Stochastic Calculus and Applications to Finance (SCAF)

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Contents

Chapter 1

Stochastic processes and Brownian motion

In this chapter we give general definitions on stochastic processes, Markov processes and continuous time martingales. We then focus on the example of Brownian motion.

1.1 Stochastic processes: general definitions and properties

In the following definitions, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given.

Definition 1.1.1. A stochastic process (or random process) $X = (X_\theta)_{\theta \in \Theta}$ with values in E (E is equipped with a σ-field $\mathcal{E}$), is a family of E-valued random variables X_θ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ (i.e. $X_\theta : \Omega \to E$ is a measurable mapping for any $\theta \in \Theta$), indexed by $\theta \in \Theta$ where Θ is a set.

If Θ is finite or countable (e.g. $\Theta = \mathbb{N}, \mathbb{Z}, \ldots$) we say that $X = (X_\theta)_{\theta \in \Theta}$ is a discrete time process.

If Θ is not countable(e.g. $\Theta = \mathbb{R}_+, \mathbb{R}, \mathbb{R}^2, \ldots$) we say that X is a continuous time process.

The space E is called the state space of the process X.

Remark 1.1.1. There are (at least) two other ways to consider a stochastic process.

1) One may see X as a bivariate mapping:

$$\begin{array}{rccc} X : & \Theta \times \Omega & \to & E \\ & (\theta, \omega) & \mapsto & X_\theta(\omega) \end{array}$$

Note that the measurability of this bivariate mapping from $\Theta \times \Omega$ to E is not clear. We will turn back to this aspect in Definition 1.1.4.

2) One may see X as a mapping from Ω to the functional space E^Θ (the set of mappings from Θ to E):

$$\begin{array}{rcllcc} X : \Omega & \to & E^\Theta & & & \\ \omega & \mapsto & X_\cdot(\omega) : & \Theta & \to & E \\ & & & \theta & \mapsto & X_\theta(\omega). \end{array}$$

Note that it is always possible to equip E^Θ with a σ-field s.t. $X : \Omega \to E^\Theta$ is measurable (see [4] Section 2.2), so that in fact X is seen as a E^Θ-valued random variable.

For $\omega \in \Omega$ the function $X_\cdot(\omega) \in E^\Theta$ is called a path (or trajectory) of the stochastic process X (it is the path associated to the randomness ω). The space E^Θ is called the paths space of the process X.

This point of view is very rich and widely used in some branches of stochastic calculus: for example it allows to construct Brownian motion in a canonical manner (see [4], again Section 2.2; here we will not go further in this direction, and introduce another construction in the forthcoming Section 1.4).

From now on we consider that $\Theta = \mathbb{R}_+$ and that the indices $t \in \mathbb{R}_+$ represent time.

Definition 1.1.2. A filtration $(\mathcal{F}_t)_{t \geq 0}$ is an increasing family of sub-σ-fields of $\mathcal{F}$ (i.e. $\forall s < t$, $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$).

Example 1.1.1. Let $X = (X_t)_{t\geq 0}$ be a continuous time stochastic process and let us consider its "natural filtration" $(\mathcal{F}_t^X)_{t\geq 0}$ defined by

$$\mathcal{F}_t^X = \sigma(X_s,\, s \leq t), \quad \forall t \geq 0$$

(note that we denote $\sigma(X_s,\, s \leq t)$ the smallest sub-σ-field of $\mathcal{F}$ s.t. each X_s, $s \leq t$ is measurable w.r.t. this σ-field).

We claim that the family $(\mathcal{F}_t^X)$ is indeed a filtration: let $s < t$; any X_u, $u \leq s$ is measurable w.r.t. $\mathcal{F}_t^X$ (as $u < t$), thus $\mathcal{F}_s^X \subset \mathcal{F}_t^X$ (as $\mathcal{F}_s^X$ is the smallest σ-field that makes the X_u's measurable for $u \leq s$).

Definition 1.1.3. A process $X = (X_t)_{t\geq 0}$ is said to be adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if for any $t \geq 0$, the random variable X_t is $\mathcal{F}_t$-measurable.

Example 1.1.2. Of course a process X is adapted to its natural filtration $(\mathcal{F}_t^X)$! Indeed X_t is $\mathcal{F}_t^X$-measurable by definition of $\mathcal{F}_t^X$ (for any $t \geq 0$; see Example 1.1.1).

Definition 1.1.4. Let $(\mathcal{F}_t)_{t\geq 0}$ a filtration. A $\mathbb{R}$-valued process $X = (X_t)_{t\geq 0}$ is progressively measurable if for any $t \geq 0$ the mapping

$$\begin{array}{rcl} [0,t] \times \Omega & \to & \mathbb{R} \\ (s,\omega) & \mapsto & X_s(\omega) \end{array}$$

is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable (here $\mathcal{B}([0,t])$ denotes the Borel σ-field of $[0,t]$).

Definition 1.1.5. A process X is said to be almost surely (a.s.) continuous (resp. left continuous (l.c.), resp. right continuous (r.c.)) if there exists $\Omega_0 \in \mathcal{F}$, with $\mathbb{P}(\Omega_0) = 1$ and such that for any $\omega \in \Omega_0$, the path $X_.(\omega)$ is continous (resp. l.c., resp. r.c.).

In other words, if X is a.s. continuous, the elements ω in Ω s.t. $X_.(\omega)$ is not continuous are included in $N \in \mathcal{F}$, with $\mathbb{P}(N) = 0$ (taking $N = \Omega_0^c$ in the above definition).

Proposition 1.1.1. *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration. Let $X = (X_t)_{t\geq 0}$ be an adapted process. Assume X is a.s. r.c. or l.c. Then X is progressively measurable.*

Proof. Cf [4], Proposition 1.1.13. □

We will use Proposition 1.1.1 later on (Chapter 3), to ensure that the stochastic integral is an adapted process.

Definition 1.1.6. Let $(\mathcal{F}_t)_{t\geq 0}$ a filtration. A random variable T with values in $\mathbb{R}_+ \cup \{+\infty\}$ is said to be a stopping time with respect to the filtration $(\mathcal{F}_t)$ (or an $(\mathcal{F}_t)$-stopping time) if for any $t \geq 0$ the event $\{T \leq t\}$ is in $\mathcal{F}_t$.

Example 1.1.3. Let $X = (X_t)_{t\geq 0}$ be an a.s. continuous process, with values in a metric space E, and $(\mathcal{F}_t^X)$ its natural filtration. Let A a closed subset of E and set

$$T = \inf\{t \geq 0 : \quad X_t \in A\}$$

(note that we use the convention $\inf \emptyset = +\infty$, so that the event $\{T = \infty\}$ corresponds to $\{X \text{ never enters the set } A \text{ on time interval } [0,\infty)\}$).

Then T is a $(\mathcal{F}_t^X)$-stopping time (cf [4] Problems 1.2.6 and 1.2.7, [6] Proposition I.4.5).

This is roughly speaking because for any $t \geq 0$,

$$\{T \leq t\} = \{X \text{ has entered the set } A \text{ before time } t\} = \{\exists s \in [0,t],\ X_s \in A\} \in \mathcal{F}_t^X,$$

as such an event can be described using the paths of X on time interval $[0,t]$.

But to establish precisely the result there are some subtleties inherent to filtrations of continuous time processes. For example if X cesses to be continuous or A to be closed, then the result is not true in general. We will not enter into these details in this document. We will in the sequel always face a.s. continuous processes, adapted to "right continuous filtrations" (see [4] p4 for a definition). Therefore the above result will always be true, even if A is open.

Definition 1.1.7. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and T a stopping time. We denote

$$\mathcal{F}_T = \{A \in \mathcal{F} :\ A \cap \{T \leq t\} \in \mathcal{F}_t,\ \forall t \geq 0\}$$

the σ-field of events determined prior the stopping time T.

Remark 1.1.2. Note that it is an exercise to show that the set $\mathcal{F}_T$ is actually a σ-field (e.g. Problem 1.2.13 in [4]).

1.2 Markov processes

From now on some knowledge of conditional expectation w.r.t. a σ-field is required (an introduction can be found in Chapter 4 of [2]).

We denote E the state space of the considered processes. The space E is assumed to be metric and $\mathcal{E}$ is then the Borel set endowed by the open sets for the underlying metric. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given, but starting from Definition 1.2.2 we may change $\mathbb{P}$ for another probability measure.

Definition 1.2.1. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $X = (X_t)_{t\geq 0}$ an adapted process. We say that X is a $(\mathcal{F}_t)$-Markov process is for any $s < t$ and any bounded measurable function $\varphi : E \to \mathbb{R}$, we have

$$\mathbb{E}[\varphi(X_t) \,|\, \mathcal{F}_s] = \mathbb{E}[\varphi(X_t) \,|\, X_s].$$

Exercise 1.2.1. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $X = (X_t)_{t\geq 0}$ an adapted process. Show that if X is a $(\mathcal{F}_t)$-Markov process then it is also a $(\mathcal{F}_t^X)$-Markov process.

Let us examine the meaning of Definition 1.2.1. In view of Exercise 1.2.1 we have for any $s < t$ and any bounded measurable function $\varphi : E \to \mathbb{R}$,

$$\mathbb{E}[\varphi(X_t) \,|\, \mathcal{F}_s^X] = \mathbb{E}[\varphi(X_t) \,|\, X_s].$$

This means that the law of X_t, $t > s$, knowing the path of X on time interval $[0, s]$, only depends on the position of X at time s: the path of X on $[0, s)$ has been forgotten. This is called the Markov property.

To go further in the definitions we introduce the notion of homogeneous Markov family.

Definition 1.2.2. A homogeneous Markov family is a filtration $(\mathcal{F}_t)_{t\geq 0}$ and an adapted process $X = (X_t)_{t\geq 0}$, defined on $(\Omega, \mathcal{F})$, together with a family of probability measures $\{\mathbb{P}^x\}_{x\in E}$ on $(\Omega, \mathcal{F})$ such that

i) For each $F \in \mathcal{F}$, the mapping $x \mapsto \mathbb{P}^x(F)$ is universally measurable (see Definition 1.5.6 in [4]).

ii) For any $x \in E$ we have $\mathbb{P}^x(X_0 = x) = 1$.

iii) For any $x \in E$, $0 \leq s < t$ and any bounded measurable function $\varphi : E \to \mathbb{R}$ we have

$$\mathbb{E}^x[\varphi(X_t) \,|\, \mathcal{F}_s] = \mathbb{E}^x[\varphi(X_t) \,|\, X_s], \quad \mathbb{P}^x - \text{a.s.}$$

(we denote $\mathbb{E}^x$ the expectation computed under $\mathbb{P}^x$).

iv) For any $x, y \in E$, $0 \leq s < t$ and any bounded measurable function $\varphi : E \to \mathbb{R}$ we have

$$\mathbb{E}^x[\varphi(X_t) \,|\, X_s = y] = \mathbb{E}^y[\varphi(X_{t-s})], \quad \text{for } \mathbb{P}^x \circ X_s - \text{a.e.} y.$$

The above definition of a homogeneous Markov family is a bit cumbersome. For conciseness it may happen that we simply say "X is a homogeneous $(\mathcal{F}_t)$-Markov process" and then consider $\mathbb{P}^x$ for varying $x \in E$ (it is also sometimes convenient to see $\mathbb{P}^x$ as $\mathbb{P}(\cdot|X_0 = x)$). Also in view of Exercise 1.2.1 it may happen that if we simply say "X is a Markov process" we by default mean that X is a $(\mathcal{F}_t^X)$-Markov process. Which filtration is considered will always be clear from the context.

