbol{\tau_s} - \boldsymbol{\tau_s}\,(\boldsymbol{\nabla_s u_s}^T)\right] - We\left[(\partial_z \boldsymbol{u_s})\,\boldsymbol{\tau_{sz}}^T + \boldsymbol{\tau_{sz}}\,(\partial_z \boldsymbol{u_s})^T\right] + \varepsilon\boldsymbol{\tau_s} - 2\varepsilon\beta\, D_s(\boldsymbol{u_s}) = 0.$$
$$\Rightarrow \varepsilon\, We\Big[\partial_t \boldsymbol{\tau_s}^{(0)} + (\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s} + u_z^{(0)}\partial_z)\boldsymbol{\tau_s}^{(0)} - (\boldsymbol{\nabla_s u_s}^{(0)})\,\boldsymbol{\tau_s}^{(0)} - \boldsymbol{\tau_s}^{(0)}\,(\boldsymbol{\nabla_s u_s}^{(0)\,T}) + \varepsilon\,\big(\partial_t\boldsymbol{\tau_s}^{(1)} + (\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s} + u_z^{(0)}\partial_z)\boldsymbol{\tau_s}^{(1)} + (\boldsymbol{u_s}^{(1)}.\boldsymbol{\nabla_s} + u_z^{(1)}\partial_z)\boldsymbol{\tau_s}^{(0)} - (\boldsymbol{\nabla_s u_s}^{(0)})\,\boldsymbol{\tau_s}^{(1)} - (\boldsymbol{\nabla_s u_s}^{(1)})\,\boldsymbol{\tau_s}^{(0)} - \boldsymbol{\tau_s}^{(1)}\,(\boldsymbol{\nabla_s u_s}^{(0)\,T}) - \boldsymbol{\tau_s}^{(0)}\,(\boldsymbol{\nabla_s u_s}^{(1)\,T})\big)\Big] - We\Big[(\partial_z \boldsymbol{u_s}^{(0)})\,\boldsymbol{\tau_{sz}}^{(0)\,T} + \boldsymbol{\tau_{sz}}^{(0)}\,(\partial_z \boldsymbol{u_s}^{(0)})^T + \varepsilon\,\big((\partial_z \boldsymbol{u_s}^{(1)})\,\boldsymbol{\tau_{sz}}^{(0)\,T} + (\partial_z \boldsymbol{u_s}^{(0)})\,\boldsymbol{\tau_{sz}}^{(1)\,T} + \boldsymbol{\tau_{sz}}^{(0)}\,(\partial_z \boldsymbol{u_s}^{(1)})^T + \boldsymbol{\tau_{sz}}^{(1)}\,(\partial_z \boldsymbol{u_s}^{(0)})^T\big)\Big] + \varepsilon\Big[\boldsymbol{\tau_s}^{(0)} + \varepsilon\boldsymbol{\tau_s}^{(1)}\Big] - 2\varepsilon\beta\,\Big[D_s(\boldsymbol{u_s}^{(0)}) + \varepsilon D_s(\boldsymbol{u_s}^{(1)})\Big] = 0.$$
$$\Rightarrow \begin{cases} (\partial_z \boldsymbol{u_s}^{(0)})\,\boldsymbol{\tau_{sz}}^{(0)\,T} + \boldsymbol{\tau_{sz}}^{(0)}\,(\partial_z \boldsymbol{u_s}^{(0)})^T = 0. \\ We\left[\partial_t \boldsymbol{\tau_s}^{(0)} + (\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s} + u_z^{(0)}\partial_z)\boldsymbol{\tau_s}^{(0)} - (\boldsymbol{\nabla_s u_s}^{(0)})\,\boldsymbol{\tau_s}^{(0)} - \boldsymbol{\tau_s}^{(0)}\,(\boldsymbol{\nabla_s u_s}^{(0)\,T})\right] + \boldsymbol{\tau_s}^{(0)} \\ -2\beta\, D_s(\boldsymbol{u_s}^{(0)}) = We\Big[(\partial_z \boldsymbol{u_s}^{(1)})\,\boldsymbol{\tau_{sz}}^{(0)\,T} + (\partial_z \boldsymbol{u_s}^{(0)})\,\boldsymbol{\tau_{sz}}^{(1)\,T} + \boldsymbol{\tau_{sz}}^{(0)}\,(\partial_z \boldsymbol{u_s}^{(1)})^T + \boldsymbol{\tau_{sz}}^{(1)}\,(\partial_z \boldsymbol{u_s}^{(0)})^T\Big] \end{cases}$$

- $$\varepsilon We\Big[\partial_t \boldsymbol{\tau_{sz}} + \big(\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z\big)\boldsymbol{\tau_{sz}} - (\boldsymbol{\nabla_s u_s} + \partial_z u_z)\boldsymbol{\tau_{sz}}\Big] - We\,(\partial_z \boldsymbol{u_s})\tau_{zz} - \varepsilon^2 We\,\boldsymbol{\tau_s}\boldsymbol{\nabla_s} u_z + \varepsilon\,\boldsymbol{\tau_{sz}} - \beta\big(\partial_z \boldsymbol{u_s} + \varepsilon^2\,\boldsymbol{\nabla_s} u_z\big) = 0.$$
$$\Rightarrow \varepsilon We\Big[\partial_t \boldsymbol{\tau_{sz}}^{(0)} + \big(\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s} + u_z^{(0)}\partial_z\big)\boldsymbol{\tau_{sz}}^{(0)} - (\boldsymbol{\nabla_s u_s}^{(0)} + \partial_z u_z^{(0)})\boldsymbol{\tau_{sz}}^{(0)} + \varepsilon\Big(\partial_t \boldsymbol{\tau_{sz}}^{(1)} + \big(\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s} + u_z^{(0)}\partial_z\big)\boldsymbol{\tau_{sz}}^{(1)} + \big(\boldsymbol{u_s}^{(1)}.\boldsymbol{\nabla_s} + u_z^{(1)}\partial_z\big)\boldsymbol{\tau_{sz}}^{(0)} - (\boldsymbol{\nabla_s u_s}^{(0)} + \partial_z u_z^{(0)})\boldsymbol{\tau_{sz}}^{(1)} - (\boldsymbol{\nabla_s u_s}^{(1)} + \partial_z u_z^{(1)})\boldsymbol{\tau_{sz}}^{(0)}\Big)\Big] - We\,\Big[(\partial_z \boldsymbol{u_s}^{(0)})\tau_{zz}^{(0)} + \varepsilon\Big((\partial_z \boldsymbol{u_s}^{(0)})\tau_{zz}^{(1)} + (\partial_z \boldsymbol{u_s}^{(1)})\tau_{zz}^{(0)}\Big)\Big] + \varepsilon\,\Big(\boldsymbol{\tau_{sz}}^{(0)} + \varepsilon\,\boldsymbol{\tau_{sz}}^{(1)}\Big) - \beta\Big[(\partial_z \boldsymbol{u_s}^{(0)} + \varepsilon\,\partial_z \boldsymbol{u_s}^{(1)}) + \varepsilon^2\,(\boldsymbol{\nabla_s} u_z^{(0)})\Big] = 0.$$
$$\Rightarrow \begin{cases} \beta\partial_z \boldsymbol{u_s}^{(0)} + We\,\partial_z \boldsymbol{u_s}^{(0)}\tau_{zz}^{(0)} = 0 \Rightarrow \partial_z \boldsymbol{u_s}^{(0)}\,(\beta + We\,\tau_{zz}^{(0)}) = 0. \\ We\Big[\partial_t \boldsymbol{\tau_{sz}}^{(0)} + \big(\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s} + u_z^{(0)}\partial_z\big)\boldsymbol{\tau_{sz}}^{(0)} - (\boldsymbol{\nabla_s u_s}^{(0)} + \partial_z u_z^{(0)})\boldsymbol{\tau_{sz}}^{(0)}\Big] - We\Big[(\partial_z \boldsymbol{u_s}^{(0)})\tau_{zz}^{(1)} \\ +(\partial_z \boldsymbol{u_s}^{(1)})\tau_{zz}^{(0)}\Big] + \boldsymbol{\tau_{sz}}^{(0)} - \beta\partial_z \boldsymbol{u_s}^{(1)} = 0. \end{cases}$$

- $$We\Big[\partial_t \tau_{zz} + \big(\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z\big)\tau_{zz} - 2(\partial_z u_z)\tau_{zz}\Big] - \varepsilon We\Big[(\boldsymbol{\nabla_s} u_z)^T.\boldsymbol{\tau_{sz}} + \boldsymbol{\tau_{sz}}^T.(\boldsymbol{\nabla_s} u_z)\Big] + \tau_{zz} - 2\beta\,\partial_z u_z = 0.$$
$$\Rightarrow We\Big[\partial_t \tau_{zz}^{(0)} + \big(\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s} + u_z^{(0)}\partial_z\big)\tau_{zz}^{(0)} - 2(\partial_z u_z)^{(0)}\tau_{zz}^{(0)} + \varepsilon\Big(\partial_t \tau_{zz}^{(1)} + \big(\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s} + u_z^{(0)}\partial_z\big)\tau_{zz}^{(1)} + \big(\boldsymbol{u_s}^{(1)}.\boldsymbol{\nabla_s} + u_z^{(1)}\partial_z\big)\tau_{zz}^{(0)} - 2(\partial_z u_z^{(1)})\tau_{zz}^{(0)} - 2(\partial_z u_z^{(0)})\tau_{zz}^{(1)}\Big)\Big] - \varepsilon We\Big[(\boldsymbol{\nabla_s} u_z)^{(0)\,T}.\boldsymbol{\tau_{sz}}^{(0)} + \boldsymbol{\tau_{sz}}^{(0)\,T}.(\boldsymbol{\nabla_s} u_z^{(0)})\Big] + (\tau_{zz}^{(0)} + \varepsilon\tau_{zz}^{(1)}) - 2\beta\,(\partial_z u_z^{(0)} + \varepsilon\partial_z u_z^{(1)}) = 0.$$
$$\Rightarrow We\Big[\partial_t \tau_{zz}^{(0)} + \big(\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s} + u_z^{(0)}\partial_z\big)\tau_{zz}^{(0)} - 2(\partial_z u_z)^{(0)}\tau_{zz}^{(0)}\Big] + \tau_{zz}^{(0)} - 2\beta\,\partial_z u_z^{(0)} = 0.$$

3.4.1.4 Kinematic equation at free surface ($z = h$)

$\partial_t h + (\boldsymbol{\nabla_s} h).\boldsymbol{u_s} - u_z = 0$
$\Rightarrow \partial_t h^{(i)} + (\boldsymbol{\nabla_s} h^{(i)}).\boldsymbol{u_s}^{(0)} - u_z^{(i)} = 0, \quad i = 0, 1, 2..., \quad$ at $z = h$

3.4.1.5 Boundary conditions

Bounday conditions at $z = h$ are:

- $-\varepsilon \boldsymbol{\sigma_s}\,(\boldsymbol{\nabla_s} h) + \boldsymbol{\sigma_{sz}} = 0.$
  $\Rightarrow -\varepsilon\Big[\boldsymbol{\sigma_s}^{(0)}\,(\boldsymbol{\nabla_s} h^{(0)}) + \varepsilon\Big(\boldsymbol{\sigma_s}^{(1)}\,(\boldsymbol{\nabla_s} h^{(0)}) + \boldsymbol{\sigma_s}^{(0)}\,(\boldsymbol{\nabla_s} h^{(1)})\Big)\Big]$
  $+\Big[\boldsymbol{\sigma_{sz}}^{(0)} + \varepsilon \boldsymbol{\sigma_{sz}}^{(1)}\Big] = 0.$

  $$\Rightarrow \begin{cases} \boldsymbol{\sigma_{sz}}^{(0)} = 0. \\ -\boldsymbol{\sigma_s}^{(0)}\,(\boldsymbol{\nabla_s} h^{(0)}) + \boldsymbol{\sigma_{sz}}^{(1)} = 0. \end{cases} \quad \text{at } z = h$$

- $-\varepsilon \boldsymbol{\sigma_{sz}}^T.(\boldsymbol{\nabla_s} h) + \sigma_{zz} = 0.$
  $\Rightarrow -\varepsilon\Big[\boldsymbol{\sigma_{sz}}^{(0)\,T}.(\boldsymbol{\nabla_s} h^{(0)})\Big] + (\sigma_{zz}^{(0)} + \varepsilon \sigma_{zz}^{(1)}) = 0.$

  $$\Rightarrow \begin{cases} \sigma_{zz}^{(0)} = 0. \\ -\boldsymbol{\sigma_{sz}}^{(0)\,T}.(\boldsymbol{\nabla_s} h^{(0)}) + \sigma_{zz}^{(1)} = 0. \end{cases} \quad \text{at } z = h$$

  Boundary conditions at the bottom are:

- $u_z = 0 \Rightarrow u_z^{(i)} = 0\ ,\ \ i = 0, 1, 2, 3...,$ at $z = 0$

- $\boldsymbol{\sigma_{sz}} - \varepsilon \alpha \boldsymbol{u_s} = -\boldsymbol{f_{as}}.$
  $\Rightarrow (\boldsymbol{\sigma_{sz}}^{(0)} + \varepsilon\, \boldsymbol{\sigma_{sz}}^{(1)}) - \varepsilon \alpha \boldsymbol{u_s}^{(0)}) = -(\boldsymbol{f_{as}}^{(0)} + \varepsilon \boldsymbol{f_{as}}^{(1)}).$

  $$\Rightarrow \begin{cases} \boldsymbol{\sigma_{sz}}^{(0)} = -\boldsymbol{f_{as}}^{(0)}. \\ \boldsymbol{\sigma_{sz}}^{(1)} - \alpha \boldsymbol{u_s}^{(0)} = -\boldsymbol{f_{as}}^{(1)}. \end{cases} \quad \text{at } z = 0$$