One way to check the Markov property for a process X is the following proposition, whose proof is left to the reader.

Proposition 1.2.1. *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $X = (X_t)_{t\geq 0}$ be an adapted process. The process X is $(\mathcal{F}_t)$-Markov if and only if for any $s < t$, any bounded measurable function $\varphi : E \to \mathbb{R}$, and any $x \in E$ we have*

$$\mathbb{E}^x[\varphi(X_t) \,|\, \mathcal{F}_s] = g(t, s, X_s)$$

where $g(t, s, \cdot)$ is a Borel measurable function.

The process X is homogeneous Markov if and only if $g(t, s, \cdot)$ depends only on $t - s$, i.e. $g(t, s, \cdot) = g(t - s, \cdot)$. In that case note that we have $g(t, x) = \mathbb{E}^x[\varphi(X_t)]$.

Definition 1.2.3. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $X = (X_t)_{t\geq 0}$ an adapted process. We say that X has the strong Markov property (or is a $(\mathcal{F}_t)$-strong Markov process) if for any $(\mathcal{F}_t)$-stopping time τ, any time $t \geq 0$, any bounded measurable function $\varphi : E \to \mathbb{R}$ and any $x \in E$, we have

$$\mathbb{E}^x[\varphi(X_t) \,|\, \mathcal{F}_\tau] = \mathbb{E}^x[\varphi(X_t) \,|\, X_\tau]$$

on the event $\{\tau \leq t\}$.

Remark 1.2.1. For a time homogeneous strong Markov process X we have, for any stopping time τ, any time $t \geq 0$, any bounded measurable function $\varphi : E \to \mathbb{R}$ and any $x \in E$,

$$\mathbb{E}^x[\varphi(X_t) \,|\, \mathcal{F}_\tau] = (U_{t-\tau}\varphi)(X_\tau)$$

on the event $\{\tau \leq t\}$, where the family of operators $(U_s)_{s\geq 0}$ is defined by $(U_s\varphi)(x) = \mathbb{E}^x[\varphi(X_s)]$, for $\varphi : E \to \mathbb{R}$ bounded and measurable.

For the proof see [4], Proposition 2.6.7. We will use this property at the end of the chapter, in order to prove the reflection principle for Brownian motion.

1.3 Continuous time martingales: first definitions

A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given and the considered martingales are $\mathbb{R}$-valued.

Definition 1.3.1. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration. A process $M = (M_t)_{t\geq 0}$ is called a $(\mathcal{F}_t)$-martingale if:
i) M is $(\mathcal{F}_t)$-adapted.
ii) For any $t \geq 0$, we have $\mathbb{E}|M_t| < \infty$.
iii) For any $0 \leq s < t$, we have $\mathbb{E}[M_t \,|\, \mathcal{F}_s] = M_s$.

Remark 1.3.1. If iii) in the above definition is replaced by $\mathbb{E}[M_t \,|\, \mathcal{F}_s] \leq M_s$ we say that M is a supermartingale; if iii) is replaced by $\mathbb{E}[M_t \,|\, \mathcal{F}_s] \geq M_s$ we say that M is a submartingale.

Remark 1.3.2. We say in short "M is a martingale" when there is no ambiguity w.r.t. the involved filtration.

Remark 1.3.3. We stress the importance in Definition 1.3.1 of the probability measure $\mathbb{P}$ that has been put on $(\Omega, \mathcal{F})$. If we alter $\mathbb{P}$ there is no reason why we would keep point iii) and M would remain a martingale. Note that we could have done an analogous remark for Markov processes. But the remark is here more relevant with martingales, because we will encounter later on the notion of change of probability measure, that will give rise to new martingales (see Girsanov theorem in Chapter 4).

Concerning martingales we have "optional sampling theorems"; we mention a few of them.

Theorem 1.3.1. *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration, M a martingale, and T and S two bounded stopping times satisfying $S \leq T \leq c < \infty$ a.s. Then*

$$\mathbb{E}[M_T \,|\, \mathcal{F}_S] = M_S \quad a.s.$$

Proof. See [4], Problem 1.3.23. □

Theorem 1.3.2. *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration, $M = (M_t)_{t\geq 0}$ a martingale and T a stopping time (possibly unbounded).*

Then $M^T = (M_{t\wedge T})_{t\geq 0}$ is a again a martingale (called the stopped martingale M^T).

Proof. See [4], Problem 1.3.24. □

Exercise 1.3.1. Some filtration is given. Let $M = (M_t)_{t\geq 0}$ be a square integrable martingale, i.e. with $\mathbb{E}|M_t|^2 < \infty$, for any $t \geq 0$. Show that

$$\mathbb{E}[(M_t - M_s)^2] = \mathbb{E}[M_t^2 - M_s^2], \quad \forall 0 \leq s < t.$$

1.4 A fundamental stochastic process: the Brownian motion

Definition 1.4.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_t)_{t\geq 0}$ a filtration. A $\mathbb{R}$-valued process $B = (B_t)_{t\geq 0}$ is called a $(\mathcal{F}_t)$-standard Brownian motion if it is adapted and satisfies
i) $B_0 = 0$, $\mathbb{P}$-a.s.
ii) For any $0 \leq s < t$ we have $B_t - B_s \sim \mathcal{N}(0, t-s)$.
iii) For any $0 \leq s < t$ the increment $B_t - B_s$ is independent from $\mathcal{F}_s$.
iv) B is a.s. continuous.

Remark 1.4.1. Point iii) of Definition 1.4.1 implies that the increments of B are independent, that is: for any $0 < t_1 < \ldots < t_n$ the random variables $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent.

Conversely, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and B a $\mathbb{R}$-valued process defined on it, satisfying Points i), ii) and iv) of Definition 1.4.1 and

iii') for any $0 < t_1 < \ldots < t_n$ the random variables $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent.

Then B is a $(\mathcal{F}^B_t)$-Brownian motion (standard). To prove that iii') implies iii) one uses the monotone class theorem (see Theorem 0.2.1 in [6]).

Note that in Definition 1.4.1 there is no reason why $(\mathcal{F}_t)$ should be the natural filtration of B. It is sometimes convenient to work with a filtration larger that $(\mathcal{F}^B_t)$, therefore the general Definition 1.4.1.

Remark 1.4.2. The word "standard" in Definition 1.4.1 refers to the fact that B starts from zero under $\mathbb{P}$.

But we may have to consider some Brownian motion starting from $x \neq 0$. Therefore we will consider the Brownian family $(\mathcal{F}_t)$, B and $\{\mathbb{P}^x\}_{x\in\mathbb{R}}$, satisfying Point i) of Definition 1.2.2, Points ii) to iv) of Definition 1.4.1, and $\mathbb{P}^x(B_0 = x) = 1$ for all $x \in \mathbb{R}$.

For conciseness we will most often say "B is a Brownian motion" and have in mind that under $\mathbb{P}^x$, $x \neq 0$, the process B is a non standard Brownian motion (it starts from $x \neq 0$). It is standard under $\mathbb{P}^0$, but we will omit the superscript when we are satisfied with standard Brownian motion and there is no ambiguity.

Exercise 1.4.1. Let B be a $(\mathcal{F}_t)$-Brownian motion (some filtration is given). Show that the process $B - x = (B_t - x)_{t\geq 0}$ is a standard Brownian motion under $\mathbb{P}^x$, for any $x \in \mathbb{R}$.

The first question about Brownian motion is: how can such a process be defined ?

There are several ways to construct the Brownian motion. Among them the canonical approach (see Section 2.2 of [4], already mentioned in Remark 1.1.1), the Hilbert analysis approach (Section 2.3 of [4]), etc...

But maybe the most intuitive one is by scaling the symmetric random walk on $\mathbb{Z}$ (Section 2.4 of [4]).

Let us recall what we mean by symmetric random walk on $\mathbb{Z}$: a sequence $(X_i)_{i\geq 1}$ of i.i.d. random variables is defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{P}(X_1 = +1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$.

Then we define the $\mathbb{Z}$-valued discrete time process $M = (M_n)_{n\geq 0}$ by

$$M_0 = 0 \quad \text{and} \quad M_n = \sum_{i=1}^n X_i, \;\; \forall n \geq 1.$$

This process M is the symmetric random walk on $\mathbb{Z}$.

Note that M is a discrete time martingale with respect to the filtration $(\mathcal{F}_n)$ defined by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(X_i,\ 1 \leq i \leq n)$, $n \geq 1$ (see Chapter 4 of [2] for a definition of discrete time martingales).

Indeed M is obviously $(\mathcal{F}_n)$-adapted, and we have for any $n \geq 0$, $\mathbb{E}|M_n| < \infty$ and

$$\mathbb{E}[M_{n+1} \,|\, \mathcal{F}_n] = \sum_{i=1}^n X_i + \mathbb{E}[X_{n+1}|\mathcal{F}_n] = M_n$$

(we have used the fact that X_{n+1} is independent from $\mathcal{F}_n$ so that $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_{n+1}] = \mathbb{E}[X_1] = 0$).

Then we define the continuous time process $B^{(n)} = (B^{(n)}_t)_{t\geq 0}$ by: $B^{(n)}_t = \frac{1}{\sqrt{n}} M_{nt}$ if nt is itself an integer; if not, we define $B^{(n)}_t$ by linear interpolation between its values at the nearest times s and u s.t. $s < t < u$ and ns and nu are integers.

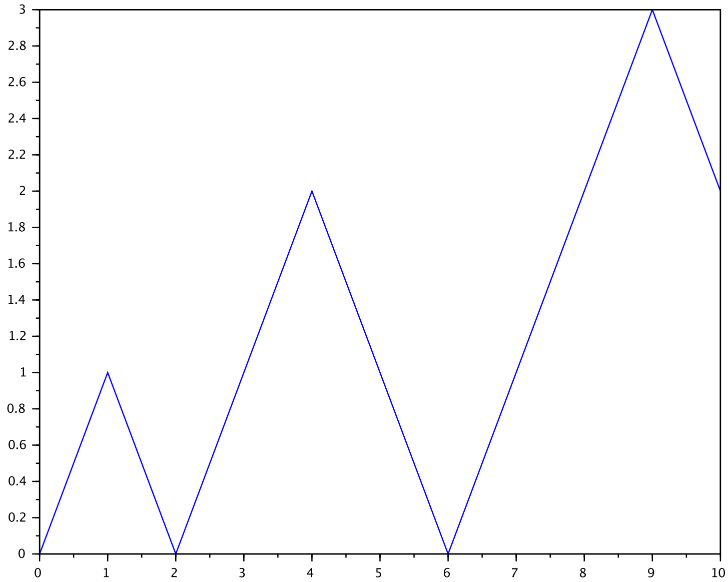

Figure 1.1: Path of $B^{(1)}$ on time interval $[0, 10]$.

Figures 1.1, 1.2 and 1.3 show, for a given path of M, the corresponding path of $B^{(n)}$, for $n = 1$ (this is simply the path of M, that has been linearized), for $n = 5$ and for $n = 1000$.

One can observe that the path of $B^{(1000)}$ looks a bit like the path of a Brownian motion (that you may have encountered on TV, internet or newspaper...).

In fact we have the following convergence result.