3.4.2 Final equations

Eventually, I have written down the final set of equations after assymptotic series expansions:
Conservation of mass:

$$\mathrm{div_s}\, \boldsymbol{u_s}^{(i)} + \partial_z u_z^{(i)} = 0 \tag{4a}$$

Conservation of momentum:

$$\partial_z \boldsymbol{\sigma_{sz}}^{(0)} = 0 \tag{4b}$$

$$Re\Big[\partial_t \boldsymbol{u_s}^{(0)} + (\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s} + u_z^{(0)}\partial_z)\boldsymbol{u_s}^{(0)}\Big] - \mathbf{div_s}\, \boldsymbol{\sigma_s}^{(0)} - \partial_z \boldsymbol{\sigma_{sz}}^{(1)} = 0 \tag{4c}$$

$$\partial_z \sigma_{zz}^{(0)} = 0 \tag{4d}$$

$$\mathrm{div_s}\, (\boldsymbol{\sigma_{sz}}^{(0)})^T + \partial_z \sigma_{zz}^{(1)} = 0 \tag{4e}$$

Constitutive equations:

$$\boldsymbol{\sigma_s}^{(0)} = -p^{(0)} I + \boldsymbol{\tau_s}^{(0)} + 2(1-\beta)\, D_s(\boldsymbol{u_s}^{(0)}) \tag{4f}$$

$$\sigma_{zz}^{(0)} = -p^{(0)} + \tau_{zz}^{(0)} + 2(1-\beta)\, \partial_z u_z^{(0)} \tag{4g}$$

$$(1-\beta)\partial_z \boldsymbol{u_s}^{(0)} = 0 \tag{4h}$$

$$\boldsymbol{\sigma_{sz}}^{(0)} = \boldsymbol{\tau_{sz}}^{(0)} + (1-\beta)\partial_z \boldsymbol{u_s}^{(1)} \tag{4i}$$

$$(\partial_z \boldsymbol{u_s}^{(0)})\,\boldsymbol{\tau_{sz}}^{(0)\,T} + \boldsymbol{\tau_{sz}}^{(0)}\,(\partial_z \boldsymbol{u_s}^{(0)})^T = 0 \tag{4j}$$

$$We\left[\partial_t \boldsymbol{\tau_s}^{(0)} + (\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s} + u_z^{(0)}\partial_z)\boldsymbol{\tau_s}^{(0)} - (\boldsymbol{\nabla_s}\boldsymbol{u_s}^{(0)})\,\boldsymbol{\tau_s}^{(0)} - \boldsymbol{\tau_s}^{(0)}\,(\boldsymbol{\nabla_s}\boldsymbol{u_s}^{(0)\,T})\right] + \boldsymbol{\tau_s}^{(0)} - 2\beta\, D_s(\boldsymbol{u_s}^{(0)})$$

$$= We\left[(\partial_z \boldsymbol{u_s}^{(1)})\,\boldsymbol{\tau_{sz}}^{(0)\,T} + (\partial_z \boldsymbol{u_s}^{(0)})\,\boldsymbol{\tau_{sz}}^{(1)\,T} + \boldsymbol{\tau_{sz}}^{(0)}\,(\partial_z \boldsymbol{u_s}^{(1)})^T + \boldsymbol{\tau_{sz}}^{(1)}\,(\partial_z \boldsymbol{u_s}^{(0)})^T\right] \tag{4k}$$

$$\partial_z \boldsymbol{u_s}^{(0)}\,(\beta + We\,\tau_{zz}^{(0)}) = 0 \tag{4l}$$

$$We\left[\partial_t \boldsymbol{\tau_{sz}}^{(0)} + \big(\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s} + u_z^{(0)}\partial_z\big)\boldsymbol{\tau_{sz}}^{(0)} - (\boldsymbol{\nabla_s}\boldsymbol{u_s}^{(0)} + \partial_z u_z^{(0)})\boldsymbol{\tau_{sz}}^{(0)}\right]$$

$$-We\left[(\partial_z \boldsymbol{u_s}^{(0)})\tau_{zz}^{(1)} + (\partial_z \boldsymbol{u_s}^{(1)})\tau_{zz}^{(0)}\right] + \boldsymbol{\tau_{sz}}^{(0)} - \beta\partial_z \boldsymbol{u_s}^{(1)} = 0 \tag{4m}$$

$$We\left[\partial_t \tau_{zz}^{(0)} + \big(\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s} + u_z^{(0)}\partial_z\big)\tau_{zz}^{(0)} - 2(\partial_z u_z)^{(0)}\tau_{zz}^{(0)}\right] + \tau_{zz}^{(0)} - 2\beta\,\partial_z u_z^{(0)} = 0 \tag{4n}$$

Kinematic equation:

$$\partial_t h^{(i)} + (\boldsymbol{\nabla_s} h^{(i)}).\boldsymbol{u_s}^{(i)} - u_z^{(i)} = 0 \tag{4o}$$

Boundary condition at $z = h$:

$$\boldsymbol{\sigma_{sz}}^{(0)} = 0 \tag{4p}$$

$$-\,\boldsymbol{\sigma_s}^{(0)}\,(\boldsymbol{\nabla_s} h^{(0)}) + \boldsymbol{\sigma_{sz}}^{(1)} = 0 \tag{4q}$$

$$\sigma_{zz}^{(0)} = 0 \tag{4r}$$

$$-\,\boldsymbol{\sigma_{sz}}^{(0)\,T}.(\boldsymbol{\nabla_s} h^{(0)}) + \sigma_{zz}^{(1)} = 0 \tag{4s}$$

Boundary condition at $z = 0$:

$$u_z^{(i)} = 0 \tag{4t}$$

$$\boldsymbol{\sigma_{sz}}^{(0)} = -\boldsymbol{f_{as}}^{(0)} \tag{4u}$$

$$\boldsymbol{\sigma_{sz}}^{(1)} - \alpha \boldsymbol{u_s}^{(0)} = -\boldsymbol{f_{as}}^{(1)} \tag{4v}$$

Where, the entire problem is closed by the initial conditions for h, $\boldsymbol{u_s}$, $\boldsymbol{\tau_s}$, τ_{zz}:
$h^{(i)}(t=0) = h_0^{(i)}$, in Ω
$\boldsymbol{u_s}^{(i)}(t=0) = \boldsymbol{u_{s\,0}}^{(i)}$ in $\Lambda(0)$
$\boldsymbol{\tau_s}^{(i)}(t=0) = \boldsymbol{\tau_{s\,0}}^{(i)}$ in $\Lambda(0)$.
$\tau_{zz}(t=0)^{(i)} = \tau_{zz\,0}^{(i)}$ in $\Lambda(0)$

3.5 Reduction

We are going to reduce the number of equations, derived from the assymptotic expansion. We have used some certain mathematical hypothesis to do so.

3.5.1 Reduced equations

- From (4d) and (4r), $\sigma_{zz}^{(0)} = 0, \quad \forall\ z$.
  $\Rightarrow p^{(0)} = \tau_{zz}^{(0)} + 2(1-\beta)\partial_z u_z^{(0)}$ (Putting $\sigma_{zz}^{(0)} = 0$ in (4g)).

- From (4h), $(1-\beta)\partial_z \boldsymbol{u_s}^{(0)} = 0$.
  If $0 < \beta < 1$, then $\partial_z \boldsymbol{u_s}^{(0)} = 0$.
  Which means: $\boldsymbol{u_s}^{(0)}$ is independent of z.

- From (4b) and (4p), $\boldsymbol{\sigma_{sz}}^{(0)} = 0, \quad \forall\ z$.
  $\boldsymbol{f_{as}}^{(0)} = 0$ (As $\boldsymbol{\sigma_{sz}}^{(0)} = -\boldsymbol{f_{as}}^{(0)}$ from (4u)).
  Assuming $\partial_z \boldsymbol{u_s}^{(1)} = 0$ from (4*i*), we find: $\boldsymbol{\tau_{sz}}^{(0)} = 0$,
  when $1-\beta \neq 0$ (discussion about the assumption is made in 3.5.3).

- From (4a), $\mathrm{div_s}\, \boldsymbol{u_s}^{(0)} = -\partial_z u_z^{(0)}$.
  $\Rightarrow \int_0^z \mathrm{div_s}\, \boldsymbol{u_s}^{(0)}\, dz = -\int_0^z \partial_z u_z^{(0)}\, dz$.
  In (4t), $u_z^{(0)} = 0$ at $z = 0$. Hence, we can say that:
  $u_z^{(0)} = -z \mathrm{div_s}\, \boldsymbol{u_s}^{(0)}$(As, $\boldsymbol{u_s}^{(0)}$ does not depend on z).

- From (4o), $\partial_t h^{(0)} + (\boldsymbol{\nabla_s} h^{(0)}).\boldsymbol{u_s}^{(0)} = -h^{(0)} \mathrm{div_s}\, \boldsymbol{u_s}^{(0)}$.
  $\Rightarrow \partial_t h^{(0)} + \mathrm{div_s}\left(\boldsymbol{u_s}^{(0)}.\, h^{(0)}\right) = 0$.

- From (4c),
  $Re \int_0^h [(\partial_t \boldsymbol{u_s}^{(0)} + \left(\boldsymbol{u_s}^{(0)} \boldsymbol{\nabla_s} + u_z^{(0)} \partial_z\right) \boldsymbol{u_s}^{0}] dz - \int_0^h [\mathbf{div_s}\, \boldsymbol{\sigma_s}^{(0)} + \partial_z \boldsymbol{\sigma_{sz}}^{(1)}]\, dz = 0$.

  By using Leibniz integral rule:
  $\int_0^h [\mathbf{div_s}\, \boldsymbol{\sigma_s}^{(0)} + \partial_z \boldsymbol{\sigma_{sz}}^{(1)}]\, dz = \mathbf{div_s} \int_0^h \boldsymbol{\sigma_s}^{(0)}\, dz - \boldsymbol{\sigma_s}^{(0)} \boldsymbol{\nabla_s} h^{(0)}$
  $+ \boldsymbol{\sigma_{sz}}^{(1)}(z = h) - \boldsymbol{\sigma_{sz}}^{(1)}(z = 0)$.

  From (4*q*), $\tilde{\boldsymbol{\sigma}}_{\boldsymbol{sz}}^{(1)} = \tilde{\boldsymbol{\sigma}}_{\boldsymbol{s}}^{(0)} \tilde{\boldsymbol{\nabla}}_{\boldsymbol{s}} \tilde{h}^{(0)}$ at $z = h$.
  From (4*v*), at $z = 0$, $\boldsymbol{\sigma_{sz}}^{(1)} = \alpha \boldsymbol{u_s}^{(0)} - \boldsymbol{f_{as}}^{(1)}$.

  Hence, $\int_0^h [\mathbf{div_s}\, \boldsymbol{\sigma_s}^{(0)} + \partial_z \boldsymbol{\sigma_{sz}}^{(1)}]\, dz = \mathbf{div_s} \int_0^h \boldsymbol{\sigma_s}^{(0)}\, dz - \alpha \boldsymbol{u_s}^{(0)} + \boldsymbol{f_{as}}^{(1)}$.

  As, Re is very small, we can deduce that:
  $-\mathbf{div_s} \int_0^h \boldsymbol{\sigma_s}^{(0)}\, dz + \alpha \boldsymbol{u_s}^{(0)} = \boldsymbol{f_{as}}^{(1)}$.

  $p^{(0)} = \tau_{zz}^{(0)} + 2(1-\beta)\partial_z u_z^{(0)}$ (from (4*g*)) because $\sigma_{zz}^{(0)} = 0$).
  $\Rightarrow p^{(0)} = \tau_{zz}^{(0)} - 2(1-\beta)\, \mathrm{div_s}\, \boldsymbol{u_s}^{(0)}$ (As, $\partial_z u_z^{(0)} = -\mathrm{div_s}\, \boldsymbol{u_s}^{(0)}$).
  So, from (4*f*), $\boldsymbol{\sigma_s}^{(0)} = \boldsymbol{\tau_s}^{(0)} - \tau_{zz}^{(0)} I + 2(1-\beta)\, [D_s(\boldsymbol{u_s}^{(0)}) + \mathrm{div_s}\, \boldsymbol{u_s}^{(0)} I]$.

  Hence, $-\mathbf{div_s} \int_0^h \left[\boldsymbol{\tau_s}^{(0)} - \tau_{zz}^{(0)} I + 2(1-\beta) \left(D_s(\boldsymbol{u_s}^{(0)}) + \mathrm{div_s}\, \boldsymbol{u_s}^{(0)} I\right)\right] dz + \alpha \boldsymbol{u_s}^{(0)} = \boldsymbol{f_{as}}^{(1)}$.

The depth average of any function f is $\hat{f}(x) = \frac{1}{h}\int_0^h f(x)dx$, where h is the depth So, taking the depth average, we can say: $\int_0^h \left(\boldsymbol{\tau_s}^{(0)} - \tau_{zz}^{(0)} I\right) dz = h\left(\hat{\boldsymbol{\tau}}_{\boldsymbol{s}}^{(0)} - \hat{\tau}_{zz}^{(0)} I\right)$.