Theorem 1.4.1 (Donsker theorem)**.** *The process $B^{(n)}$ converges in distribution, as $n \to \infty$ to a process B satisfying Points i), ii), iii') and iv) in Definition 1.4.1 and Remark 1.4.1.*

Proof. See Theorem 2.4.17 and 2.4.20 in [4]. □

Therefore B is a $(\mathcal{F}_t^B)$-standard Brownian motion (Remark 1.4.1) defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

Note that the fact that the increments of B are gaussian (while the ones of $B^{(n)}$ are not) is due to the central limit theorem.

Note also that when we say that $B^{(n)}$ converges in distribution to B this is in the sense of the convergence of laws of continuous processes. We do not enter into details and refer again to [4].

We now explore some properties of the Brownian motion. In the sequel a filtration $(\mathcal{F}_t)$ is given and B is a $(\mathcal{F}_t)$-Brownian motion.

Proposition 1.4.1. *B is a homogeneous $(\mathcal{F}_t)$-Markov process.*

Proof. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a Borel bounded function and $0 \leq s < t$. We have for any $x \in \mathbb{R}$

$$\mathbb{E}^x[\varphi(B_t) \,|\, \mathcal{F}_s] = \mathbb{E}^x[\varphi(B_t - B_s + B_s) \,|\, \mathcal{F}_s].$$

But $B_t - B_s$ is independent from $\mathcal{F}_s$ and B_s is $\mathcal{F}_s$-measurable, thus (properties of conditional expectation)

$$\mathbb{E}^x[\varphi(B_t - B_s + B_s) \,|\, \mathcal{F}_s] = F(B_s)$$

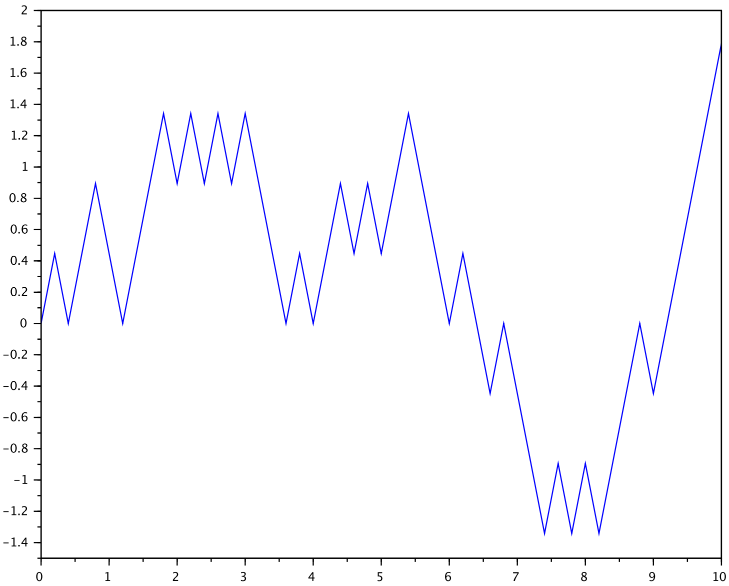

Figure 1.2: Path of $B^{(5)}$ on time interval $[0, 10]$.

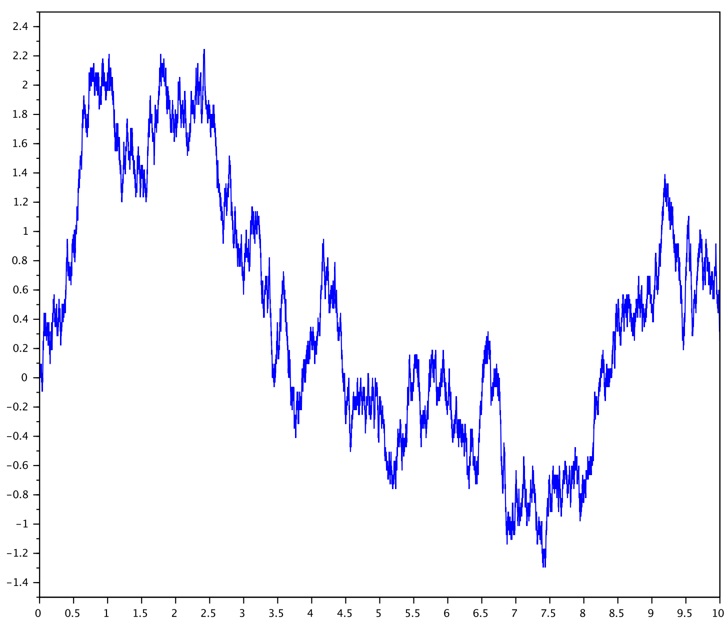

Figure 1.3: Path of $B^{(1000)}$ on time interval $[0, 10]$.

where $F(y) = \mathbb{E}^x[\varphi(B_t - B_s + y)]$. Denoting $p(u,z) = \frac{1}{\sqrt{2\pi u}} e^{-\frac{z^2}{2u}}$ and using Point ii) of Definition 1.4.1 we have

$$F(y) = \int_{\mathbb{R}} \varphi(z+y) p(t-s, z) dz.$$

Thus $F(y) = F(t-s, y)$ and we have

$$\mathbb{E}^x[\varphi(B_t) \,|\, \mathcal{F}_s] = F(t-s, B_s).$$

Therefore the result by Proposition 1.2.1. □

Proposition 1.4.2. *The process B is an $(\mathcal{F}_t)$-martingale.*

Proof. To fix ideas let us work under $\mathbb{P} = \mathbb{P}^0$, under which B is standard (but the result remains true under $\mathbb{P}^x$, $x \neq 0$).

The process B is $(\mathcal{F}_t)$-adapted by definition and we have for any $t > 0$, $B_t = B_t - B_0 \sim \mathcal{N}(0,t)$, thus $\mathbb{E}|B_t| < \infty$. Let us check the martingale property. We have for any $s < t$,

$$\mathbb{E}[B_t|\mathcal{F}_s] = \mathbb{E}[B_t - B_s + B_s|\mathcal{F}_s] = \mathbb{E}[B_t - B_s|\mathcal{F}_s] + \mathbb{E}[B_s|\mathcal{F}_s] = \mathbb{E}[B_t - B_s] + B_s = B_s.$$

Here we have used the fact that B_s is $\mathcal{F}_s$-measurable so that $\mathbb{E}[B_s|\mathcal{F}_s] = B_s$, the fact that $B_t - B_s$ is independent from $\mathcal{F}_s$ so that $\mathbb{E}[B_t - B_s|\mathcal{F}_s] = \mathbb{E}[B_t - B_s]$ and finally the fact that $B_t - B_s \sim \mathcal{N}(0, t-s)$ so that $\mathbb{E}[B_t - B_s] = 0$. □

The two above propositions show that B is both a Markov process and a martingale. But note that not all Markov processes are martingales, and not all martingales enjoy the Markov property.

We now turn to properties of the Brownian paths.

Proposition 1.4.3. *Assume B is standard. We have*
i) (Symmetry property): $(-B_t)_{t\geq 0}$ is again a standard Brownian motion.
ii) (Scaling property): For any $c > 0$ the process $(c^{-1}B_{c^2t})_{t\geq 0}$ is again a standard Brownian motion, for the filtration $(\mathcal{F}_{c^2t})_{t\geq 0}$.
iii) (Inversion of time): The process $\hat{B}$ defined by $\hat{B}_0 = 0$ and $\hat{B}_t = tB_{1/t}$, $t > 0$ is again a standard Brownian motion, for its natural filtration $(\hat{\mathcal{F}}_t)_{t\geq 0}$.

Proof. Points i) and ii) are left to the reader. We give some elements for the proof of Point iii).

We have $\hat{B}_0 = 0$ by definition. Let $t > s > 0$ we prove that $\hat{B}_t - \hat{B}_s \sim \mathcal{N}(0, t-s)$. We have

$$\hat{B}_t - \hat{B}_s = tB_{\frac{1}{t}} - sB_{\frac{1}{s}} = tB_{\frac{1}{t}} - s(B_{\frac{1}{s}} - B_{\frac{1}{t}} + B_{\frac{1}{t}}) = (t-s)B_{\frac{1}{t}} - s(B_{\frac{1}{s}} - B_{\frac{1}{t}})$$

where $B_{\frac{1}{t}} = B_{\frac{1}{t}} - B_0 \sim \mathcal{N}(0, \frac{1}{t})$ and $B_{\frac{1}{s}} - B_{\frac{1}{t}} \sim \mathcal{N}(0, \frac{1}{s} - \frac{1}{t})$ are independant normal (gaussian) variables. Thus $\hat{B}_t - \hat{B}_s$ is gaussian with

$$\mathbb{E}[\hat{B}_t - \hat{B}_s] = (t-s)\mathbb{E}[B_{\frac{1}{t}}] - s\mathbb{E}[B_{\frac{1}{s}} - B_{\frac{1}{t}}] = 0$$

and

$$\mathbb{V}\mathrm{ar}[\hat{B}_t - \hat{B}_s] = (t-s)^2\frac{1}{t} + s^2(\frac{1}{s} - \frac{1}{t}) = (t^2 - 2st + s^2)\frac{1}{t} + s - \frac{s^2}{t} = t - 2s + \frac{s^2}{t} + s - \frac{s^2}{t} = t - s.$$

Thus $\hat{B}_t - \hat{B}_s \sim \mathcal{N}(0, t-s)$. The proof that for any $0 < t_1 < \ldots < t_n$ the random variables $\hat{B}_{t_1}$, $\hat{B}_{t_2} - \hat{B}_{t_1}, \ldots, \hat{B}_{t_n} - \hat{B}_{t_{n-1}}$ are independent is left to the reader. It implies (Remark 1.4.1) that $\hat{B}_t - \hat{B}_s$ is independent from $\hat{\mathcal{F}}_s$ for any $0 \leq s < t$ (note that $\hat{B}$ is obviously $(\hat{\mathcal{F}}_t)$-adapted).

From the (a.s.) continuity of B it is clear that $\hat{B}_t = tB_{1/t}$ is continuous (a.s.) at any time $t > 0$. It remains to see that $\lim_{t\downarrow 0} \hat{B}_t = 0$. The proof of this point is postponed to Proposition 1.4.6. □

Proposition 1.4.4 (Translated Brownian motion)**.** *Assume B is standard and let $h > 0$. Then $(B_{t+h} - B_h)_{t\geq 0}$ is again a standard Brownian motion (for its natural filtration).*

Figure 1.4: A Brownian path on time interval $[0, 10]$ and the graph of $t \mapsto -t$.

Proof. We have $B_{0+h} - B_h = 0$ and the a.s. continuity of $t \mapsto B_{t+h} - B_h$ is clear. For any $t > s$ we have $(B_{t+h} - B_h) - (B_{s+h} - B_h) = B_{t+h} - B_{s+h} \sim \mathcal{N}(0, t-s)$. For any $0 < t_1 < \ldots < t_n$ we have that $B_{t_1+h} - B_h, (B_{t_2+h} - B_h) - (B_{t_1+h} - B_h) = B_{t_2+h} - B_{t_1+h}, \ldots, (B_{t_n+h} - B_h) - (B_{t_{n-1}+h} - B_h) = B_{t_n+h} - B_{t_{n-1}+h}$ are independent. $\square$

Proposition 1.4.5 (Behavior at infinity)**.** *Assume B is standard. We have*

i)

$$\limsup_{t\to\infty} B_t = +\infty \quad a.s. \quad \text{and} \quad \liminf_{t\to\infty} B_t = -\infty \quad a.s.$$

ii)

$$\lim_{t\to\infty} \frac{B_t}{t} = 0 \quad a.s.$$

Proof. See Proposition 1.4.1 in [5] and Problem 2.9.3 in [4]. $\square$

This proposition means that the standard Brownian motion explores the whole real line $\mathbb{R}$, but slower than the identity function $t \mapsto t$ (see Figure 1.4).