Hence:

$$-\,\mathbf{div_s}\left(h\Big[\hat{\boldsymbol{\tau}}_{\boldsymbol{s}}^{(0)} - \hat{\tau}_{zz}^{(0)} I + 2(1-\beta)\left(D_s(\boldsymbol{u_s}^{(0)}) + \mathrm{div_s}\,\boldsymbol{u_s}^{(0)} I\right)\Big]\right) + \alpha \boldsymbol{u_s}^{(0)} = \boldsymbol{f_{as}}^{(1)} \quad (5)$$

- From (4k),
  $We\left[\partial_t \boldsymbol{\tau_s}^{(0)} + (\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s} + u_z^{(0)}\partial_z)\boldsymbol{\tau_s}^{(0)} - (\boldsymbol{\nabla_s u_s}^{(0)})\,\boldsymbol{\tau_s}^{(0)} - \boldsymbol{\tau_s}^{(0)}\,(\boldsymbol{\nabla_s u_s}^{(0)\,T})\right] + \boldsymbol{\tau_s}^{(0)}$
  $-\,2\beta\,D_s(\boldsymbol{u_s}^{(0)}) = 0$. (because, $\partial_z \boldsymbol{u_s}^{(0)} = \partial_z \boldsymbol{u_s}^{(1)} = 0$).
  $\Rightarrow We\left[\partial_{\tilde{t}} \boldsymbol{\tau_s}^{(0)} + \mathrm{div_s}(\boldsymbol{\tau_s}^{(0)}\,\boldsymbol{u_s}^{(0)}) + \partial_z(u_z^{(0)}\boldsymbol{\tau_s}^{(0)}) - (\boldsymbol{\nabla_s u_s}^{(0)})\,\boldsymbol{\tau_s}^{(0)} - \boldsymbol{\tau_s}^{(0)}\,(\boldsymbol{\nabla_s u_s}^{(0)\,T})\right]$
  $+\,\boldsymbol{\tau_s}^{(0)} - 2\beta\,D_s(\boldsymbol{u_s}^{(0)}) = 0$.
  $\Rightarrow We\int_0^h \left[\partial_t \boldsymbol{\tau_s}^{(0)} + \mathrm{div_s}(\boldsymbol{\tau_s}^{(0)}\,\boldsymbol{u_s}^{(0)}) + \partial_z(u_z^{(0)}\boldsymbol{\tau_s}^{(0)}) - (\boldsymbol{\nabla_s u_s}^{(0)})\,\boldsymbol{\tau_s}^{(0)} - \boldsymbol{\tau_s}^{(0)}\,(\boldsymbol{\nabla_s u_s}^{(0)\,T})\right] dz$
  $+\int_0^h \boldsymbol{\tau_s}^{(0)}\,dz - 2\beta \int_0^h D_s(\boldsymbol{u_s}^{(0)})\,dz = 0$.
  Using Leibnitz integral and the value $u_z^{(0)} = -h\ \mathrm{div_s}\,\boldsymbol{u_s}^{(0)}$ at $z = h$, we get:
  $We\left[\partial_t \int_0^h \boldsymbol{\tau_s}^{(0)}\,dz + \mathrm{div_s}\int_0^h (\boldsymbol{\tau_s}^{(0)}\,\boldsymbol{u_s}^{(0)})\,dz - \int_0^h (\boldsymbol{\nabla_s u_s}^{(0)})\,\boldsymbol{\tau_s}^{(0)}\,dz - \int_0^h \boldsymbol{\tau_s}^{(0)}\,(\boldsymbol{\nabla_s u_s}^{(0)\,T})\,dz\right]$
  $+\int_0^h \boldsymbol{\tau_s}^{(0)}\,dz - 2\beta \int_0^h D_s(\boldsymbol{u_s}^{(0)})\,dz = 0$.
  $\Rightarrow We\left[\partial_{\tilde{t}}\,(h\hat{\boldsymbol{\tau}}_{\boldsymbol{s}}^{(0)}) + \mathrm{div_s}\,(h\,\hat{\boldsymbol{\tau}}_{\boldsymbol{s}}^{(0)}\,\boldsymbol{u_s}^{(0)}) - h\,(\boldsymbol{\nabla_s u_s}^{(0)}\,\hat{\boldsymbol{\tau}}_{\boldsymbol{s}}^{(0)}) - h\,\hat{\boldsymbol{\tau}}_{\boldsymbol{s}}^{(0)}\,(\boldsymbol{\nabla_s u_s}^{(0)\,T})\right]$
  $+\,h\,\hat{\boldsymbol{\tau}}_{\boldsymbol{s}}^{(0)} - 2\beta\,h\,D_s(\boldsymbol{u_s}^{(0)}) = 0$ (As $\boldsymbol{u_s}^{(0)}$ is independent of z).
  $\Rightarrow We\left[\partial_t\,(\hat{\boldsymbol{\tau}}_{\boldsymbol{s}}^{(0)}) + (\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s})\hat{\boldsymbol{\tau}}_{\boldsymbol{s}}^{(0)} - \boldsymbol{\nabla_s u_s}^{(0)}\,\hat{\boldsymbol{\tau}}_{\boldsymbol{s}}^{(0)} - \hat{\boldsymbol{\tau}}_{\boldsymbol{s}}^{(0)}\,\boldsymbol{\nabla_s u_s}^{(0)\,T}\right] + \hat{\boldsymbol{\tau}}_{\boldsymbol{s}}^{(0)}$
  $-\,2\beta\,D_s(\boldsymbol{u_s}^{(0)}) = 0$ (As, $\partial_t h^{(0)} + \mathrm{div_s}\left(\boldsymbol{u_s}^{(0)}.\,h^{(0)}\right) = 0$).

$$We\,\overset{\triangledown\,(\mathbf{0})}{\hat{\boldsymbol{\tau}}_{\boldsymbol{s}}} + \hat{\boldsymbol{\tau}}_{\boldsymbol{s}}^{(0)} - 2\beta\,D_s(\boldsymbol{u_s}^{(0)}) = 0 \quad (6)$$

(where, $\left[\partial_t\,(\hat{\boldsymbol{\tau}}_{\boldsymbol{s}}^{(0)}) + (\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s})\hat{\boldsymbol{\tau}}_{\boldsymbol{s}}^{(0)} - \boldsymbol{\nabla_s u_s}^{(0)}\,\hat{\boldsymbol{\tau}}_{\boldsymbol{s}}^{(0)} - \hat{\boldsymbol{\tau}}_{\boldsymbol{s}}^{(0)}\,\boldsymbol{\nabla_s u_s}^{(0)\,T}\right] = \overset{\triangledown\,(\mathbf{0})}{\hat{\boldsymbol{\tau}}_{\boldsymbol{s}}}$).

- Similarly, from (4n), we can say that:

$$We\Big(\partial_t \hat{\tau}_{zz}^{(0)} + (\boldsymbol{u_s}^{(0)}.\boldsymbol{\nabla_s})\hat{\tau}_{zz}^{(0)} + 2\,\mathrm{div}_{\tilde{s}}(\boldsymbol{u_s}^{(0)})\,\hat{\tau}_{zz}^{(0)}\Big) + \hat{\tau}_{zz}^{(0)} + 2\beta\,\mathrm{div_s}\,\boldsymbol{u_s}^{(0)} = 0. \quad (7)$$

3.5.2 Closed system of equations

So, finally we get a closed system with equations. Here we are going to ignore the orders (like (0), (1)) of terms and the hats (like $\hat{\boldsymbol{\tau}}_{\boldsymbol{s}}^{(0)}$, $\hat{\tau}_{zz}^{(0)}$) and denote them in an usual way for simplicity. So, we get four equations and four unknown variables h, $\boldsymbol{u_s}$, $\boldsymbol{\tau_s}$, τ_{zz}, depending upon t, x, y, such that:

$$\partial_t h + \mathrm{div_s}\big(\boldsymbol{u_s}\, h\big) = 0. \tag{8}$$

$$-\,\mathbf{div_s}\Big[h\,\Big(\boldsymbol{\tau_s} - \tau_{zz} I + 2(1-\beta)(D_s(\boldsymbol{u_s}) + \mathrm{div_s}(\boldsymbol{u_s}) I)\Big)\Big] + \alpha\, \boldsymbol{u_s} = \boldsymbol{f_{as}}. \tag{9}$$

$$We\,\overset{\triangledown}{\boldsymbol{\tau}}_{\boldsymbol{s}} + \boldsymbol{\tau_s} - 2\beta D(\boldsymbol{u_s}) = 0. \tag{10}$$

$$We\Big(\partial_t \tau_{zz} + (\boldsymbol{u_s}.\boldsymbol{\nabla_s})\tau_{zz} + 2\,\mathrm{div_s}(\boldsymbol{u_s})\tau_{zz}\Big) + \tau_{zz} + 2\beta\,\mathrm{div_s}(\boldsymbol{u_s}) = 0. \tag{11}$$

We have We as the Weissenberg number, $\beta = \frac{\eta_p}{\eta_p + \eta_0}$ as the retardation number of Oldroyd model. α is the dimensionless friction coefficient. The Right-Hand-Side of the equation (10) $\boldsymbol{f_{as}}$ is the dimensionless active force.

3.5.3 Assumptions

Here, we have made some strong assumptions to reach the final set of equations for the closed system.

1. The first strong assumption, we made for the friction coefficient is: $c_f = \varepsilon \hat{c}_f$ and $\alpha = \frac{\hat{c}_f U}{\Sigma}$, a dimensionless term. The reason behind the assumption is to generate feasible reduced system of equations. The equation ${\boldsymbol{\sigma_{sz}}}^{(1)} - \alpha {\boldsymbol{u_s}}^{(0)} = -{\boldsymbol{f_{as}}}^{(1)}$ will rather change into ${\boldsymbol{\sigma_{sz}}}^{(0)} - \alpha {\boldsymbol{u_s}}^{(0)} = -{\boldsymbol{f_{as}}}^{(0)}$ if $c_f \neq \varepsilon \hat{c}_f$. As, ${\boldsymbol{\sigma_{sz}}}^{(0)} = 0, \quad \forall z$, So, $\alpha {\boldsymbol{u_s}}^{(0)} = {\boldsymbol{f_{as}}}^{(0)}$ at $z = 0$. Now, if ${\boldsymbol{f_{as}}}^{(0)} = 0$, then ${\boldsymbol{u_s}}^{(0)} = 0$ at $z = 0$ because $\alpha \neq 0$. We know that $\partial_z {\boldsymbol{u_s}}^{(0)} = 0, \quad \forall z$. So, we would get the planar velocity ${\boldsymbol{u_s}}^{(0)} = 0, \quad \forall z$, which is absurd.

2. We have assumed that $\partial_z {\boldsymbol{u_s}}^{(1)} = 0$.
From the equation ${\boldsymbol{\sigma_{sz}}}^{(0)} = {\boldsymbol{\tau_{sz}}}^{(0)} + (1-\beta)\partial_z {\boldsymbol{u_s}}^{(1)}$, we get, ${\boldsymbol{\tau_{sz}}}^{(0)} = -(1-\beta)\partial_z {\boldsymbol{u_s}}^{(1)}$ because ${\boldsymbol{\sigma_{sz}}}^{(0)} = 0, \quad \forall z$. As, multiple partial derrivatives of ${\boldsymbol{\tau_{sz}}}^{(0)}$ or $\partial_z {\boldsymbol{u_s}}^{(1)}$ is equal with zero (equation 4m). Hence, both terms are constants. Here, taking constant equal with zero, $\partial_z {\boldsymbol{u_s}}^{(1)} = 0$. As $\beta \neq 1$, hence, ${\boldsymbol{\tau_{sz}}}^{(0)} = 0$ too. The assumption could have been made conversely but it's our freedom of choice to be theoritically more accurate.

3. The choice $\beta \neq 1$ is made because our work is in Oldroyd model. We can choose $\beta = 1$ which will change it to Upper Convected Maxwell model. We can analyse the case for $\beta = 1$ $(\eta_p = \eta, \eta_0 = 0)$ which would also show some significant results.

   - The dimensionless constitutive equations will be changed.

$$\boldsymbol{\sigma_s} = -p\, I + \boldsymbol{\tau_s}$$

$$\tilde{\sigma}_{zz} = -p + \tau_{zz}$$

$$\boldsymbol{\sigma_{sz}} = \boldsymbol{\tau_{sz}}$$

$$\varepsilon\, We \left[\partial_t \boldsymbol{\tau_s} + (\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z \partial_z)\boldsymbol{\tau_s} - (\boldsymbol{\nabla_s u_s})\,\boldsymbol{\tau_s} - \boldsymbol{\tau_s}\,(\boldsymbol{\nabla_s {u_s}^T})\right] - We\left[(\partial_z \boldsymbol{u_s})\,{\boldsymbol{\tau_{sz}}}^T + \boldsymbol{\tau_{sz}}\,(\partial_z \boldsymbol{u_s})^T\right]$$
$$+\varepsilon \boldsymbol{\tau_s} - 2\varepsilon\, D_s(\boldsymbol{u_s}) = 0$$

$$\varepsilon We\Big[\partial_t \boldsymbol{\tau_{sz}} + \big(\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z\big)\boldsymbol{\tau_{sz}} - (\boldsymbol{\nabla_s u_s} + \partial_z u_z)\boldsymbol{\tau_{sz}}\Big] - We\,(\partial_z \boldsymbol{u_s})\tau_{zz} - \varepsilon^2 We\, \boldsymbol{\tau_s \nabla_s} u_z + \varepsilon\, \boldsymbol{\tau_{sz}}$$
$$-\big(\partial_z \boldsymbol{u_s} + \varepsilon^2\, \boldsymbol{\nabla_s} u_z\big) = 0$$

$$We\Big[\partial_t \tau_{zz} + \big(\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z\big)\tau_{zz} - 2(\partial_z u_z)\tau_{zz}\Big] - \varepsilon We\Big[(\boldsymbol{\nabla_s} u_z)^T.\boldsymbol{\tau_{sz}} + {\boldsymbol{\tau_{sz}}}^T.(\boldsymbol{\nabla_s} u_z)\Big] + \tau_{zz}$$
$$-\partial_z u_z = 0$$

   - As $\boldsymbol{\sigma_{sz}} = \boldsymbol{\tau_{sz}}$, after assymptotic expansion:
${\boldsymbol{\sigma_{sz}}}^{(0)} = 0 \Rightarrow {\boldsymbol{\tau_{sz}}}^{(0)} = 0, \quad \forall z$ (No assumption is required).
From the equation $\partial_z {\boldsymbol{u_s}}^{(0)}\,(1 + We\,\tau_{zz}^{(0)}) = 0$, we get:
$\partial_z {\boldsymbol{u_s}}^{(0)} = 0, \quad \forall z$ (As $We\,\tau_{zz}^{(0)} = -1$ is impossible).