Proposition 1.4.6. *i) In the context of Proposition 1.4.3-iii) we have $\lim_{t\downarrow 0} \hat{B}_t = 0$.*

ii) (Nowhere differentiability of Brownian motion): we have for any $t_0 \geq 0$,

$$\limsup_{t\downarrow 0} \left| \frac{B_{t_0+t} - B_{t_0}}{t} \right| = +\infty \quad a.s.$$

Proof. i) Performing a change of variable we have $\lim_{t\downarrow 0} tB_{\frac{1}{t}} = \lim_{u\uparrow\infty} \frac{B_u}{u} = 0$, thanks to Proposition 1.4.5-ii).

ii) By Proposition 1.4.5-i) we have $\limsup_{t\downarrow 0} |\frac{\hat{B}_t}{t}| = \limsup_{t\downarrow 0} |B_{\frac{1}{t}}| = +\infty$.

In fact it is possible to show that we have the property $\limsup_{t\downarrow 0} |\frac{B_t}{t}| = +\infty$ for any Brownian motion B (REF?).

Thus we have this property in particular for $(B_{t+t_0} - B_{t_0})_{t\geq 0}$ (Proposition 1.4.4), which leads to

$$\limsup_{t\downarrow 0} \left| \frac{B_{t_0+t} - B_{t_0}}{t} \right| = +\infty \quad \text{a.s.}$$

$\square$

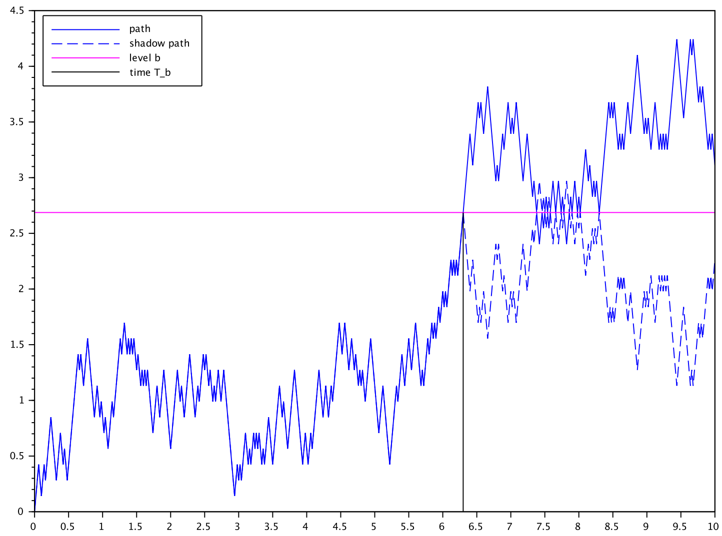

Figure 1.5: A Brownian path on time interval $[0, 10]$ and its "shadow path" (reflected around the axis $y = b$ after time T_b). Here $b = 2.68$.

We finish this section by stating and proving the reflection principle for Brownian motion.

Proposition 1.4.7 (Reflection principle)**.** *Let $b \geq 0$ and set $T_b = \inf\{t \geq 0 : \ B_t = b\}$. We have for any $t \geq 0$,*

$$\mathbb{P}^0(T_b \leq t) = 2\mathbb{P}^0(B_t > b) = \mathbb{P}^0(|B_t| > b). \tag{1.4.1}$$

The reflection principle allows for example to compute the law of T_b.

Exercise 1.4.2. Show that

$$\mathbb{P}^0(T_b \in dt) = \frac{b}{\sqrt{2\pi t^3}} \exp\big(-\frac{b^2}{2t}\big)\, dt.$$

Note that the last part of (1.4.1) is simply due to

$$\mathbb{P}^0(|B_t| > b) = \mathbb{P}^0(B_t > b) + \mathbb{P}^0(B_t < -b) = \mathbb{P}^0(B_t > b) + \mathbb{P}^0(-B_t > b) = 2\mathbb{P}^0(B_t > b)$$

(using Proposition 1.4.3-i)).

The idea to prove the first part of (1.4.1) is to write

$$\mathbb{P}^0(T_b \leq t) = \mathbb{P}^0(T_b \leq t\,;\, B_t > b) + \mathbb{P}^0(T_b \leq t\,;\, B_t \leq b).$$

But as $\{B_t > b\} \subset \{T_b \leq t\}$ we have $\mathbb{P}^0(T_b \leq t\,;\, B_t > b) = \mathbb{P}^0(B_t > b)$.

So that we are done if we prove that

$$\mathbb{P}^0(T_b \leq t\,;\, B_t \leq b) = \mathbb{P}^0(T_b \leq t\,;\, B_t > b). \tag{1.4.2}$$

Consider Figure 1.5. Heuristically we will get (1.4.2) if the shadow path has the same probability to occur than the initial path. One feels this has a chance to be true because B is Markov.

In fact we have better: B is strong Markov and will will use this to mathematically prove (1.4.2).

Proposition 1.4.8. *The Brownian motion B enjoys the strong Markov property.*

Proof. See [4], Theorem 2.6.15. $\square$

We thus write

$$\mathbb{P}^0(T_b \le t\,;\, B_t > b) = \mathbb{E}^0\Big[\mathbb{E}^0\big(\mathbf{1}_{T_b\le t}\mathbf{1}_{B_t>b}\,|\,\mathcal{F}_{T_b}\big)\Big] = \mathbb{E}^0\Big[\mathbf{1}_{T_b\le t}\mathbb{P}^0(B_t > b\,|\,\mathcal{F}_{T_b})\Big] = \mathbb{E}^0\Big[\mathbf{1}_{T_b\le t}\mathbb{P}^0(B_t > b\,|\,B_{T_b})\Big]$$

(note that at the third equality we have used the fact that $\{T_b \le t\} \in \mathcal{F}_{T_b}$; indeed let $u \ge 0$, one may check that $\{T_b \le t\} \cap \{T_b \le u\}$ is in $\mathcal{F}_u$ by noticing that $\{T_b \le t\} \cap \{T_b \le u\} = \{T_b \le u\} \in \mathcal{F}_u$ if $u \le t$, and that $\{T_b \le t\} \cap \{T_b \le u\} = \{T_b \le t\} \in \mathcal{F}_t \subset \mathcal{F}_u$ if $t < u$).

We now use Remark 1.2.1. We have (note that here $(U_s f)(x) = \mathbb{E}^x[f(B_s)]$ for any $s \ge 0$, $x \in \mathbb{R}$)

$$\mathbb{P}^0(B_t > b\,|\,B_{T_b}) = \mathbb{E}^0[\mathbf{1}_{B_t>b}|\,B_{T_b}] = (U_{t-T_b}\mathbf{1}_{(b,+\infty)})(B_{T_b}) = (U_{t-T_b}\mathbf{1}_{(b,+\infty)})(b),$$

on the event $\{T_b \le t\}$, thus

$$\mathbb{P}^0(T_b \le t\,;\, B_t > b) = \int_0^t (U_{t-s}\mathbf{1}_{(b,+\infty)})(b)\,\mathbb{P}^0(T_b \in ds).$$

Note now that

$$\begin{aligned}(U_s\mathbf{1}_{(b,+\infty)})(b) &= \mathbb{P}^b(B_s > b) = \mathbb{P}^b(B_s - b > 0) = \mathbb{P}^0(B_s > 0)\\ &= \mathbb{P}^0(B_s < 0) = \mathbb{P}^b(B_s - b < 0) = \mathbb{P}^b(B_s < b) = (U_s\mathbf{1}_{(-\infty,b)})(b)\end{aligned}$$

(we have used Exercise 1.4.1). Thus

$$\mathbb{P}^0(T_b \le t\,;\, B_t > b) = \int_0^t (U_{t-s}\mathbf{1}_{(-\infty,b)})(b)\,\mathbb{P}^0(T_b \in ds) = \mathbb{E}^0\Big[\mathbf{1}_{T_b\le t}\mathbb{P}^0(B_t < b\,|\,\mathcal{F}_{T_b})\Big] = \mathbb{P}^0(T_b \le t\,;\, B_t < b).$$

In fact this establishes (1.4.2) because the law of (T_b, B_t) has a density w.r.t. the Lebesgue measure.

Chapter 2

Processes of finite variation and quadratic variation of martingales

In this chapter we recall some elements about functions of finite variation and introduce the notion of process of finite variation. We prove that a continuous time martingale is not of finite variation, unless it is constant, and introduce the notion of quadratic variation of martingales.

2.1 Functions of finite variation

Note that all the considered functions will by default be right continuous, so that we will rarely recall this assumption.

Let $-\infty < a < b < +\infty$. We call a set $\Delta_n = \{t_0^n, \dots, t_n^n\}$ with $t_0^n = a < t_1^n < \dots < t_n^n = b$ a subdivision of the interval $[a, b]$, of size n.

We call $|\Delta_n| := \sup_{i=1,\dots,n} |t_i^n - t_{i-1}^n|$ the step of Δ_n.

Definition 2.1.1. Let $f : [a, b] \to \mathbb{R}$ a function, we call the total variation of f on $[a, b]$ the quantity

$$V_{[a,b]}(f) = \sup_{\Delta_n \in \mathcal{S}} \Big\{ \sum_{i=1}^{n} |f(t_i^n) - f(t_{i-1}^n)| \Big\}$$

where $\mathcal{S}$ is the set of all possible subdivisions of $[a, b]$ (of all possible sizes).

If $V_{[a,b]}(f) < +\infty$, we say that f is of finite variation (FV) on $[a, b]$.

Let $f : \mathbb{R}_+ \to \mathbb{R}$. If for any $T > 0$ the function $f_{|[0,T]}$ is of FV on $[0, T]$, we say that f is of FV on $\mathbb{R}_+$.

Property/Example 2.1.1. 1) If $f : [a, b] \to \mathbb{R}$ is increasing then $V_{[a,b]}(f) = f(b) - f(a) < \infty$.

Indeed, for any subdivision Δ_n of $[a, b]$, we have $\sum_{i=1}^n |f(t_i) - f(t_{i-1})| = \sum_{i=1}^n (f(t_i) - f(t_{i-1})) = f(b) - f(a)$ (note that we will often drop the superscript n on the t_i^n's in the sequel).

2) If $f \in C^1([a, b])$ then f is of FV.

Indeed, for any subdivision Δ_n of $[a, b]$ we have,

$$\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| = \sum_{i=1}^{n} \Big| \int_{t_{i-1}}^{t_i} f'(s)ds \Big| \leq \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |f'(s)|ds = \int_a^b |f'(s)|ds.$$

Thus $V_{[a,b]}(f) \leq \int_a^b |f'(s)|ds < \infty$.

3) A function f is of FV if and only if $f = f_1 - f_2$ with f_1 and f_2 two increasing functions.