Rest of the analyses will be more or less same as before and finally, we will obtain the closed system of equations:

$$\partial_t h + \mathrm{div_s}\big(\boldsymbol{u_s}\, h\big) = 0.$$

$$-\mathbf{div_s}\Big[h\,\Big(\boldsymbol{\tau_s} - \tau_{zz} I + 2(1-\beta)(D_s(\boldsymbol{u_s}) + \mathrm{div_s}(\boldsymbol{u_s}) I)\Big)\Big] + \alpha\, \boldsymbol{u_s} = \boldsymbol{f_{as}}.$$

$$We\Big(\partial_t \boldsymbol{\tau_s} + (\boldsymbol{u_s}.\boldsymbol{\nabla_s})\boldsymbol{\tau_s} - (\boldsymbol{\nabla_s}\boldsymbol{u_s})\boldsymbol{\tau_s} - \boldsymbol{\tau_s}(\boldsymbol{\nabla_s}\boldsymbol{u_s})^T\Big) + \boldsymbol{\tau_s} - 2D(\boldsymbol{u_s}) = 0.$$

$$We\Big(\partial_t \tau_{zz} + (\boldsymbol{u_s}.\boldsymbol{\nabla_s})\tau_{zz} + 2\,\mathrm{div_s}(\boldsymbol{u_s})\tau_{zz}\Big) + \tau_{zz} + 2\mathrm{div_s}(\boldsymbol{u_s}) = 0.$$

4 Numerical methods

4.1 Variational formulation and finite element method approximation

So, finally the last step before doing numerical solutions. Variational formulation is required to make the continuous system of partial differential equations suitable towards doing numerical solutions. We will define a discontinuous finite element space to approximate the continuous system.
We have considered a special case: $We = 0$ because the case $We \neq 0$ is very complicated to check for numerical results.
From the closed system of equations, with $We = 0$, we have to find h, $\boldsymbol{u_s}$.

When $We = 0$, using equation (10), we can say that: $\boldsymbol{\tau_s} = 2\beta\, D_s(\boldsymbol{u_s})$ and from equation (11), we can say that: $\tau_{zz} = -2\beta\,\mathrm{div_s}(\boldsymbol{u_s})$. Now, replacing the values of $\boldsymbol{\tau_s}$, τ_{zz} in the equation (9), we get:

$$-\,\mathbf{div_s}\Big[h\Big(2(D_s(\boldsymbol{u_s}) + \mathrm{div_s}(\boldsymbol{u_s})I)\Big)\Big] + \alpha\boldsymbol{u_s} = \boldsymbol{f_{as}}. \tag{12}$$

Hence, we found the equations (8) and (12) for doing Variational formulation.
The system is closed by the boundary conditions on $\partial\Omega$:

$$\boldsymbol{u_s}.\boldsymbol{\nu} = 0. \tag{13a}$$

$$(\boldsymbol{\sigma_s}\,\boldsymbol{\nu}).\boldsymbol{t} = 0. \tag{13b}$$

where $\boldsymbol{\nu}$ is the outward unit normal vector at the boundary and $\boldsymbol{t}$ denotes the tangent vector by which the cell flows around the obstacle.
The initial conditions are $\boldsymbol{u_s}(t=0) = \boldsymbol{u_{s0}}$ and $h(t=0) = h_0$.

1. For all $\boldsymbol{v_s} \in H_0^1(\Omega)$, multiplying the PDE equation (12) by $\boldsymbol{v_s}$ and taking integration over Ω, we can say that:
   $-\int_\Omega \left[\mathbf{div_s}\left[2h\Big(D_s(\boldsymbol{u_s}) + \mathrm{div_s}(\boldsymbol{u_s})I\Big)\right].\boldsymbol{v_s}\,dx + \int_\Omega \alpha\boldsymbol{u_s}.\boldsymbol{v_s}\,dx \;=\; \int_\Omega \boldsymbol{f_{as}}.\boldsymbol{v_s}\,dx.$
   By using Green's formula, we obtain:
   $\int_\Omega 2h\,D_s(\boldsymbol{u_s}) : \boldsymbol{\nabla_s}(\boldsymbol{v_s})\,dx + \int_\Omega 2h\,(\mathrm{div_s}(\boldsymbol{u_s})I) : \boldsymbol{\nabla_s}(\boldsymbol{v_s})\,dx + \int_\Omega \alpha\boldsymbol{u_s}.\boldsymbol{v_s}\,dx + \int_{\partial\Omega}\gamma_1(\boldsymbol{u_s})\gamma_0(\boldsymbol{v_s})\,d\Gamma = \int_\Omega \boldsymbol{f_{as}}.\boldsymbol{v_s}\,dx.$
   Now, $\int_{\partial\Omega}\gamma_1(\boldsymbol{u_s})\gamma_0(\boldsymbol{v_s})\,d\Gamma = 0$ at the boundary conditions.
   So, we can say that:
   $$\int_\Omega 2h\,D_s(\boldsymbol{u_s}) : \boldsymbol{\nabla_s}(\boldsymbol{v_s})dx + \int_\Omega 2h\,(\mathrm{div_s}(\boldsymbol{u_s})I) : \boldsymbol{\nabla_s}(\boldsymbol{v_s})dx + \int_\Omega \alpha\boldsymbol{u_s}.\boldsymbol{v_s}dx = \int_\Omega \boldsymbol{f_{as}}.\boldsymbol{v_s}dx. \tag{14}$$
   For any matrices A, $B \in \mathbb{R}^{2\times 2}$, where A is symmetric ($A = A^T$), we know that:
   $A : B = \sum_{i,j} A_{i,j}\,B_{i,j}$.
   $\Rightarrow A : B^T = \sum_{i,j} A_{i,j}\,B_{j,i}$.
   $\Rightarrow A : B^T = \sum_{i,j} A_{j,i}\,B_{j,i} = A : B$.
   So, $A : B = \; A : \left(\frac{B+B^T}{2}\right)$.
   In the above equation, $D_s(\boldsymbol{u_s})$ is symmetric, So we can say that: $D_s(\boldsymbol{u_s}) : \boldsymbol{\nabla_s}(\boldsymbol{v_s}) = D_s(\boldsymbol{u_s}) : \frac{\boldsymbol{\nabla_s}(\boldsymbol{v_s})+\boldsymbol{\nabla_s}(\boldsymbol{v_s})^T}{2}$.
   As $\frac{\boldsymbol{\nabla_s}(\boldsymbol{v_s})+\boldsymbol{\nabla_s}(\boldsymbol{v_s})^T}{2} = D_s(\boldsymbol{v_s})$, hence $D_s(\boldsymbol{u_s}) : \boldsymbol{\nabla_s}(\boldsymbol{v_s}) = D_s(\boldsymbol{u_s}) :\, D_s(\boldsymbol{v_s})$.
   Similarly, for $(\mathrm{div_s}(\boldsymbol{u_s})I) : \boldsymbol{\nabla_s}(\boldsymbol{v_s})$, where $(\mathrm{div_s}(\boldsymbol{u_s})I)$ is symmentric, we can say that:
   $(\mathrm{div_s}(\boldsymbol{u_s})I) : \boldsymbol{\nabla_s}(\boldsymbol{v_s}) = (\mathrm{div_s}(\boldsymbol{u_s})I) : D_s(\boldsymbol{v_s}) = \mathrm{div_s}(\boldsymbol{u_s}).\Big(I : D_s(\boldsymbol{v_s})\Big)$.
   Now, $I : D_s(\boldsymbol{v_s}) = \mathrm{div_s}(\boldsymbol{v_s})$.
   Hence, $(\mathrm{div_s}(\boldsymbol{u_s})I) : \boldsymbol{\nabla_s}(\boldsymbol{v_s}) = \mathrm{div_s}(\boldsymbol{u_s}).\mathrm{div_s}(\boldsymbol{v_s})$.
   So, From equation (14), we can say that the variational formulation is:
   $$\int_\Omega \left[2h\left(D_s(\boldsymbol{u_s}) : D_s(\boldsymbol{v_s}) + \mathrm{div_s}(\boldsymbol{u_s}).\mathrm{div_s}(\boldsymbol{v_s})\right) + \alpha\,\boldsymbol{u_s}.\boldsymbol{v_s}\right]dx = \int_\Omega \boldsymbol{f_{as}}.\boldsymbol{v_s}dx. \tag{15}$$
   We can rewrite the variational formulation in a standard form:
   Let $V \subset H_0^1(\Omega)$, a Hilbert space. We have a bilinear form on $V \times V$,
   $a(\boldsymbol{u_s},\,\boldsymbol{v_s}) = \int_\Omega \left[2h\Big(\,D_s(\boldsymbol{u_s}) : D_s(\boldsymbol{v_s}) + \mathrm{div_s}(\boldsymbol{u_s})\,\mathrm{div_s}(\boldsymbol{v_s})\Big) + \alpha\,\boldsymbol{u_s}.\boldsymbol{v_s}\right]dx.$
   and a linear form on V,
   $l(\boldsymbol{v_s}) = \int_\Omega \boldsymbol{f_{as}}.\boldsymbol{v_s}\,dx.$
   Hence, $\forall \boldsymbol{v_s} \in V$, $a(\boldsymbol{u_s},\,\boldsymbol{v_s}) = l(\boldsymbol{v_s})$.
   The discontinuous finite element space is defined by:
   $V_h = \{\boldsymbol{v_{sh}} \in L^2(\Omega); \boldsymbol{v_{sh|K}} \in P_k,\ \forall\ K \in T_h\}$, where $k \geq 0$ is the polynomial degree.
   This leads to a discrete version a_h of the bilinear form a, defined $\forall\ \boldsymbol{u_{sh}},\ \boldsymbol{v_{sh}}\ \in V_h$.
   So finally, the discrete variational formulation writes:
   Find $\boldsymbol{u_{sh}}\ \in V_h$ such that:
   $a_h(\boldsymbol{u_{sh}},\,\boldsymbol{v_{sh}}) \;=\; l_h(\boldsymbol{v_{sh}}),\quad \forall\ \boldsymbol{v_{sh}}\ \in V_h.$

2. Using equation (9), we can say that:
   $\partial_t h + \mathrm{div_s}\big(\boldsymbol{u_s}h\big) = 0.$

$\Rightarrow \partial_t h + (\boldsymbol{u_s}.\boldsymbol{\nabla_s})h + h\,\mathrm{div_s}(\boldsymbol{u_s}) = 0.$
Hence, by approximating with respect to time, we get:
$\frac{h^{n+1}-h^n}{\Delta t} + (\boldsymbol{u_s}^n.\boldsymbol{\nabla_s})h^{n+1} + h^{n+1}\,\mathrm{div_s}(\boldsymbol{u_s}^n) = 0$
where h^n is known and we need to find $h^{n+1} \in H^1_0$.
$\Rightarrow \frac{h^{n+1}}{\Delta t} + (\boldsymbol{u_s}^n.\boldsymbol{\nabla_s})h^{n+1} + h^{n+1}\,\mathrm{div_s}(\boldsymbol{u_s}^n) = \frac{h^n}{\Delta t}.$
So, finally,

$$h^{n+1}\Big(\frac{1}{\Delta t} + \mathrm{div_s}(\boldsymbol{u_s}^n)\Big) + (\boldsymbol{u_s}^n.\boldsymbol{\nabla_s})h^{n+1} = \frac{h^n}{\Delta t}. \tag{16}$$

Let $\boldsymbol{u_s} \in W^{1,\infty}(\Omega)^d$ and introduce the space: $X = \{\varphi \in L^2(\Omega); (\boldsymbol{u_s}.\boldsymbol{\nabla_s})\varphi \in L^2(\Omega)^d\}$ and, for all $h,\ \varphi \in X$
$c(h,\varphi) = \int_\Omega \big(\boldsymbol{u_s}.\boldsymbol{\nabla_s} h\,\varphi + (\frac{1}{\Delta t} + \mathrm{div_s}(\boldsymbol{u_s}))\big)h\varphi\,dx + \int_{\partial\Omega} max(0, -\boldsymbol{u_s}.\boldsymbol{n})h\varphi\,ds.$
$l(\varphi) = \int_\Omega \frac{h}{\Delta t}\varphi\,dx$. Then, the variational formulation writes:
(F V): find $h \in X$ such that
$c(h,\varphi) = l(\varphi), \quad \forall\varphi \in X.$
The term $max(0, -\boldsymbol{u_s}.\boldsymbol{n}) = \frac{(|\boldsymbol{u_s}.\boldsymbol{n}| - \boldsymbol{u_s}.\boldsymbol{n})}{2}$ is positive and vanishes everywhere except on $\partial\Omega_-$.
The discontinuous finite element space is defined by:
$X_h = \{\varphi_h \in L^2(\Omega); \varphi_{h|K} \in P_k,\ \forall\, K \in T_h\}$, where $k \geq 0$ is the polynomial degree.
$X_h \not\subset X$ and that the $\boldsymbol{\nabla_s} h_h$ term has no more sense for discontinous functions $h_h \in X_h$.
We can introduce the broken gradient $\boldsymbol{\nabla_{sh}}$ as a convenient notation.
$(\boldsymbol{\nabla_{sh}} h_h)_{|K} = \boldsymbol{\nabla_s}(h_{h|K}), \quad \forall K \in T_h.$
Thus,

$$\int_\Omega \boldsymbol{u_s}.\boldsymbol{\nabla_{sh}} h_h\,\varphi_h dx = \sum_{K\in T_h}\int_K \boldsymbol{u_s}.\boldsymbol{\nabla_{sh}} h_h\,\varphi_h dx \ \ \forall\ h_h.\varphi_h \in X_h.$$