The necessary condition is clear as

$$|f(t_i) - f(t_{i-1})| \leq |f_1(t_i) - f_1(t_{i-1})| + |f_2(t_i) - f_2(t_{i-1})| = (f_1(t_i) - f_1(t_{i-1})) + (f_2(t_i) - f_2(t_{i-1})).$$

For the sufficient condition see the Appendix (Proposition 6.1.1).

Now consider μ a positive measure on $\mathbb{R}_+$ and set $f(t) = \mu([0,t])$. The function f is increasing and thus of FV.

If μ is a signed measure, i.e. $\mu = \mu_1 - \mu_2$ with μ_1, μ_2 two positive measures, then $f(t) = \mu([0,t]) = \mu_1([0,t]) - \mu_2([0,t])$ is the difference of two increasing functions and therefore of FV.

In fact the converse is true. More precisely we have the following result.

Theorem 2.1.1. *There is a one-to-one correspondance between the r.c. functions f of FV and the signed measures μ on $\mathbb{R}_+$, via the equality*

$$f(t) = \mu([0,t]), \quad t \geq 0.$$

Proof. REF? $\square$

We are then led to the concept of Stieltjes integral.

Let f of FV on $\mathbb{R}_+$ and μ_f the corresponding signed measure. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ a Borel function s.t.

$$\int_0^t |\varphi|(s)\, |\mu_f|(ds) < +\infty, \quad \forall t \geq 0$$

(here we have denoted $|\mu_f|$ the positive measure defined by $|\mu_f| = \mu_{f_1} + \mu_{f_2}$ where $\mu_f = \mu_{f_1} - \mu_{f_2}$ is the decomposition of μ_f; note that μ_{f_1} and μ_{f_2} correspond to the increasing functions f_1 and f_2 in the decomposition $f = f_1 - f_2$ of f).

Then we note

$$\int_0^t \varphi(s)\, df(s) := \int_{(0,t]} \varphi(s)\, \mu_f(ds), \quad t \geq 0$$

the Stieltjes integral of φ against f at time t. We may consider the function $\int_0^\cdot \varphi(s)\, df(s) \,:\, t \mapsto \int_0^t \varphi(s)\, df(s)$ and call it the Stieltjes integral of φ against f.

Note that $\int_0^t df(s) = f(t) - f(0)$. In the sequel we will often note $df(s)$ for $\mu_f(ds)$.

Property/Example 2.1.2. 1) The function $\int_0^\cdot \varphi(s)\, df(s)$ is itself of FV.

Indeed we have $\int_0^t \varphi(s) df(s) = \mu_f^\varphi((0,t])$ with $\mu_f^\varphi(A) = \int_A \varphi(s) df(s)$ for any $A \in \mathcal{B}(\mathbb{R}_+)$. And using the decompositions $\varphi = \varphi^+ - \varphi^-$ and $df = df_1 - df_2$ one may check that μ_f^φ is a signed measure. Therefore the result, considering $\mu_f^\varphi((0,t]) = \mu_f^\varphi([0,t]) - \mu_f^\varphi(\{0\})$ and Theorem 2.1.1.

2) (Associativity of the Stieltjes integral) Let $a : \mathbb{R}_+ \to \mathbb{R}$ of FV and $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}$ having the required integrability. One sets $A(t) = \int_0^t \psi(s) da(s)$ for any $t \geq 0$. Then $\int_0^t \phi(s) dA(s) = \int_0^t \phi(s)\psi(s)\, da(s)$ for any $t \geq 0$.

Indeed $A(t) = \mu_a^\psi(]0,t])$ with $\mu_a^\psi(B) = \int_B \psi(s)\, da(s)$ for any $B \in \mathcal{B}(\mathbb{R}_+)$ (that is the measure μ_a^ψ has density ψ w.r.t. the measure $da(s)$). Thus, for any $t \geq 0$,

$$\int_0^t \phi(s) dA(s) = \int_{]0,t]} \phi(s) \mu_a^\psi(ds) = \int_{]0,t]} \phi(s)\psi(s) da(s) = \int_0^t \phi(s)\psi(s) da(s).$$

3) If φ is continuous then $\int_0^T \varphi(s) df(s) = \lim_{|\Delta_n| \downarrow 0} \sum_{i=1}^n \varphi(t_{i-1}^n)(f(t_i^n) - f(t_{i-1}^n))$ (the limit is taken over subdivisions $|\Delta_n|$ of $[0,T]$, $0 < T < \infty$).

4) If f is of class C^1 and $f(0) = 0$ then $\int_0^t \varphi(s) df(s) = \int_0^t \varphi(s) f'(s) ds$ for any $t \geq 0$.

Indeed, $\mu_f([0,t]) = f(t) = \int_0^t f'(s) ds$ for any t, which shows that $\mu_f(dt) = f'(t) dt$.

Exercise 2.1.1. Show that for $f : \mathbb{R}_+ \to \mathbb{R}$ of FV (with $f = f_1 - f_2$ with f_1, f_2 increasing) we have $V_{[0,t]}(f) \leq \int_0^t |df|(s)$, where $|df|$ denotes $df_1 + df_2$.

2.2 Processes of finite variation

From now on and till the end of the chapter the encountered processes are $\mathbb{R}$-valued and defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A filtration $(\mathcal{F}_t)_{t \geq 0}$ is given.

Definition 2.2.1. An $(\mathcal{F}_t)$-adapted and a.s. r.c. process $X = (X_t)_{t \geq 0}$ is of FV if, for almost every $\omega \in \Omega$, the mapping $t \mapsto X_t(\omega)$ is of FV.

Proposition 2.2.1. *Let $X = (X_t)_{t\geq 0}$ be a process of FV, a.s. continuous and adapted.*

Let $H = (H_t)_{t\geq 0}$ be a progressively measurable process such that for almost every $\omega \in \Omega$, the function $H.(\omega)$ is integrable against $X.(\omega)$ in the Stieltjes sense.

The process $H \cdot X$ defined for almost every $\omega \in \Omega$ by

$$(H \cdot X)_t(\omega) = \int_0^t H_s(\omega) dX_s(\omega), \quad \forall t \geq 0$$

(here $\int_0^t H_s(\omega)dX_s(\omega)$ is the Stieltjes integral of $H.(\omega)$ against $X.(\omega)$ at time t) is called the Stieltjes integral of H against X.

This process $H \cdot X$ is of FV, a.s. continuous and adapted.

Proof. See [6] p119 for some details.

The idea is that for a.e. ω the process $\int_0^{\cdot} H_s(\omega)dX_s(\omega)$ is of FV by Property/Example 2.1.2-1). To see the continuity one uses the continuity of the integral. To show that $H \cdot X$ is adapted is maybe the most tricky part. It is here that the fact that H is progressively measurable comes into play. $\square$

Proposition 2.2.2. *Let $M = (M_t)_{t\geq 0}$ be a continuous martingale with $M_0 = 0$. If M is of FV then $M_t = 0$ a.s., for any $t \geq 0$.*

Proof. Note that this proof makes use of Exercises 2.2.1 and 2.2.2 that are proposed just after.

We consider, for any $n \in \mathbb{N}$, the stopping time $T_n = \inf\{t \geq 0,\, V_{[0,t]}(M) \geq n\}$.

For any n we consider the stopped martingale $M^{T_n} = (M_{t\wedge T_n})_{t\geq 0}$ (see Theorem 1.3.2).

For a while we fix n and denote $X = M^{T_n}$ for conciseness. Then we fix $t > 0$. We have for any subdivision Δ_p of $[0, t]$,

$$\mathbb{E}(X_t^2) = \mathbb{E}\Big[\sum_{i=1}^p (X_{t_i}^2 - X_{t_{i-1}}^2)\Big] = \mathbb{E}\Big[\sum_{i=1}^p (X_{t_i} - X_{t_{i-1}})^2\Big].$$

Here we have used Exercise 1.3.1 at the second inequality (and note that Exercise 2.2.1 guarantees that $\mathbb{E}(X_t^2) < \infty$).

Thus we have $\mathbb{E}(X_t^2) \leq \mathbb{E}\big[\sup_i |X_{t_i} - X_{t_{i-1}}|\ \sum_{i=1}^p |X_{t_i} - X_{t_{i-1}}|\big]$, but $\sum_{i=1}^p |X_{t_i} - X_{t_{i-1}}| \leq V_{[0,t]}(X) = V_{[0,t\wedge T_n]}(M) \leq n$.

Thus $\mathbb{E}(X_t^2) \leq n\mathbb{E}\big[\sup_i |X_{t_i} - X_{t_{i-1}}|\big]$. And by the a.s. continuity of X we have

$$\sup_i |X_{t_i} - X_{t_{i-1}}| \xrightarrow[|\Delta_p|\downarrow 0]{} 0 \text{ a.s.}$$

As $\sup_i |X_{t_i} - X_{t_{i-1}}| \leq V_{[0,t]}(X) \leq n$ we can use the dominated convergence theorem to claim that

$$\mathbb{E}(X_t^2) \leq n\mathbb{E}\big[\sup_i |X_{t_i} - X_{t_{i-1}}|\big] \xrightarrow[|\Delta_p|\downarrow 0]{} 0.$$

Thus $\mathbb{E}(X_t^2) = 0$ and $X_t = 0$ a.s., i.e. we have shown that

$$M_{t\wedge T_n} = 0 \text{ a.s.}, \quad \forall n \in \mathbb{N}, \forall t > 0. \tag{2.2.1}$$

To achieve the proof we now fix $t > 0$ again. As $T_n \uparrow \infty$ a.s. (Exercise 2.2.2) for a.e. ω there exists $N_t(\omega)$ great enough s.t. $T_{N_t(\omega)}(\omega) > t$.

Thus for a.e. ω we have $M_t(\omega) = M_{t\wedge T_{N_t(\omega)}(\omega)}(\omega)$. But this quantity is equal to zero thanks to (2.2.1). $\square$

Exercise 2.2.1. In the context and with the notations of the proof of Proposition 2.2.2, show that X is bounded, and thus square-integrable (for a certain fixed n).

Exercise 2.2.2. Let $Y = (Y_t)_{t\geq 0}$ be a process of FV and define $\tau_n = \inf\{t \geq 0,\, V_{[0,t]}(Y) \geq n\}$. Show that $\tau_n \uparrow \infty$ a.s., as $n \uparrow \infty$.

(*Hint:* Show that if $\tau_n \leq B < \infty$ then $V_{[0,B]}(Y) = +\infty$.)

A consequence of Proposition 2.2.2 is that a continuous martingale M with $M_0 = 0$ which is not constantly equal to zero (as the Brownian motion B for example !) if not of FV.

Therefore $\int_0^{\cdot} H_s dMs$ cannot be defined in the Stieltjes sense: we will have to define the Itô integral, this will be the topic of Chapter 3.

2.3 Quadratic variation of martingales

Theorem 2.3.1 (Doob-Meyer decomposition)**.** *Let $M = (M_t)_{t\geq 0}$ be a square-integrable continuous martingale. There is a unique increasing process, continuous and adapted, denoted $\langle M\rangle = (\langle M\rangle_t)_{t\geq 0}$, such that $\langle M\rangle_0 = 0$ and $M^2 - \langle M\rangle$ is a martingale.*

Proof. For the existence we refer to Theorem 1.4.10 and Definition 1.5.3 in [4].

But it is quite easy to check uniqueness: let A and $\hat{A}$ be two increasing, continuous and adapted processes, with $A_0 = \hat{A}_0 = 0$, and s.t. $M^2 - A$ and $M^2 - \hat{A}$ are martingales.

Then $A - \hat{A} = A - M^2 - (\hat{A} - M^2)$ is a continuous martingale, starting from zero (by linear combination). But $A - \hat{A}$ is of FV, as the difference of two increasing processes. Thus $A - \hat{A} \equiv 0$ by Proposition 2.2.2. □

Example 2.3.1. For a standard Brownian motion B we have $\langle B\rangle_t = t$, for all $t \geq 0$.

Indeed, notice first that, as $B_t \sim \mathcal{N}(0,t)$, we have $\mathbb{E}(B_t^2) = \mathbb{V}\mathrm{ar}(B_t) = t < \infty$ for any $t > 0$, so that B is actually a square-integrable martingale.

Then the process $(B_t^2 - t)_{t\geq 0}$ is integrable, adapted and continuous. Let us check that it satisfies the martingale property. We have for any $0 \leq s < t$

$$\begin{aligned}
\mathbb{E}(B_t^2 - t\,|\,\mathcal{F}_s) &= \mathbb{E}\big[(B_t - B_s)^2 + 2B_tB_s - B_s^2 - s - (t-s)\,\big|\,\mathcal{F}_s\big] \\
&= \mathbb{E}[(B_t - B_s)^2] - (t-s) - s - B_s^2 + 2B_s\mathbb{E}(B_t|\mathcal{F}_s) \\
&= B_s^2 - s.
\end{aligned}$$

Here we have used the facts that $B_t - B_s$ is independent from $\mathcal{F}_s$ and is distributed along $\mathcal{N}(0, t-s)$, that B_s is $\mathcal{F}_s$-measurable and that B is a martingale.

Thus $(B_t^2 - t)_{t\geq 0}$ is a martingale. The (deterministic) process $(t)_{t\geq 0}$ obviously satisfies all the requirements of $\langle B\rangle$. Thus by the uniqueness property in Theorem 2.3.1 we get $\langle B\rangle_t = t$ (and note that the above computation provide the existence of $\langle B\rangle$ in the case of the Brownian motion B).

For a square-integrable continuous martingale M the process $\langle M\rangle$ is called the "bracket" or the "quadratic variation" of M.

Indeed, for a subdivision Δ_n of $[0,t]$ we denote $Q_t^{\Delta_n}(X) = \sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})^2$, for any process X. We say that X is of finite quadratic variation if for any $t \geq 0$ there exists $Q_t < \infty$ a.s. such that $Q_t^{\Delta_n}(X) \to Q_t$ in probability as $|\Delta_n| \downarrow 0$. We have the following result.

Theorem 2.3.2. *Let $M = (M_t)_{t\geq 0}$ be a continuous square-integrable martingale. We have for any $t \geq 0$,*

$$\sup_{s\leq t} |Q_s^{\Delta_n}(M) - \langle M\rangle_s| \xrightarrow[|\Delta_n|\downarrow 0]{\mathbb{P}} 0. \tag{2.3.1}$$

In particular M is of finite quadratic variation and $Q_t = \langle M\rangle_t$ for any $t \geq 0$.

Proof. See Theorem IV.1.8 in [6]. □

Remark 2.3.1. This is possible to understand why we have such a result by examining the case of Brownian motion (again; see Example 2.3.1).

Let $0 = t_0^n < \ldots < t_n^n = t$ be a subdivision of $[0,t]$ and B be a standard Brownian motion.

We have

$$\mathbb{E}\Big|\sum_{i=1}^n (B_{t_i} - B_{t_{i-1}})^2 - t\Big|^2 = \mathbb{E}\Big|\sum_{i=1}^n \big((B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\big)\Big|^2 = \mathbb{E}\Big|\sum_{i=1}^n Z_i^n\Big|^2$$

where $Z_i^n = (B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})$. Note that the Z_i^n's are centered (as $\mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2] = t_i - t_{i-1}$) and square-integrable with $\mathbb{E}|Z_i^n|^2 = 2(t_i - t_{i-1})^2$ (using $\mathbb{E}|X|^{2k} = \frac{(2k)!}{2^k k!}\sigma^{2k}$ for $X \sim \mathcal{N}(0,\sigma^2)$).

In addition the Z_i^n's are independent, thanks to the independence of the Brownian increments. Thus

$$\mathbb{E}\Big|\sum_{i=1}^n (B_{t_i} - B_{t_{i-1}})^2 - t\Big|^2 = \sum_{i=1}^n \mathbb{E}|Z_i^n|^2 = 2\sum_{i=1}^n (t_i - t_{i-1})^2 \leq 2t \sup_i |t_i - t_{i-1}| \xrightarrow[|\Delta_n|\downarrow 0]{} 0.$$

This shows that $Q_t^{\Delta_n}(B) \to t = \langle B \rangle_t$ in the L^2 sense when $|\Delta_n| \downarrow 0$. This is not the convergence stated in (2.3.1), but gives an insight why we have some convergence of $Q_t^{\Delta_n}(B)$ to $\langle B \rangle_t$.

Property 2.3.1. *If a process X is continuous and of FV then it is of null quadratic variation.*

Proof. We have

$$Q_t^{\Delta_n}(X) \leq \sup_i |X_{t_i} - X_{t_{i-1}}| \sum_{i=1}^n |X_{t_i} - X_{t_{i-1}}| \leq V_{[0,t]}(X) \sup_i |X_{t_i} - X_{t_{i-1}}| \xrightarrow[|\Delta_n| \downarrow 0]{} 0.$$

□

Definition 2.3.1 (Bracket of two martingales)**.** Let M, N be two continuous square-integrable martingales. We set

$$\langle M, N \rangle := \frac{1}{2}\Big[\langle M + N \rangle - \langle M \rangle - \langle N \rangle\Big].$$

This is the "(crossed) bracket of M with N".

Property 2.3.2. *1) $\langle M, N \rangle$ is the unique continuous and adapted process, of FV, starting from zero, such that $MN - \langle M, N \rangle$ is a martingale.*

2) We have

$$\langle M, N \rangle_t = \lim_{|\Delta_n| \downarrow 0} |_{\mathbb{P}} \sum_{i=1}^n (M_{t_i} - M_{t_{i-1}})(N_{t_i} - N_{t_{i-1}}).$$

3) For any progressively measurable process H that is integrable against $\langle M, N \rangle$ we have

$$\int_0^t H_s d\langle M, N \rangle_s = \lim_{|\Delta_n| \downarrow 0} |_{\mathbb{P}} \sum_{i=1}^n H_{t_{i-1}} (M_{t_i} - M_{t_{i-1}})(N_{t_i} - N_{t_{i-1}}).$$

Proof. You may check 1) as an exercise. For 2) and 3) see [6] and [4]. □

Exercise 2.3.1. Show that $(M, N) \mapsto \langle M, N \rangle$ is bilinear and symmetric.

Note that of course $\langle M, M \rangle = \langle M \rangle$. We will use one or the other notation in the sequel.

To finish with, we have the following property.

Property 2.3.3. *Let $M = (M_t)_{t \geq 0}$ be a continuous square-integrable martingale with $M_0 = 0$ and $\langle M, M \rangle \equiv 0$. Then $M \equiv 0$.*

Proof. We have for any $t \geq 0$ that $\mathbb{E}[M_t^2 - \langle M, M \rangle_t] = \mathbb{E}[M_0^2 - \langle M, M \rangle_0] = 0$, as a martingale is constant in expectation and $M_0^2 - \langle M, M \rangle_0 = 0$. Thus $\mathbb{E}[M_t^2] = \mathbb{E}[\langle M, M \rangle_t] = 0$, for any $t \geq 0$. Thus $M_t = 0$ a.s. for any $t \geq 0$. □

Chapter 3

Stochastic integration and Itô formula

In this chapter we build the (Itô) stochastic integral and present the Itô formula (or Itô lemma). Those are the two main blocks of Stochastic Calculus. Note that the Itô formula is presented with the formalism used in [6], but the proofs follow more often the spirit of [4].

3.1 Stochastic integration

In this whole chapter some time horizon $0 < T < \infty$ is fixed. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ are given.

We aim at giving a sense to $\int_0^t H_s dM_s$, $0 \leq t \leq T$, where M is a square-integrable martingale and H a progressively measurable process satisfying some integrability conditions. From Proposition 2.2.2 we already know that this cannot be done in the Stieltjes sense (unless M is constant).

We introduce some notations. We denote $\mathcal{M}_2$ the space of continuous square integrable martingales, starting from zero (i.e. with $M_0 = 0$). It is equipped with the norm

$$||\cdot|| : M \mapsto ||M|| = \sqrt{\mathbb{E}(M_T^2)} = \sqrt{\mathbb{E}(\langle M\rangle_T)}.$$

Exercise 3.1.1. Show that for any $0 \leq t \leq T$ we have $\mathbb{E}(M_t^2) \leq ||M||^2$. This imply that any element of $\mathcal{M}_2$ is bounded in L^2.

In fact the normed space $(\mathcal{M}_2, ||\cdot||)$ is a Banach space (see Proposition IV.1.22 in [6]). This fact is crucial and will be used later on (Theorem 3.1.2).

Pick M in $\mathcal{M}_2$. We denote $\Pi_2(M)$ the space of progressively measurable processes H satisfying

$$||H||_M^2 := \mathbb{E}\int_0^T H_s^2 d\langle M\rangle_s < \infty.$$

Note that again the space $(\Pi_2(M), ||\cdot||_M)$ is a Banach space (see a remark p137 in [6]; this is in fact just due to the properties of $L^p(E, \mathcal{E}, \mu)$ spaces, for any measured space $(E, \mathcal{E}, \mu)$; here the considered measure is in some sense $d\langle M\rangle_s \otimes d\mathbb{P}$). But we will not use this fact later on.

Step 1. We denote $b\Pi_1$ the space of simple processes, those are processes H of the form

$$H_t = Y_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=1}^n Y_i \mathbf{1}_{]t_{i-1},t_i]}(t), \quad 0 \leq t \leq T,$$

where $t_0 = 0 < t_1 < \ldots < t_n = T$ is a subdivision of $[0, T]$, Y_0 is $\mathcal{F}_0$-measurable, Y_i is $\mathcal{F}_{t_{i-1}}$-measurable for any $1 \leq i \leq n$, and $|Y_i| \leq C < \infty$ a.s. for any $0 \leq i \leq n$.

Note that as H in $b\Pi_1$ is l.c. and adapted it is progressively measurable (Proposition 1.1.1). Besides, the boundedness of the Y_i's implies that $\mathbb{E}\int_0^T H_s^2 d\langle M\rangle_s < \infty$.

Therefore $b\Pi_1 \subset \Pi_2(M)$, for any $M \in \mathcal{M}_2$.

For $M \in \mathcal{M}_2$ and $H \in b\Pi_1$ we now define the process $H \cdot M$ by

$$(H \cdot M)_t = \sum_{i=1}^{n} Y_i (M_{t_i \wedge t} - M_{t_{i-1} \wedge t}). \tag{3.1.1}$$

Theorem 3.1.1. *1) For any $M \in \mathcal{M}_2$ and any $H \in b\Pi_1$ the process $H \cdot M$ is in $\mathcal{M}_2$.*

2) Let $M \in \mathcal{M}_2$ fixed. The application

$$\begin{array}{llll} (\cdot M): & b\Pi_1 & \to & \mathcal{M}_2 \\ & H & \mapsto & H \cdot M \end{array}$$

is linear.