This leads to a discrete version c_h of the bilinear form c, defined for all h_h , $\varphi_h \in X_h$ by

$$c_h(h_h,\varphi_h) = \int_\Omega \big(\boldsymbol{u_s}.\boldsymbol{\nabla_{sh}} h_h\,\varphi_h + (\frac{1}{\Delta t} + \mathrm{div_{sh}}(\boldsymbol{u_s}))\big)h_h\,\varphi_h\,dx + \int_{\partial\Omega} max(0, -\boldsymbol{u_s}.\boldsymbol{n})h_h\varphi_h ds$$

$$+ \sum_{S\in\mathbb{S}_h^{(i)}}\int_s (-\boldsymbol{u_s}.\boldsymbol{n}\,[h_h]\{\varphi_h\} + \frac{\delta}{2}|\boldsymbol{u_s}.\boldsymbol{n}|[h_h][\varphi_h])ds.$$

The last term involves a sum over $\mathbb{S}_h^i$, the set of internal sides of the mesh T_h. Each internal side $S \in \mathbb{S}_h^i$ has two possible orientations: one is choosen definitively. Let $\boldsymbol{n}$ be the normal to the oriented side S: as S is an internal side, there exists two elements K_- and K_+ such that $S = \partial K_- \cap \partial K_+$ and $\boldsymbol{n}$ is the outward unit normal of K_- on $\partial K_- \cap S$ and the inward unit normal of K_+ on $\partial K_+ \cap S$. For all $h_h \in X_h$, where h_h is in general discontinuous accross the internal side S. We define on S the inner value $h_h^- = h_{h|K_-}$ of h as the restriction $h_{h|K_-}$ of h_h in K_- along $\partial K_- \cap S$. Conversely, we define the outer value $h_h^+ = h_{h|K_+}$. We also denote on S the jump $[h_h] = h_h^- - h_h^+$ and the average $\{h_h\} = \frac{(h_h^- + h_h^+)}{2}$. The last term in the definition of c_h is ponderated by a coefficient $\delta \geq 0$. Choosing $\delta = 0$ correspond to the so-called centered flux approximation, while $\delta = 1$ is the upwinding flux approximation. The case $\delta = 1$

and $k = 0$ (piecewise constant approximation) leads to the popular upwinding finite volume scheme. Finally, the discrete variational formulation writes:
$(FV)_h$: find $h_h \in X_h$ such that
$c_h(h_h, \varphi_h) = l(\varphi_h) \ \ \forall \varphi_h \in X_h$.

4.2 Results

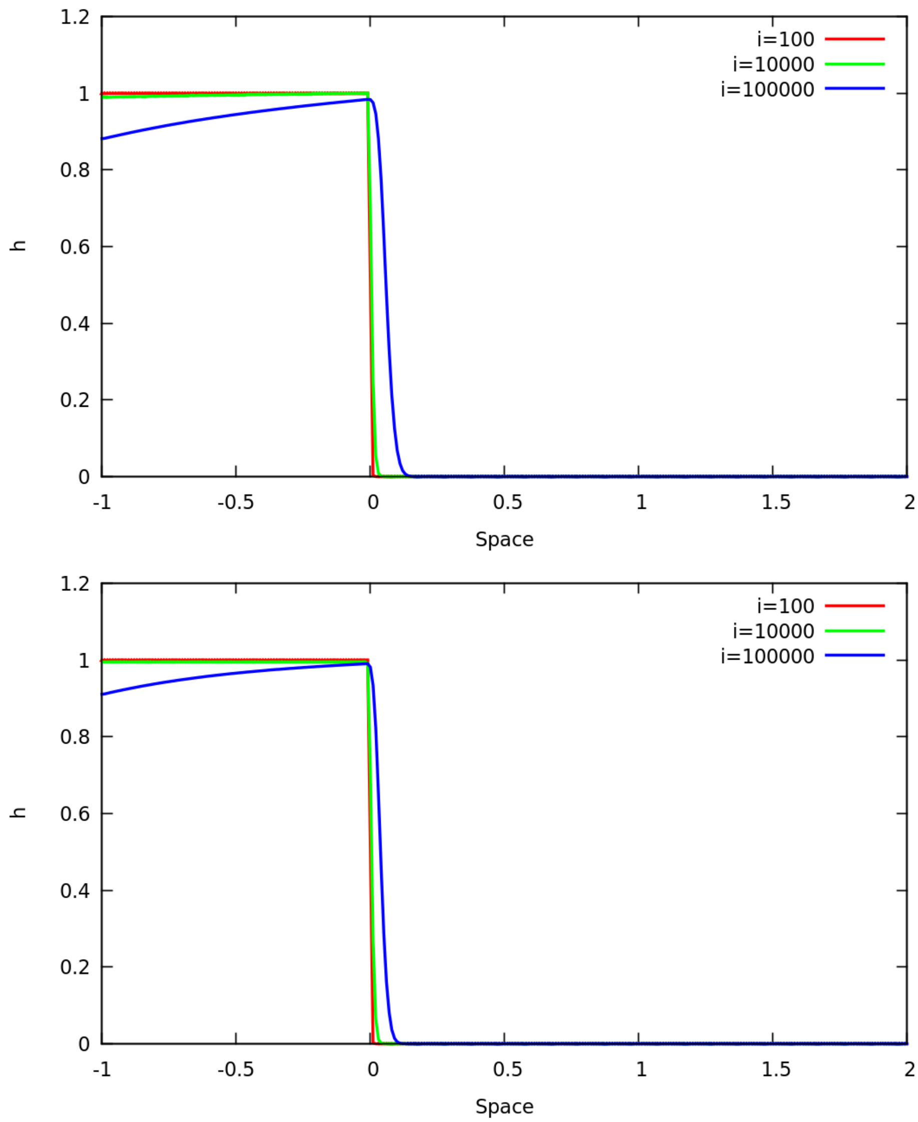


Figure 2: Figures show the change of the profile depth h for different time steps (100, 10000, 100000). Figure on top has the friction coefficient $\alpha = 5$ and figure in bottom has the friction coefficient $\alpha = 10$. The front is moving forward with timesteps.

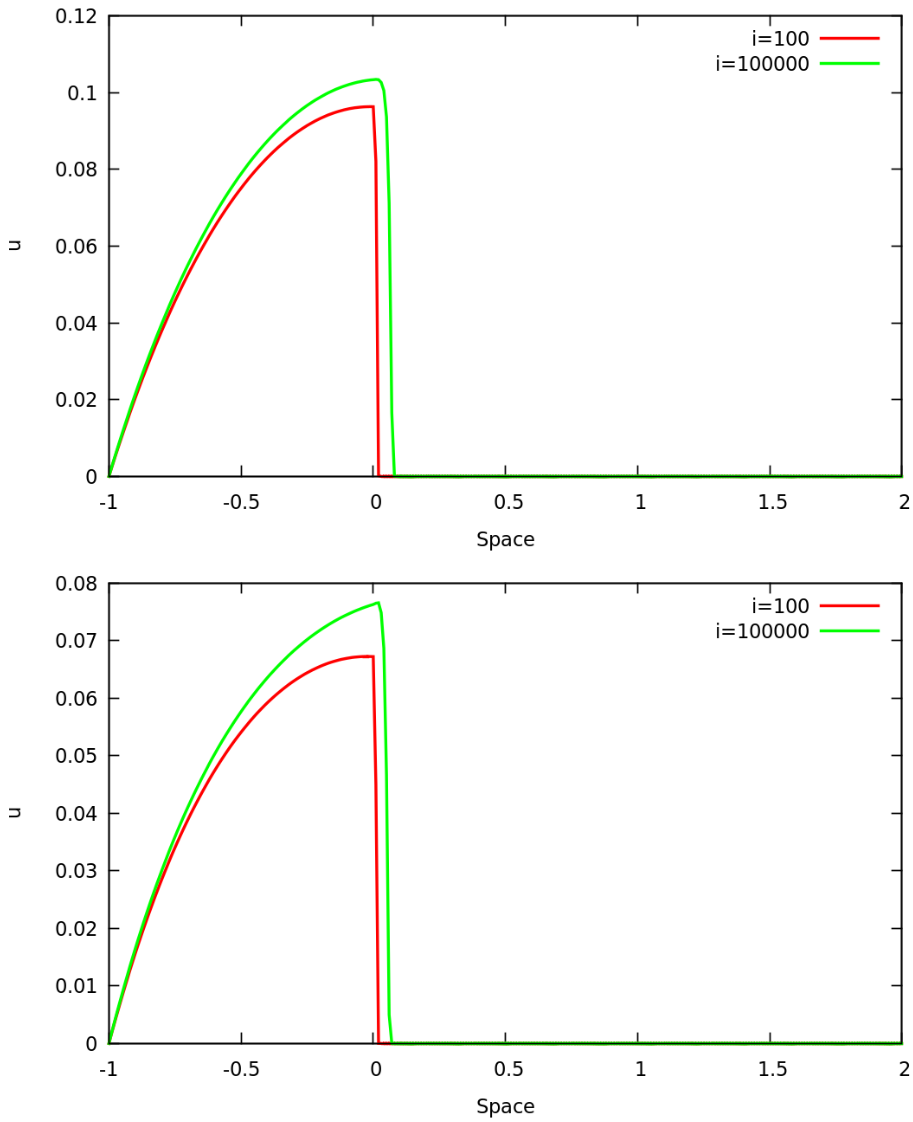


Figure 3: Figures show the change of the profile planar velocity u for different time steps (100 and 100000). Figure on top has the friction coefficient $\alpha = 5$ and figure in bottom has the friction coefficient $\alpha = 10$. The front is moving forward with timesteps.

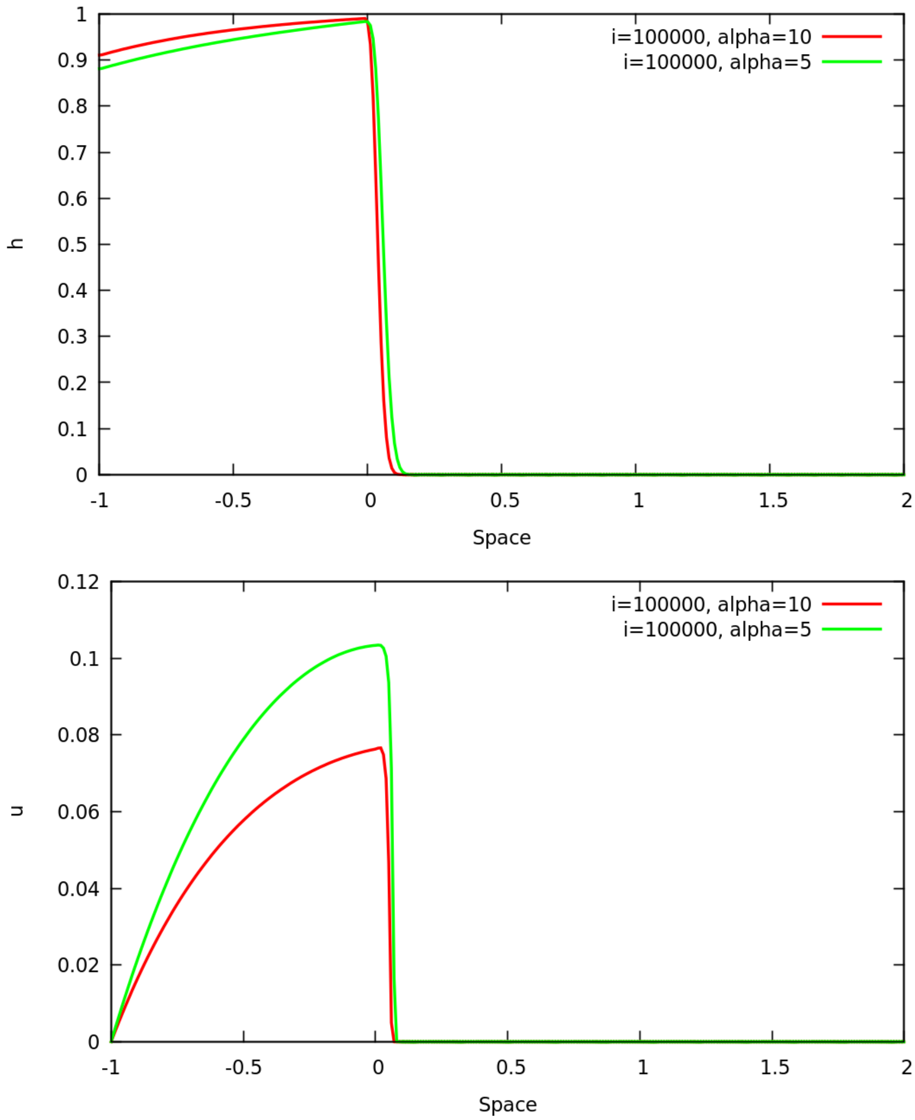


Figure 4: Figure on top compares the depth h for friction coefficient 5 and 10 at a particular time step (100000). Figure on bottom compares the planar velocity u for friction coefficient 5 and 10 at a particular time step (100000)

4.2.0.1 Discussion of results

The initial conditions used to obtain the numerical results are:
$h(t = 0) = h_0$, in Ω
$\boldsymbol{u}(t = 0) = \boldsymbol{u_0} = 0$ in $\Lambda(0)$
The numerical results are derived for the special case $We = 0$, the case for $We \neq 0$ would be arduous and numerically much more complex. So, we rest this case. The function for active force is chosen inversely proportional to depth ($\boldsymbol{f_{as}} \propto \frac{1}{h}$). As, we can see in the numerical results the planar velocity u rises upto a certain point and then finally falls to zero with increasing time period also the front is moving forward by increasing the timesteps. The peak velocity of u depends on the friction coefficient α. If α is high, the peak velocity of u is low because friction reduces planar velocity and make the movement slow (see Figs, 3 and 4(bottom) for comparision). The profile depth h shows an upward inclination of collective cells on top with increasing time period and then come to zero after certain time. We can see that h decreases for increasing timesteps. Depth h is high for higher values of friction coefficient at every timestep and the front is ahead when friction coefficient is low (see Figs, 2 and 4(top) for comparision). Collective cells get advance movement with increasing time. Due to the migration, the depth of cells change. After, a certain time period the migration is stopped. As, the process is happening continuously, we get a moving front at every progressing time step and also comparatively lower depth of cells at the commencement. If I could add more timesteps then accordingly the front would have moved further in the space and will generate better results. Same explanation for planar velocity (u), it changes with progressing timesteps and finally becomes zero when migration is stopped.