3) For any $M, N \in \mathcal{M}_2$ and any $H, K \in b\Pi_1$ we have

$$\langle H \cdot M, K \cdot N \rangle_t = \int_0^t H_s K_s d\langle M, N \rangle_s, \quad \forall 0 \le t \le T,$$

where the above integral is understood in the Stieltjes sense.

4) We have for any $0 \le t \le T$, any $M, N \in \mathcal{M}_2$ and any $H, K \in b\Pi_1$,

$$\mathbb{E}[(H \cdot M)_t (K \cdot N)_t] = \mathbb{E} \int_0^t H_s K_s d\langle M, N \rangle_s$$

and in particular

$$\mathbb{E}[(H \cdot M)_t^2] = \mathbb{E} \int_0^t H_s^2 d\langle M \rangle_s.$$

Proof. 1) From the definition (3.1.1) one sees that $H \cdot M$ is continuous and starts from zero, and it is easy to check it is square integrable (thanks in particular to the boundedness of the Y_i's that define H).

The fact that $H \cdot M$ is adapted is clear, one checks the martingale property. By linearity of the conditional expectation it is enough to check that each of the processes $\Big(Y_i(M_{t_i \wedge t} - M_{t_{i-1} \wedge t})\Big)_{0 \le t \le T}$ verifies the martingale property. Let $1 \le i \le n$ fixed.

Let $0 \le s < t \le T$. There are several cases to treat separately.

<u>If $s \le t_{i-1}$</u>: One has $Y_i(M_{t_i \wedge s} - M_{t_{i-1} \wedge s}) = 0$.

a) If $t < t_{i-1}$ then $Y_i(M_{t_i \wedge t} - M_{t_{i-1} \wedge t}) = 0$ and thus $\mathbb{E}[Y_i(M_{t_i \wedge t} - M_{t_{i-1} \wedge t}) \,|\, \mathcal{F}_s] = 0$.

b) If $t \ge t_{i-1}$, then

$$\begin{aligned} \mathbb{E}[Y_i(M_{t_i \wedge t} - M_{t_{i-1} \wedge t}) \,|\, \mathcal{F}_s] &= -\mathbb{E}[Y_i M_{t_{i-1}} | \mathcal{F}_s] + \mathbb{E}\big[\mathbb{E}(Y_i M_{t_i \wedge t} | \mathcal{F}_{t_{i-1}}) | \mathcal{F}_s\big] \\ &= -\mathbb{E}[Y_i M_{t_{i-1}} | \mathcal{F}_s] + \mathbb{E}\big[Y_i \mathbb{E}(M_{t_i \wedge t} | \mathcal{F}_{t_{i-1}}) | \mathcal{F}_s\big] \\ &= -\mathbb{E}[Y_i M_{t_{i-1}} | \mathcal{F}_s] + \mathbb{E}\big[Y_i M_{t_{i-1}} | \mathcal{F}_s\big] \\ &= 0, \end{aligned}$$

using the fact that Y_i is $\mathcal{F}_{t_{i-1}}$-measurable and that M is a martingale.

Thus

$$\mathbb{E}[Y_i(M_{t_i \wedge t} - M_{t_{i-1} \wedge t}) \,|\, \mathcal{F}_s] = Y_i(M_{t_i \wedge s} - M_{t_{i-1} \wedge s}) = 0.$$

<u>If $s \ge t_i$</u>: One has $Y_i(M_{t_i \wedge s} - M_{t_{i-1} \wedge s}) = Y_i M_{t_i} - Y_i M_{t_{i-1}}$ and

$$\mathbb{E}[Y_i(M_{t_i \wedge t} - M_{t_{i-1} \wedge t}) \,|\, \mathcal{F}_s] = \mathbb{E}[Y_i(M_{t_i} - M_{t_{i-1}}) \,|\, \mathcal{F}_s] = Y_i(M_{t_i} - M_{t_{i-1}}),$$

where we have used the fact that $Y_i(M_{t_i} - M_{t_{i-1}})$ is $\mathcal{F}_{t_i}$- and thus $\mathcal{F}_s$-measurable.

Thus

$$\mathbb{E}[Y_i(M_{t_i \wedge t} - M_{t_{i-1} \wedge t}) \,|\, \mathcal{F}_s] = Y_i(M_{t_i \wedge s} - M_{t_{i-1} \wedge s}) = Y_i(M_{t_i} - M_{t_{i-1}}).$$

If $t_{i-1} < s < t_i$: One has $Y_i(M_{t_i\wedge s} - M_{t_{i-1}\wedge s}) = Y_i(M_s - M_{t_{i-1}})$ and

$$\begin{aligned}
\mathbb{E}[Y_i(M_{t_i\wedge t} - M_{t_{i-1}\wedge t})\,|\,\mathcal{F}_s] &= \mathbb{E}[Y_i M_{t_i\wedge t} - Y_i M_{t_{i-1}}\,|\,\mathcal{F}_s] \\
&= Y_i \mathbb{E}[M_{t_i\wedge t}\,|\,\mathcal{F}_s] - Y_i M_{t_{i-1}} \\
&= Y_i(M_s - M_{t_{i-1}}),
\end{aligned}$$

using successively the facts that Y_i and $Y_i M_{t_{i-1}}$ are $\mathcal{F}_{t_{i-1}}$- and thus $\mathcal{F}_s$-measurable, and that M is a martingale. Thus,

$$\mathbb{E}[Y_i(M_{t_i\wedge t} - M_{t_{i-1}\wedge t})\,|\,\mathcal{F}_s] = Y_i(M_{t_i\wedge s} - M_{t_{i-1}\wedge s}) = Y_i(M_s - M_{t_{i-1}}).$$

To sum up, in any case we have $\mathbb{E}[Y_i(M_{t_i\wedge t} - M_{t_{i-1}\wedge t})\,|\,\mathcal{F}_s] = Y_i(M_{t_i\wedge s} - M_{t_{i-1}\wedge s})$, we have checked the martingale property.

2) The linearity is left to the reader.

3) We aim at showing that the process defined by $Z_t := (H\cdot M)_t (K\cdot N)_t - \int_0^t H_s K_s d\langle M, N\rangle_s$ is a martingale. Indeed the result will then follow from the uniqueness part of Property 2.3.2-1).

By linearity arguments it is enough to prove the result for $H_t = Y_i \mathbf{1}_{]t_{i-1},t_i]}(t)$ and $K_t = Y_j' \mathbf{1}_{]t_{j-1},t_j]}(t)$ (we recall that in particular Y_i is $\mathcal{F}_{t_{i-1}}$-measurable and Y_j' is $\mathcal{F}_{t_{j-1}}$-measurable).

If $i < j$: Then $\int_0^t H_s K_s d\langle M, N\rangle_s = 0$, as $HK \equiv 0$. Consider now

$$(H\cdot M)_t (K\cdot N)_t = (Y_i M_{t_i\wedge t} - Y_i M_{t_{i-1}\wedge t})(Y_j' N_{t_j\wedge t} - Y_j' N_{t_{j-1}\wedge t}).$$

If $t < t_{j-1}$ this quantity is equal to zero. If $t \geq t_{j-1}$ it is equal to

$$Y_i(M_{t_i} - M_{t_{i-1}})Y_j'(N_{t_j\wedge t} - N_{t_{j-1}}) = \Big(Y_i(M_{t_i} - M_{t_{i-1}})Y_j'\mathbf{1}_{]t_{j-1},t_j]}(\cdot)\cdot N\Big)_t.$$

Thus this quantity can be seen as the integral against N of the simple process $J_t = Y_i(M_{t_i} - M_{t_{i-1}})Y_j'\mathbf{1}_{]t_{j-1},t_j]}(t)$ (note that $Y_i(M_{t_i} - M_{t_{i-1}})Y_j'$ is $\mathcal{F}_{t_{j-1}}$-measurable). In fact, by the definition (3.1.1), it also true for $t < t_{j-1}$.

Thus $Z_t = (H\cdot M)_t(K\cdot N)_t = (J\cdot N)_t$, $0 \leq t \leq T$. Thus Z is a martingale according to Point 1).

If $i = j$: Then

$$Z_s = \begin{cases} 0 & \text{if} \quad s < t_{i-1} \\ Y_i Y_i'[(M_s - M_{t_{i-1}})(N_s - N_{t_{i-1}}) - (\langle M,N\rangle_s - \langle M,N\rangle_{t_{i-1}})] & \text{if} \quad t_{i-1} \leq s \leq t_i \\ Y_i Y_i'[(M_{t_i} - M_{t_{i-1}})(N_{t_i} - N_{t_{i-1}}) - (\langle M,N\rangle_{t_i} - \langle M,N\rangle_{t_{i-1}})] & \text{if} \quad s > t_i \end{cases}$$

We check the martingale property of Z for $t_{i-1} \leq s < t \leq t_i$ (other cases are a bit easier and left to the reader). We have

$$\begin{aligned}
Z_t - Z_s &= Y_i Y_i' \big[\, M_t N_t - M_{t_{i-1}} N_t - M_t N_{t_{i-1}} + M_{t_{i-1}} N_{t_{i-1}} - (\langle M,N\rangle_t - \langle M,N\rangle_{t_{i-1}}) \\
&\qquad - M_s N_s + M_{t_{i-1}} N_s + M_s N_{t_{i-1}} - M_{t_{i-1}} N_{t_{i-1}} + (\langle M,N\rangle_s - \langle M,N\rangle_{t_{i-1}})\big] \\
&= Y_i Y_i' \big[\, M_t N_t - M_s N_s - M_{t_{i-1}}(N_t - N_s) - N_{t_{i-1}}(M_t - M_s) - (\langle M,N\rangle_t - \langle M,N\rangle_s)\big].
\end{aligned}$$

But Y_i, Y_i', $M_{t_{i-1}}$ and $N_{t_{i-1}}$ are $\mathcal{F}_{t_{i-1}}$- and thus $\mathcal{F}_s$-measurable, thus

$$\begin{aligned}
\mathbb{E}[Z_t - Z_s|\mathcal{F}_s] &= Y_i Y_i' \big\{\, \mathbb{E}[M_t N_t - M_s N_s - (\langle M,N\rangle_t - \langle M,N\rangle_s)|\mathcal{F}_s] \\
&\qquad - M_{t_{i-1}}\mathbb{E}[N_t - N_s|\mathcal{F}_s] - N_{t_{i-1}}\mathbb{E}[M_t - M_s|\mathcal{F}_s]\big\} \\
&= 0
\end{aligned}$$

as M, N and $MN - \langle M, N\rangle$ are martingales.

4) This comes from the fact that $\mathbb{E}(Z_t) = \mathbb{E}(Z_0) = 0$. □

Note that Point 4) of Theorem 3.1.1 implies that

$$||H \cdot M|| = ||H||_M \quad \forall H \in b\Pi_1, \ \forall M \in \mathcal{M}_2. \tag{3.1.2}$$

This is an isometry property. We will now extend the construction of the stochastic integral to $\Pi_2(M)$, preserving this isometry property.

Step 2. To start with we have the following result.

Lemma 3.1.1. *For any $M \in \mathcal{M}_2$ the space $b\Pi_1$ is dense in $\Pi_2(M)$.*

Proof. See Proposition 3.2.8 in [4]. □

We then have the following theorem.