4.2.0.2 Code

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
Float h0 (const point& x) { return (x[0] <= 0) ? 1 : 0; }
point f (const Float& h, const point& grad_h) {
 Float norm_grad_h = norm(grad_h);
 if (h < 0.1 || norm_grad_h < 1e-15) {
   return point(0,0);
 } else {

   return (1/h)*point(1,0);
 }
}
int main(int argc, char**argv) {
  environment rheolef (argc, argv);
  geo omega (argv[1]);
  Float alpha  = (argc > 2) ? atof(argv[2]) : 10;
  Float tf     = (argc > 3) ? atof(argv[3]) : 0.1;
  size_t n_max = 100000;
```

```
size_t k_max = 10;
Float tol = 1e-7;
derr << "# alpha = " << alpha << endl
     << "# tf    = " << tf << endl;
Float delta_t = tf/n_max;

quadrature_option qopt;
qopt.set_family ("gauss_lobatto");
qopt.set_order  (1);
space Vh (omega, "P1","vector");
Vh[0].block ("left");
Vh[0].block ("right");
if (omega.dimension() == 2) {
  Vh[1].block ("top");
  Vh[1].block ("bottom");
}
trial u (Vh); test v (Vh);
space Hh (omega, "P0");
trial phi (Hh); test psi (Hh);
space Xh (omega, "P1","vector");
trial uu (Xh); test vv (Xh);
form muu = integrate (dot(uu,vv), qopt);
solver smuu (muu.uu());
Float t = 0;
field h = interpolate(Hh, h0);
field uh (Vh, 0);
field gh (Xh, 0);
branch even_h("t","h");
//branch even_u("t","u","g");
branch even_u("t","u");
odiststream out_h ("h.branch");
odiststream out_u ("u.branch");
out_h << even_h(t,h);
//out_u << even_u(t,field(uh[0]),field(gh[0]));
out_u << even_u(t,field(uh[0]));
t = delta_t;
for (size_t n = 1; n <= n_max; ++n, t += delta_t) {
  field h_prev_n = h;
  derr << "# n = " << n << " t = " << t << endl
       << "# k rel_err_l2 er_rel_linf" << endl;
  for (size_t k = 0; k < k_max; ++k) {
    field h_prev_k = h;
    field uh_prev_k = uh;
    // 1) viscous problem
    form ah = integrate (h_prev_k*(2*ddot(D(u),D(v))+div(u)*div(v)) + alpha*dot(u,v),
```

```
      field zh = integrate (-h*div(vv))
               + integrate ("boundary", h*dot(vv,normal()));
      gh.set_u() = smuu.solve (zh.u());
      field lh = integrate (dot(compose(f,h,gh), v), qopt);
      solver sah (ah.uu());
      uh.set_u() = sah.solve(lh.u());
      // 2) transport problem
      form ch = integrate (dot(uh,grad_h(phi))*psi + (1/delta_t + div(uh))*phi*psi)
              + integrate ("boundary", max(0, -dot(uh,normal()))*phi*psi)
              + integrate ("internal_sides",
       - dot(uh,normal())*jump(phi)*average(psi)
                    + 0.5*abs(dot(uh,normal()))*jump(phi)*jump(psi));
      field kh = integrate ((1/delta_t)*h_prev_n*psi);
      solver sch (ch.uu());
      h.set_u() = sch.solve(kh.u());
      Float rel_err_linf = field(h - h_prev_k).max_abs()
                         + field(uh - uh_prev_k).max_abs();
      Float rel_err_l2 = sqrt(integrate(omega,sqr(h - h_prev_k),qopt)
                           + integrate(omega,norm2(uh - uh_prev_k),qopt));
      derr << k << " " << rel_err_l2 << " " << rel_err_linf << endl;
      if (rel_err_l2 < tol) break;
    }
    out_h << even_h(t,h);
    //out_u << even_u(t,field(uh[0]),field(gh[0]));
    out_u << even_u(t,field(uh[0]));
  }
}
```

5 Conclusion

In this report, we have mathematically analysed the collective cell migration. Initially, the problem is defined by the 3D Oldroyd model because it is widely used models for viscous flows. Further, we have used a reduced model to show the motion of thin layers of viscoelastic fluids below a free surface and over a flat surface. The flow is driven by an external force. More specifically, we have shown that for the given boundary conditions and under scaling choices (see 3.3.1), the solution of the incompressible Oldroyd system of equations can be approximated by the solutions to the reduced model (see 3.5.2) in some asymptotic region. We think that our asymptotic regime is physically meaningful, and model make sense in the domain of modelling collective cell migration as viscoelastic flows. The variational formulation is done on the basis of reduced closed system of equations. We consider the special case $Weissenberg(We) = 0$ for variational formulation and genrated the discontinuous finite element space to obtain numerical results, albeit rather non zero Weissenberg number could show the comparable contributions of both elastic and viscous effects in the process of collective cell migration. In the final results, we have explored the behavior of parameters h

(layer depth) and $\boldsymbol{u_s}$ (planar velocity) under some certain values of friction coefficient (α). Due to lack of knowledge, there is flexibility in choosing the active force $\boldsymbol{f_{as}}$ as a suitable smooth function to fit for numerical approximation. According to my knowledge, the project is a pioneering work and I am glad to take part in this topic. I give my effort in the research project but I know it is not enough to obtain the summit of its pedagogical studies. There is still a lot to explore, learn and obtain. I believe that the project will open doors for further research work in future.

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6 Appendix

6.1 Expansion

1. Expanding the equation (1a), we get:
   $\partial_x u_x + \partial_y u_y + \partial_z u_z = 0.$

2. Expanding the equation (1b), we get:

   The xx, xy, yy, xz, yz, zz components are:
   $\sigma_{xx} = -p + \tau_{xx} + 2\eta_0\,\partial_x u_x$
   $\sigma_{xy} = \tau_{xy} + \eta_0\,(\partial_x u_y + \partial_y u_x)$
   $\sigma_{yy} = -p + \tau_{yy} + 2\eta_0\,\partial_y u_y$
   $\sigma_{xz} = \tau_{xz} + \eta_0\,(\partial_x u_z + \partial_z u_x)$
   $\sigma_{yz} = \tau_{yz} + \eta_0\,(\partial_y u_z + \partial_z u_y)$
   $\sigma_{zz} = -p + \tau_{zz} + 2\eta_0\,\partial_z u_z$

3. Expanding the equation (1c), we get:

   (a) The x component is:
   $\rho\Big(\partial_t u_x + u_x\partial_x\, u_x + u_y\partial_y\, u_x + u_z\partial_z\, u_x\Big) - \partial_x\sigma_{xx} - \partial_y\sigma_{xy} - \partial_z\sigma_{xz} = 0.$

   (b) The y component is:
   $\rho\Big(\partial_t u_y + u_x\partial_x\, u_y + u_y\partial_y\, u_y + u_z\partial_z\, u_y\Big) - \partial_x\sigma_{xy} - \partial_y\sigma_{yy} - \partial_z\sigma_{yz} = 0.$

   (c) The z component is:
   $\rho\Big(\partial_t u_z + u_x\partial_x u_z + u_y\partial_y u_z + u_z\partial_z u_z\Big) - \partial_x\sigma_{xz} - \partial_y\sigma_{yz} - \partial_z\sigma_{zz} = 0.$

4. Expanding the equation (1d), we get:

   (a) The xx component is:
   $\lambda\Big(\partial_t\tau_{xx} + u_x\partial_x\tau_{xx} + u_y\partial_y\tau_{xx} + u_z\partial_z\tau_{xx} - 2\partial_x u_x\ \tau_{xx} - 2\partial_y u_x\ \tau_{xy} - 2\partial_z u_x\ \tau_{xz}\Big) + \tau_{xx}$
   $= 2\eta_p\,\partial_x u_x.$

   (b) The xy component is:
   $\lambda\Big(\partial_t\tau_{xy} + u_x\partial_x\tau_{xy} + u_y\partial_y\tau_{xy} + u_z\partial_z\tau_{xy} - \partial_x u_x\tau_{xy} - \partial_y u_x\tau_{yy} - \partial_z u_x\tau_{yz} - \partial_x u_y\tau_{xx}$
   $- \partial_y u_y\tau_{xy} - \partial_z u_y\tau_{xz}\Big) + \tau_{xy} = \eta_p\,(\partial_x u_y + \partial_y u_x).$

(c) The yy component is:
$$\lambda\Big(\partial_t\tau_{yy} + u_x\partial_x\tau_{yy} + u_y\partial_y\tau_{yy} + u_z\partial_z\tau_{yy} - 2\,\partial_x u_y\tau_{xy} - 2\,\partial_y u_y\tau_{yy} - 2\,\partial_z u_y\tau_{yz}\Big) + \tau_{yy}$$
$$= \eta_p\,(\partial_x u_y + \partial_y u_x).$$

(d) The xz component is:
$$\lambda\Big(\partial_t\tau_{xz} + u_x\partial_x\tau_{xz} + u_y\partial_y\tau_{xz} + u_z\partial_z\tau_{xz} - \partial_x u_x\tau_{xz} - \partial_y u_x\tau_{yz} - \partial_z u_x\tau_{zz} - \partial_x u_z\tau_{xx}$$
$$- \partial_y u_z\tau_{xy} - \partial_z u_z\tau_{xz}\Big) + \tau_{xz} = \eta_p\,(\partial_x u_z + \partial_z u_x).$$

(e) The yz component is:
$$\lambda\Big(\partial_t\tau_{yz} + u_x\partial_x\tau_{yz} + u_y\partial_y\tau_{yz} + u_z\partial_z\tau_{yz} - \partial_x u_y\tau_{xz} - \partial_y u_y\tau_{yz} - \partial_z u_y\tau_{zz} - \partial_x u_z\tau_{xy}$$
$$- \partial_y u_z\tau_{yy} - \partial_z u_z\tau_{yz}\Big) + \tau_{yz} = \eta_p\,(\partial_z u_y + \partial_y u_z).$$

(f) The zz component is:
$$\lambda\Big(\partial_t\tau_{zz} + u_x\partial_x\tau_{zz} + u_y\partial_y\tau_{zz} + u_z\partial_z\tau_{zz} - 2\partial_x u_z\,\tau_{xz} - 2\partial_y u_z\,\tau_{yz} - 2\partial_z u_z\,\tau_{zz}\Big) + \tau_{zz}$$
$$= 2\eta_p\,\partial_z u_z.$$

5. Expanding equation (1f), we get:
$$\boldsymbol{\sigma}\,\boldsymbol{n} = 0 \Rightarrow \begin{pmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}\end{pmatrix} . \begin{pmatrix}-\partial_x h\\ -\partial_y h\\ 1\end{pmatrix} = 0.$$
$$\Rightarrow \left.\begin{aligned} -\sigma_{xx}\partial_x h - \sigma_{xy}\partial_y h + \sigma_{xz} = 0.\\ -\sigma_{xy}\partial_x h - \sigma_{yy}\partial_y h + \sigma_{yz} = 0.\\ -\sigma_{xz}\partial_x h - \sigma_{yz}\partial_y h + \sigma_{zz} = 0.\end{aligned}\right\}\text{ at z=h.}$$

6. Expanding equation (1g), we get:
$\boldsymbol{u}.(0,0,-1) = 0 \Rightarrow u_z = 0$ at $z = 0$.

7. Expanding equation (1h), we get:
$\sigma_{nt} + c_f\,u_t = \boldsymbol{f_a} \Rightarrow \boldsymbol{\sigma}\,\boldsymbol{n} - \big((\boldsymbol{\sigma}\,\boldsymbol{n}).\boldsymbol{n}\big)\boldsymbol{n} + c_f\,u_t = \boldsymbol{f_a} \Rightarrow \boldsymbol{\sigma}\,\boldsymbol{n} = \big((\boldsymbol{\sigma}\,\boldsymbol{n}).\boldsymbol{n}\big)\boldsymbol{n} - c_f\,u_t + \boldsymbol{f_a}$.
The LHS is:
$$\boldsymbol{\sigma n} = \begin{pmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}(t,x,z)\\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}\end{pmatrix}\begin{pmatrix}0\\0\\-1\end{pmatrix} = \begin{pmatrix}-\sigma_{xz}\\ -\sigma_{yz}\\ -\sigma_{zz}\end{pmatrix}.$$
From the RHS, we can say that:
$$\Big((\boldsymbol{\sigma n}).\boldsymbol{n}\Big)\boldsymbol{n} - c_f\boldsymbol{u}_t + \boldsymbol{f}_a$$
$$= \begin{pmatrix}-\sigma_{xz}\\ -\sigma_{yz}\\ -\sigma_{zz}\end{pmatrix} . \begin{pmatrix}0\\0\\-1\end{pmatrix} n - \begin{pmatrix}c_f\,u_x\\ c_f\,u_y\\ 0\end{pmatrix} + \begin{pmatrix}f_{ax}\\ f_{ay}\\ 0\end{pmatrix}$$
$$= \sigma_{zz}\begin{pmatrix}0\\0\\-1\end{pmatrix} - \begin{pmatrix}c_f\,u_x\\ c_f\,u_y\\ 0\end{pmatrix} + \begin{pmatrix}f_{ax}\\ f_{ay}\\ 0\end{pmatrix}$$
$$= \begin{pmatrix}-c_f\,u_x + f_{ax}\\ -c_f\,u_y + f_{ay}\\ -\sigma_{zz}\end{pmatrix}.$$

From LHS=RHS, we can say that:
$\sigma_{xz} = c_f u_x - f_{ax} \Rightarrow \sigma_{xz} - c_f u_x = -f_{ax}$ at $z = 0$.
and $\sigma_{yz} = c_f u_y - f_{ay} \Rightarrow \sigma_{yz} - c_f u_y = -f_{ay}$ at $z = 0$.

Where, the entire problem is closed by the initial conditions for h, $\boldsymbol{u}$, $\boldsymbol{\tau}$:
$h(t=0) = h_0$, in Ω
$\boldsymbol{u}(t=0) = \boldsymbol{u_0}$ in $\Lambda(0)$
$\boldsymbol{\tau}(t=0) = \boldsymbol{\tau_0}$ in $\Lambda(0)$.

6.2 Change of form

- The equation of mass conservation is:
  $\operatorname{div} \boldsymbol{u} = 0 \Rightarrow \partial_x u_x + \partial_y u_y + \partial_z u_z = 0 \Rightarrow \operatorname{div}_s \boldsymbol{u_s} + \partial_z u_z = 0$.

- Changing the form of equations for conservation of momentum are are shown below:
  Combininig first two equations from 3, we get:
  $\partial_t \begin{pmatrix} u_x \\ u_y \end{pmatrix} + (u_x\partial_x + u_y\partial_y) \begin{pmatrix} u_x \\ u_y \end{pmatrix} + u_z\partial_z \begin{pmatrix} u_x \\ u_y \end{pmatrix} - \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} . \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix} - \partial_z \begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \end{pmatrix} = 0.$
  Hence, we can say that:
  $\rho\Big(\partial_t u_s + (\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z)\, \boldsymbol{u_s}\Big) - \boldsymbol{\nabla_s}.\boldsymbol{\sigma_s} - \partial_z \boldsymbol{\sigma_{sz}} = 0.$
  $\Rightarrow \rho\Big(\partial_t u_s + (\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z)\, \boldsymbol{u_s}\Big) - \mathbf{div_s}\, \boldsymbol{\sigma_s} - \partial_z \boldsymbol{\sigma_{sz}} = 0.$
  Considering the last equation from 3, we get:
  $\rho\Big(\partial_t u_z + (u_x\partial_x + u_y\partial_y)u_z + u_z\partial_z u_z\Big) - \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} . \begin{pmatrix} \sigma_{xz} & \sigma_{yz} \end{pmatrix} - \partial_z \sigma_{zz} = 0.$
  $\Rightarrow \rho\Big(\partial_t u_z + (\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\, \partial_z) u_z\Big) - \operatorname{div}_s \boldsymbol{\sigma_{sz}}^T - \partial_z\, \sigma_{zz} = 0.$

- We know that $\boldsymbol{\nabla u} = \begin{pmatrix} \partial_x u_x & \partial_y u_x & \partial_z u_x \\ \partial_x u_y & \partial_y u_y & \partial_z u_y \\ \partial_x u_z & \partial_y u_z & \partial_z u_z \end{pmatrix} = \begin{pmatrix} \boldsymbol{\nabla_s u_s} & \partial_z \boldsymbol{u_s} \\ (\boldsymbol{\nabla_s} u_z)^T & \partial_z u_z \end{pmatrix}.$
  Hence, $\boldsymbol{\nabla u}^T = \begin{pmatrix} (\boldsymbol{\nabla_s u_s})^T & \boldsymbol{\nabla_s} u_z \\ \partial_z \boldsymbol{u_s}^T & \partial_z u_z \end{pmatrix}.$
  So, $\boldsymbol{\nabla u} + \boldsymbol{\nabla u}^T = 2D(\boldsymbol{u}) = \begin{pmatrix} \boldsymbol{\nabla_s u_s} + (\boldsymbol{\nabla_s u_s})^T & \partial_z \boldsymbol{u_s} + \boldsymbol{\nabla_s} u_z \\ (\boldsymbol{\nabla_s} u_z)^T + \partial_z \boldsymbol{u_s}^T & 2\partial_z u_z \end{pmatrix}.$

  So, $2\, D(\boldsymbol{u}) = \begin{pmatrix} 2\, D_s(\boldsymbol{u_s}) & \partial_z \boldsymbol{u_s} + \boldsymbol{\nabla_s} u_z \\ (\boldsymbol{\nabla_s} u_z)^T + \partial_z \boldsymbol{u_s}^T & 2\partial_z u_z \end{pmatrix}.$
  where $\boldsymbol{\nabla_s u_s} + (\boldsymbol{\nabla_s u_s})^T = 2\, D_s(\boldsymbol{u_s})$.
  The change of forms for constitutive equations after are shown below:
  Considering the first three equations from 2, (We are considering the second equation two times because of the symmetric nature of total stress tensor $\boldsymbol{\sigma}$), we can say that:
  $\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix} = -p \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{xy} & \tau_{yy} \end{pmatrix} + \eta_0 \begin{pmatrix} 2\partial_x u_x & \partial_y u_x + \partial_x u_y \\ \partial_y u_x + \partial_x u_y & 2\partial_y u_y \end{pmatrix}.$

$\Rightarrow \boldsymbol{\sigma_s} = -pI + \boldsymbol{\tau_s} + 2\eta_0\, D_s(\boldsymbol{u_s})$.
Considering the fourth and fifth equations from 2, we can say:
$\begin{pmatrix}\sigma_{xz}\\ \sigma_{yz}\end{pmatrix} = \begin{pmatrix}\tau_{xz}\\ \tau_{yz}\end{pmatrix} + \eta_0 \begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix} u_z + \eta_0 \partial_z \begin{pmatrix}u_x\\ u_y\end{pmatrix}$.
$\Rightarrow \boldsymbol{\sigma_{sz}} = \boldsymbol{\tau_{sz}} + \eta_0\,(\partial_z \boldsymbol{u_s} + \boldsymbol{\nabla_s} u_z)$.

Considering the first three equations from 4, (We are considering the second equation two times because of the symmetric nature of elastic tensor $\boldsymbol{\tau}$), we can say that:
$\lambda\Big[\partial_t \begin{pmatrix}\tau_{xx} & \tau_{xy}\\ \tau_{xy} & \tau_{yy}\end{pmatrix} + (u_x\partial_x + u_y\partial_y + u_z\partial_z)\begin{pmatrix}\tau_{xx} & \tau_{xy}\\ \tau_{xy} & \tau_{yy}\end{pmatrix} - \begin{pmatrix}\partial_x u_x & \partial_y u_x\\ \partial_x u_y & \partial_y u_y\end{pmatrix}\begin{pmatrix}\tau_{xx} & \tau_{xy}\\ \tau_{xy} & \tau_{yy}\end{pmatrix}$
$- \begin{pmatrix}\tau_{xx} & \tau_{xy}\\ \tau_{xy} & \tau_{yy}\end{pmatrix}\begin{pmatrix}\partial_x u_x & \partial_x u_y\\ \partial_y u_x & \partial_y u_y\end{pmatrix} - \partial_z \begin{pmatrix}u_x\\ u_y\end{pmatrix}\begin{pmatrix}\tau_{xz} & \tau_{yz}\end{pmatrix} - \begin{pmatrix}\tau_{xz}\\ \tau_{yz}\end{pmatrix}\partial_z\begin{pmatrix}u_x & u_y\end{pmatrix}\Big] + \begin{pmatrix}\tau_{xx} & \tau_{xy}\\ \tau_{xy} & \tau_{yy}\end{pmatrix}$
$= 2\,\eta_p\, D_s(\boldsymbol{u_s})$.
So, changing the form of equation, we can rewrite:
$\lambda\Big[\partial_t\boldsymbol{\tau_s} + \big(\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z\big)\boldsymbol{\tau_s} - (\boldsymbol{\nabla_s u_s})\boldsymbol{\tau_s} - \boldsymbol{\tau_s}(\boldsymbol{\nabla_s u_s})^T - (\partial_z\boldsymbol{u_s})\boldsymbol{\tau_{sz}}^T - \boldsymbol{\tau_{sz}}(\partial_z\boldsymbol{u_s})^T\Big]$
$+\boldsymbol{\tau_s} = 2\eta_p\, D_s(\boldsymbol{u_s})$.

Considering the fourth and fifth equations from 4, we can say:
$\lambda\Big[\partial_t\begin{pmatrix}\tau_{xz}\\ \tau_{yz}\end{pmatrix} + \big(u_x\partial_x + u_y\partial_y + u_z\partial_z\big)\begin{pmatrix}\tau_{xz}\\ \tau_{yz}\end{pmatrix} - \begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix}\begin{pmatrix}u_x\\ u_y\end{pmatrix} - \partial_z u_z\begin{pmatrix}\tau_{xz}\\ \tau_{yz}\end{pmatrix} - \partial_z\begin{pmatrix}u_x\\ u_y\end{pmatrix}\tau_{zz}$
$- \begin{pmatrix}\tau_{xx} & \tau_{xy}\\ \tau_{xy} & \tau_{yy}\end{pmatrix}\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix} u_z\Big] = \eta_p\big(\partial_z\begin{pmatrix}u_x\\ u_y\end{pmatrix} + \begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix} u_z\big) - \begin{pmatrix}\tau_{xz}\\ \tau_{yz}\end{pmatrix}$.
So, changing the form of the equation, we can rewrite:
$\lambda\Big[\partial_t\boldsymbol{\tau_{sz}} + \big(\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z\big)\boldsymbol{\tau_{sz}} - (\boldsymbol{\nabla_s u_s} + \partial_z u_z)\boldsymbol{\tau_{sz}} - (\partial_z\boldsymbol{u_s})\tau_{zz} - \boldsymbol{\tau_s}\boldsymbol{\nabla_s} u_z\Big] + \boldsymbol{\tau_{sz}}$
$= \eta_p\,\big(\partial_z\boldsymbol{u_s} + \boldsymbol{\nabla_s} u_z\big)$.

Considering the last equation from 4, we can say that:
$\lambda\Big[\partial_t\tau_{zz} + \big(u_x\partial_x + u_y\partial_y + u_z\partial_z\big)\tau_{zz} - 2(\partial_z u_z)\tau_{zz} - \begin{pmatrix}\partial_x & \partial_y\end{pmatrix} u_z . \begin{pmatrix}\tau_{xz}\\ \tau_{yz}\end{pmatrix} - \begin{pmatrix}\tau_{xz} & \tau_{yz}\end{pmatrix} . \begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix} u_z\Big]$
$= 2\eta_p\,\partial_z u_z - \tau_{zz}$.
So, changing the form of the equation, we can rewrite:
$\lambda\Big[\partial_t\tau_{zz} + \big(\boldsymbol{u_s}.\boldsymbol{\nabla_s} + u_z\partial_z\big)\tau_{zz} - 2(\partial_z u_z)\tau_{zz} - (\boldsymbol{\nabla_s} u_z)^T.\boldsymbol{\tau_{sz}} - \boldsymbol{\tau_{sz}}^T.(\boldsymbol{\nabla_s} u_z)\Big] + \tau_{zz} = 2\eta_p\,\partial_z u_z$.

- The kinematic condition at free surface ($z = h$) is:
  $\partial_t h + u_x\,\partial_x h + u_y\,\partial_y h - u_z = 0 \Rightarrow \partial_t h + (\boldsymbol{\nabla_s} h).\boldsymbol{u_s} - u_z = 0$.
- Change of forms for boundary conditions at free surface ($z = h$) are shown below:
  Considering the first two equations from 5, we can say that:
  $-\begin{pmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{xy} & \sigma_{yy}\end{pmatrix}\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix} h + \begin{pmatrix}\sigma_{xz}\\ \sigma_{yz}\end{pmatrix} = 0 \Rightarrow -\boldsymbol{\sigma_s}\,(\boldsymbol{\nabla_s} h) + \boldsymbol{\sigma_{sz}} = 0$.

  Considering the last equation from 5, we can say that:

$$-\begin{pmatrix}\sigma_{xz} & \sigma_{yz}\end{pmatrix}\begin{pmatrix}\partial_x \\ \partial_y\end{pmatrix} h + \sigma_z z = 0$$
$$\Rightarrow -\boldsymbol{\sigma_{sz}}^T\left(\boldsymbol{\nabla_s} h\right) + \sigma_{zz} = 0.$$

- Boundary condition at the bottom ($z = 0$) becomes:
  $u_z = 0$ and
  $\begin{pmatrix}\sigma_{xz} \\ \sigma_{yz}\end{pmatrix} - c_f\begin{pmatrix}u_x \\ u_y\end{pmatrix} = \begin{pmatrix}f_{ax} \\ f_{ay}\end{pmatrix}.$
  $\Rightarrow \boldsymbol{\sigma_{sz}} - c_f\boldsymbol{u_s} = -\boldsymbol{f_{as}}.$

Problem 1. We need to show that:
$\int_0^{h(t,x)}(\partial_x u_x(t,x,z)+\partial_z u_z(t,x,z))dz = \partial_t h(t,x)+\partial_x\int_0^{h(t,x)} u_x(t,x,z)dz = \partial_t h(t,x)+\partial_x[h(t,x)\hat{u}_x(t,x,z)] = 0$,
where $\hat{u}_x(t,x,z) = \frac{1}{h(t,x)}\int_0^{h(x,t)} u_x(t,x,z)dz$ and $\nabla.u(t,x,z) = 0$.
The boundary conditions are:

1. $u(t,x,z).n = 0$ when $z = 0$.
2. $\partial_t h(t,x) + u_x(t,x,z)\partial_x(h(t,x)) = u_z(t,x,z)$ when $z = h(t,x)$.

Proof. As *div* $u(t,x,z) = 0$, Hence, we can say that $\int_0^{h(t,x)}(\partial_x u_x(t,x,z)+\partial_z u_z(t,x,z))dz = 0$.
We use Leibniz's integral rule for differentiating the integral $\frac{\partial}{\partial x}\int_0^{h(t,x)} u_x(t,x,z)dz$.
So,

$$\frac{\partial}{\partial x}\int_0^{h(t,x)} u_x(t,x,z)dz = u_x(t,x,h(t,x))\partial_x(h(t,x)) - u_x(t,x,0)\partial_x(0) + \int_0^{h(t,x)}\partial_x u_x(t,x,z)dz$$

$$\Rightarrow \int_0^{h(t,x)}\partial_x u_x(t,x,z)dz = \partial_x\int_0^{h(t,x)} u_x(t,x,z)dz - u_x(t,x,h(t,x))\partial_x(h(t,x))..................(1)$$

Further,

$$\int_0^{h(t,x)}\partial_z u_z(t,x,z)dz = u_z(t,x,h(t,x)) - u_z(t,x,0)........(2)$$

From the boundary conditions, we can say that:

- $\partial_t h(t,x) + u_x(t,x,h(t,x))\partial_x(h(t,x)) = u_z(t,x,h(t,x))$
- $\big(u_x(t,x,0), u_z(t,x,0)\big).(n_x, n_z) = 0.$
  where $n_x = 0$ and $n_z = -1$

$$\Rightarrow u_z(t,x,0) = 0$$

Using the value of $u_z(t,x,h(t,x))$ and $u_z(t,x,0)$ in equation (2), we get:

$$\int_0^{h(t,x)} \partial_z u_z(t,x,z)dz = \partial_t h(t,x) + u_x(t,x,h(t,x))\partial_x(h(t,x))...............(3)$$

Summing equation (1) and equation (3), we will get:

$$\int_0^{h(t,x)} \partial_x u_x(t,x,z)dz + \int_0^{h(t,x)} \partial_z u_z(t,x,z)dz$$

$$= \partial_x \int_0^{h(t,x)} u_x(t,x,z)dz + \partial_t h(t,x) + u_x(t,x,h(t,x))\partial_x(h(t,x)) - u_x(t,x,h(x))\partial_x h(x)$$

$$\Rightarrow \int_0^{h(t,x)} \Big(\partial_x u_x(t,x,z) + \partial_z u_z(t,x,z)\Big)dz = \partial_t h(t,x) + \partial_x \int_0^{h(t,x)} u_x(t,x,z)dz.......(4)$$

As, $\hat{u}_x(t,x,z) = \frac{1}{h(t,x)} \int_0^{h(t,x)} u_x(t,x,z)dz$
Hence, $\int_0^{h(t,x)} u_x(t,x,z)dz = h(t,x)\hat{u}_x(t,x,z)$.
Using this value in equation (4), we get:

$$\int_0^{h(t,x)} \Big(\partial_x u_x(t,x,z) + \partial_z u_z(t,x,z)\Big)dz = \partial_t h(t,x) + \partial_x(h(t,x)\hat{u}_x(t,x,z)).............(5)$$

Comparing equation (4) and equation (5) and using the fact that $\int_0^{h(t,x)} \Big(\partial_x u_x(t,x,z) + \partial_z u_z(t,x,z)\Big)dz = 0$, we conclude that:

$$\int_0^{h(t,x)} \Big(\partial_x u_x(t,x,z) + \partial_z u_z(t,x,z)\Big)dz = \partial_t h(t,x) + \partial_x \int_0^{h(t,x)} u_x(t,x,z)dz$$

$$= \partial_t h(t,x) + \partial_x(h(t,x)\hat{u}_x(t,x,z)) = 0$$

□

Problem 2. We need to show that:

$$\rho\partial_t\Bigg(\int_0^{h(t,x)} u_x(t,x,z)dz\Bigg) + \partial_x\Bigg(\int_0^{h(t,x)} \Big(\rho u_x^2(t,x,z) + p(t,x,z) - \tau_{xx}(t,x,z)\Big)dz\Bigg) =$$

$$\int_0^{h(t,x)} f_x(t,x,z)dz - \tau_{xz}(t,x,0)$$

Where the boundary conditions are:

- $u(t,x,z).n = 0$ where $z = 0$ and $n = (0,-1)$. So, $u_z(t,x,0) = 0$.
- $\partial_t h(t,x) + u_x(t,x,z)\partial_x h(t,x) = u_z(t,x,z)$ when $z = h(t,x)$.

- $(p(t,x,z)I-\tau(t,x,z)).(-\partial_x h(t,x),1)=0$ when $z=h(t,x)$. As we know that $\tau(t,x,z)=\begin{pmatrix}\tau_{xx}(t,x,z) & \tau_{xz}(t,x,z)\\ \tau_{xz}(t,x,z) & \tau_{zz}(t,x,z)\end{pmatrix}$.
  Hence, $(p(t,x,z)I-\tau(t,x,z))=\begin{pmatrix}p(t,x,z)-\tau_{xx}(t,x,z) & -\tau_{xz}(t,x,z)\\ -\tau_{xz}(t,x,z) & p-\tau_{zz}(t,x,z)\end{pmatrix}$.
  $\therefore$ $(p(t,x,h)I-\tau(t,x,h)).(-\partial_x h(t,x),1)=0$ implies
  $(p(t,x,h)-\tau_{xx}(t,x,h))\partial_x h(t,x)-\tau_{xz}(t,x,h)=0$ (Considering the projection on $X-axis$)

Proof. Consider the equation of Conservation of momentum:

$$\rho\Big(\partial_t u(t,x,z)+\Big(u(t,x,z).\nabla\Big)u(t,x,z)\Big)=-\nabla p(t,x,z)+div\tau(t,x,z)+f(t,x,z)$$

Assuming the projection on $X-axis$ only, we get:

$$\rho\Big(\partial_t u_x(t,x,z)+u_x(t,x,z)\partial_x u_x(t,x,z)+u_z(t,x,z)\partial_z u_x(t,x,z)\Big)+\partial_x p(t,x,z)-(\partial_x\tau_{xx}(t,x,z)+\partial_z\tau_{xz}(t,x,z))$$

$$=f_x$$

$$\int_0^{h(t,x)}\Big(\partial_t u_x(t,x,z)+u_x(t,x,z)\partial_x u_x(t,x,z)+u_z(t,x,z)\partial_z u_x(t,x,z)\Big)dz$$

$$=\int_0^{h(t,x)}\Big(\partial_t u_x(t,x,z)+\partial_x u_x^2(t,x,z)+\partial_z\big(u_x(t,x,z)u_z(t,x,z)\big)\Big)dz..........(1)$$

$\Big($ As $\partial_x u_x^2(t,x,z)+\partial_z\big(u_x(t,x,z)u_z(t,x,z)\big)=u_x(t,x,z)\partial_x u_x(t,x,z)+u_x(t,x,z)\partial_x u_x(t,x,z)+u_x(t,x,z)\partial_z u_z(t,x,z)+u_z(t,x,z)\partial_z u_x(t,x,z)=u_x(t,x,z)\partial_x u_x(t,x,z)+u_z(t,x,z)\partial_z u_x(t,x,z)$, using the equation for conservation of mass $\partial_x u_x(t,x,z)+\partial_z u_z(t,x,z)=0$ $\Big)$.
Using Leibniz integral rule:

$$\partial_t\int_0^{h(t,x)}u_x(t,x,z)dz=\int_0^{h(t,x)}\partial_t u_x(t,x,z)dz+u_x(t,x,h)\partial_t h(t,x)$$

$$\Rightarrow\int_0^{h(t,x)}\partial_t u_x(t,x,z)dz=\partial_t\int_0^{h(t,x)}u_x(t,x,z)dz-u_x(t,x,h)\partial_t h(t,x)..........(2)$$

$$\partial_x\int_0^{h(t,x)}u_x^2(t,x,z)dz=\int_0^{h(t,x)}\partial_x u_x^2(t,x,z)dz+u_x^2(t,x,h)\partial_x h(t,x)$$

$$\Rightarrow\int_0^{h(t,x)}\partial_x u_x^2(t,x,z)dz=\partial_x\int_0^{h(t,x)}u_x^2(t,x,z)dz-u_x^2(t,x,h)\partial_x h(t,x)..........(3)$$

$$\int_0^{h(t,x)}\partial_z\big(u_x(t,x,z)u_z(t,x,z)\big)dz=u_x(t,x,h)u_z(t,x,h)-u_x(t,x,0)u_z(t,x,0)$$

$$\Rightarrow\int_0^{h(t,x)}\partial_z\big(u_x(t,x,z)u_z(t,x,z)\big)dz=\Big(\partial_t h(t,x)+u_x(t,x,h)\partial_x h(t,x)\Big)u_x(t,x,h)$$

$$\Rightarrow \int_0^{h(t,x)} \partial_z\big(u_x(t,x,z)u_z(t,x,z)\big)dz = \partial_t h(t,x)u_x(t,x,h) + u_x^2(t,x,h)\partial_x h(t,x)..........(4)$$

$\Big($ As we know that $\partial_t h(t,x) + u_x(t,x,h)\partial_x h(t,x) = u_z(t,x,h)$ and $u_z(t,x,0) = 0\Big)$

Summing equation (2), (3), (4), we get:

$$\int_0^{h(t,x)} \Big(\partial_t u_x(t,x,z) + \partial_x u_x^2(t,x,z) + \partial_z\big(u_x(t,x,z)u_z(t,x,z)\big)\Big)dz$$

$$= \partial_t \int_0^{h(t,x)} u_x(t,x,z)dz + \partial_x \int_0^{h(t,x)} u_x^2(t,x,z)dz..........(5)$$

We know that:

$$\partial_t u_x(t,x,z)+u_x(t,x,z)\partial_x u_x(t,x,z)+u_z(t,x,z)\partial_z u_x(t,x,z)+\partial_x p(t,x,z)-(\partial_x \tau_{xx}(t,x,z)+\partial_z \tau_{xz}(t,x,z)) = f_x$$

Hence,

$$\int_0^{h(t,x)} \rho\Big(\partial_t u_x(t,x,z) + u_x(t,x,z)\partial_x u_x(t,x,z) + u_z(t,x,z)\partial_z u_x(t,x,z)\Big)dz$$

$$+\int_0^{h(t,x)} \partial_x\Big(p(t,x,z) - \tau_{xx}(t,x,z)\Big)dz - \int_0^{h(t,x)} \partial_z \tau_{xz}(t,x,z))dz = \int_0^{h(t,x)} f_x dz$$

or we can say that:

$$\int_0^{h(t,x)} \rho\Big(\partial_t u_x(t,x,z) + \partial_x u_x^2(t,x,z) + \partial_z\big(u_x(t,x,z)u_z(t,x,z)\big)\Big)dz$$

$$+\int_0^{h(t,x)} \partial_x\Big(p(t,x,z) - \tau_{xx}(t,x,z)\Big)dz - \int_0^{h(t,x)} \partial_z \tau_{xz}(t,x,z))dz = \int_0^{h(t,x)} f_x dz$$

$$\Rightarrow \partial_t \rho \int_0^{h(t,x)} u_x(t,x,z)dz + \partial_x \rho \int_0^{h(t,x)} u_x^2(t,x,z)dz + \int_0^{h(t,x)} \partial_x\Big(p(t,x,z) - \tau_{xx}(t,x,z)\Big)dz$$

$$- \int_0^{h(t,x)} \partial_z \tau_{xz}(t,x,z))dz = \int_0^{h(t,x)} f_x dz.....(6)$$

Using Leibniz integral rule:

$$\int_0^{h(t,x)} \partial_x\Big(p(t,x,z) - \tau_{xx}(t,x,z)\Big)dz = \partial_x \int_0^{h(t,x)} \Big(p(t,x,z) - \tau_{xx}(t,x,z)\Big)dz -$$

$$\Big(p(t,x,h) - \tau_{xx}(t,x,h)\Big)\partial_x h(t,x)$$

$$\int_0^{h(t,x)} \partial_z \tau_{xz}(t,x,z)dz = \tau_{xz}(t,x,h) - \tau_{xz}(t,x,0) = -\Big(p(t,x,h) - \tau_{xx}(t,x,h)\Big)\partial_x h(t,x) - \tau_{xz}(t,x,0)$$

Substracting this two equations:

$$\int_0^{h(t,x)} \partial_x\Big(p(t,x,z) - \tau_{xx}(t,x,z)\Big)dz - \int_0^{h(t,x)} \partial_z \tau_{xz}(t,x,z)dz =$$

$$\partial_x \int_0^{h(t,x)} \Big(p(t,x,z) - \tau_{xx}(t,x,z)\Big) dz) + \tau_{xz}(t,x,0)$$

Using this value in equation (6), we get:

$$\rho\partial_t \int_0^{h(t,x)} u_x(t,x,z)dz + \rho\partial_x \int_0^{h(t,x)} u_x^2(t,x,z)dz + \partial_x \int_0^{h(t,x)} \Big(p(t,x,z) - \tau_{xx}(t,x,z)\Big) dz) + \tau_{xz}(t,x,0)$$

$$= \int_0^{h(t,x)} f_x(t,x,z)dz$$

$$\Rightarrow \rho\partial_t \left(\int_0^{h(t,x)} u_x(t,x,z)dz \right) + \partial_x \left(\int_0^{h(t,x)} \Big(\rho u_x^2(t,x,z)dz + p(t,x,z) - \tau_{xx}(t,x,z)\Big) dz \right) =$$

$$\int_0^{h(t,x)} f_x(t,x,z)dz - \tau_{xz}(t,x,0) ....(proved)$$

□