Theorem 3.1.2. *1) Let $M \in \mathcal{M}_2$. The application*

$$\begin{array}{rccc} (\cdot M): & b\Pi_1 & \to & \mathcal{M}_2 \\ & H & \mapsto & H \cdot M \end{array}$$

extends in an isometry $I_M : \Pi_2(M) \to \mathcal{M}_2$.
We note $I_M(H) =: (H \cdot M)$, $H \in \Pi_2(M)$.
2) For any $M, N \in \mathcal{M}_2$, any $H \in \Pi_2(M)$ and any $K \in \Pi_2(N)$ we have

$$\langle H \cdot M, K \cdot N \rangle_t = \int_0^t H_s K_s d\langle M, N \rangle_s, \quad 0 \le t \le T.$$

Proof. 1) Let $H \in \Pi_2(M)$, there is a sequence $(H^n)_n$ in $b\Pi_1$ s.t. $||H^n - H||_M \to 0$ as $n \to \infty$. Thus $(H^n)_n$ is Cauchy in $\Pi_2(M)$ (as it is convergent). By linearity and isometry (Eq. (3.1.2)) we thus have

$$||H^n \cdot M - H^m \cdot M|| = ||(H^n - H^m) \cdot M|| = ||H^n - H^m||_M \xrightarrow[m,n\to\infty]{} 0$$

Thus $(H^n \cdot M)_n$ is Cauchy in $\mathcal{M}_2$, and thus convergent, to an element $I_M(H)$ of $\mathcal{M}_2$. But by continuity of the norm we have

$$||I_M(H)|| = \lim_{n\to\infty} ||H^n \cdot M|| = \lim_{n\to\infty} ||H^n||_M = ||H||_M,$$

and thus the application I_M is an isometry.
2) Again limiting arguments: see [4], Section 3.2.B. □

Thus for $M \in \mathcal{M}_2$ and $H \in \Pi_2(M)$ we have constructed $H \cdot M$, which we call the stochastic integral (or the Itô integral) of H against M, and will often denote

$$\int_0^\cdot H_s dM_s$$

(and for any $0 \le t \le T$ we denote $\int_0^t H_s dM_s = \left(\int_0^\cdot H_s dM_s \right)_t = (H \cdot M)_t$).

Proposition 3.1.1. *Let $M \in \mathcal{M}_2$ and $H \in \Pi_2(M)$, then $H \cdot M$ is the unique element in $\mathcal{M}_2$ satisfying*

$$\forall N \in \mathcal{M}_2, \quad \langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle.$$

Proof. Let X another element of $\mathcal{M}_2$ satisfying

$$\forall N \in \mathcal{M}_2, \quad \langle X, N \rangle = H \cdot \langle M, N \rangle.$$

Then

$$\forall N \in \mathcal{M}_2, \quad \langle H \cdot M - X, N \rangle = \langle H \cdot M, N \rangle - \langle X, N \rangle = 0.$$

In particular, as $H \cdot M - X$ is in $\mathcal{M}_2$ we have

$$\langle H \cdot M - X, H \cdot M - X \rangle_t = 0 \text{ a.s.} \quad \forall 0 \le t \le T.$$

Thus $H \cdot M - X \equiv 0$ (using Property 2.3.3). □

The above property of the stochastic integral allows to prove the important following result.

Proposition 3.1.2 (Associativity of the stochastic integral)**.** *Let $M \in \mathcal{M}_2$, $K \in \Pi_2(M)$ and $H \in \Pi_2(K \cdot M)$. Then $HK \in \Pi_2(M)$ and $(HK) \cdot M = H \cdot (K \cdot M)$.*

Proof. Using $\langle K \cdot M\rangle_t = \langle K \cdot M, K \cdot M\rangle_t = \int_0^t K_s^2 d\langle M\rangle_s$ (Point 2) of Theorem 3.1.2) and the associativity of the Stieltjes integral (Property 2.1.2-2)) one gets, under the assumption $H \in \Pi_2(K \cdot M)$,

$$\mathbb{E}\int_0^T H_s^2 K_s^2 d\langle M\rangle_s = \mathbb{E}\int_0^T H_s^2 d\langle K \cdot M\rangle_s < +\infty,$$

therefore HK is in $\Pi_2(M)$.

We now prove that

$$\forall N \in \mathcal{M}_2, \quad \langle H \cdot (K \cdot M), N\rangle = \int_0^\cdot H_s K_s d\langle M, N\rangle_s \tag{3.1.3}$$

As by Proposition 3.1.1 the process $(HK) \cdot M$ is the unique element in $\mathcal{M}_2$ to satisfy $\langle (HK) \cdot M, N\rangle = \int_0^\cdot H_s K_s d\langle M, N\rangle_s$, $\forall N \in \mathcal{M}_2$, we will get the desired result.

We have for any $N \in \mathcal{M}_2$, using $\langle K \cdot M, N\rangle = K \cdot \langle M, N\rangle$ and the associativity of Stieltjes integral again,

$$\langle H \cdot (K \cdot M), N\rangle = \int_0^\cdot H_s d\langle K \cdot M, N\rangle_s = \int_0^\cdot H_s K_s d\langle M, N\rangle_s,$$

therefore (3.1.3). The proof is completed. □

Remark 3.1.1. To sum up we write in the integral form some of the above encountered properties, that we shall often use when doing stochastic calculus.

⋆) Point 2) of Theorem 3.1.2 may be rewritten: for any $M, N \in \mathcal{M}_2$, any $H \in \Pi_2(M)$ and any $K \in \Pi_2(N)$,

$$\Big\langle \int_0^\cdot H_s dM_s\,, \int_0^\cdot K_s dN_s \Big\rangle_t = \int_0^t H_s K_s d\langle M, N\rangle_s, \quad 0 \le t \le T.$$

⋆) The associativity of the Itô integral may be rewritten: for $M \in \mathcal{M}_2$, $K \in \Pi_2(M)$ and $H \in \Pi_2(K \cdot M)$ one has

$$\int_0^t H_s d\Big(\int_0^\cdot K_u dM_u\Big)_s = \int_0^t H_s K_s dM_s, \quad 0 \le t \le T.$$

3.2 Itô formula

Definition 3.2.1. We call a ($\mathbb{R}$-valued) semimartingale a $\mathbb{R}$-valued process $X = (X_t)$ of the form

$$X_t = X_0 + A_t + M_t, \quad \forall 0 \le t \le T,$$

where X_0 is some $\mathbb{R}$-valued and $\mathcal{F}_0$-measurable random variable representing the initial position of X, A is some continuous and adapted process of FV, with $A_0 = 0$, and M is some element of $\mathcal{M}_2$ (in particular $M_0 = 0$).

Definition 3.2.2. Let X be a semimartingale with martingale part M and FV part A. Let $H \in \Pi_2(M)$ and integrable against A, we denote

$$\int_0^\cdot H_s dX_s := \int_0^\cdot H_s dA_s + \int_0^\cdot H_s dM_s,$$

where the first integral is understood in the Stieltjes sense and the second in the Itô sense.

Definition 3.2.3. Let $X = X_0 + A + M$ and $Y = Y_0 + C + N$ be two semimartingales. We call the bracket of X with Y the symmetric quantity

$$\langle X, Y\rangle := \langle M, N\rangle.$$

Note that Definition 3.2.3 comes from the fact that for any subdivision Δ_n,

$$Q_t^{\Delta_n}(X,Y) = Q_t^{\Delta_n}(A+M,C+N) = Q_t^{\Delta_n}(M,N) + Q_t^{\Delta_n}(A,N) + Q_t^{\Delta_n}(M,C) + Q_t^{\Delta_n}(A,C)$$

and that $Q_t^{\Delta_n}(M,N) \to \langle M,N\rangle_t$ as $|\Delta_n| \downarrow 0$ (Property 2.3.1-2)), while the three other terms tend to zero.

Indeed they are the crossed quadratic variation of a continuous process with a process of FV. So that for example (this is similar to Property 2.3.1)

$$|Q_t^{\Delta_n}(A,N)| \leq \sum_i |A_{t_i} - A_{t_{i-1}}| \times |N_{t_i} - N_{t_{i-1}}| \leq \sup_i |N_{t_i} - N_{t_{i-1}}| \, V_{[0,t]}(A) \xrightarrow[|\Delta_n|\downarrow 0]{} 0,$$

as A is of FV and $\sup_i |N_{t_i} - N_{t_{i-1}}|$ tends to zero by continuity of M.

With all these notations and definitions we can now state the Itô rule.

Theorem 3.2.1 (Itô rule, Itô formula)**.** *Let $X^1, \ldots, X^p$ be continuous semimartingales and $f \in C^2(\mathbb{R}^p;\mathbb{R})$ such that*

$$\forall 1 \leq i \leq p, \quad \mathbb{E}\int_0^T |\partial_{x_i} f(X_s)|^2 d\langle X^i\rangle_s < +\infty$$

(we have denoted $X_s = (X_s^1, \ldots, X_s^p)^T$ for any $0 \leq s \leq T$). Then

$$f(X_t) = f(X_0) + \sum_{i=1}^p \int_0^t \partial_{x_i} f(X_s) dX_s^i + \frac{1}{2}\sum_{i,j=1}^p \int_0^t \partial^2_{x_i x_j} f(X_s) d\langle X^i, X^j\rangle_s, \quad \forall 0 \leq t \leq T.$$

In order to give the main ideas of the proof of Theorem 3.2.1 we will need the following result.

Proposition 3.2.1 (Dominated convergence for the stochastic integral)**.** *Let $X = A + M$ be a semimartingale and $(H^n)_n$ a sequence of integrable processes (against X; in particular the H^n's are progressively measurable). Assume $H_s^n(\omega) \to H_s(\omega)$, as $n \to \infty$, for any (s,ω). Assume H is bounded and assume $|H_s^n(\omega)| \leq C < +\infty$ for any (s,ω) and any n. Then*

$$\sup_{s \leq t} \Big| \int_0^s H_u^n dX_u - \int_0^s H_u dX_u \Big| \xrightarrow[n\to\infty]{\mathbb{P}} 0 \tag{3.2.1}$$

Proof. We do not prove the result fully, by give only the great lines. Thanks to the dominated convergence for the Stieltjes integral $\int_0^\cdot H_s^n dA_s$ converges to $\int_0^\cdot H_s dA_s$ a.s. We now turn to the martingale part.

Thanks to the isometry of Itô integral we get for any $0 \leq t \leq T$

$$\mathbb{E}\Big|\int_0^t H_s^n dM_s - \int_0^t H_s dM_s\Big|^2 = \mathbb{E}\Big|\int_0^t (H_s^n - H_s) dM_s\Big|^2 = \mathbb{E}\int_0^t |H_s^n - H_s|^2 d\langle M\rangle_s.$$

As $|H^n - H|$ is bounded and converges pointwise to zero one may conclude by dominated convergence (for the integral against $d\langle M\rangle_s \otimes d\mathbb{P}$) that $\mathbb{E}\int_0^t |H_s^n - H_s|^2 d\langle M\rangle_s \to 0$, as $n \to \infty$. That is to say $\int_0^t H_s^n dM_s$ converges to $\int_0^t H_s dM_s$ in $L^2(\mathbb{P})$.

It remains to pass to (3.2.1) (in the spirit of Theorem IV.1.8 in [6]). □

Corollary 3.2.1. *Let X a semimartingale and H a continuous adapted integrable bounded process. Then for any $0 \leq t \leq T$ and any sequence $(\Delta_n)_n$ of subdivisions of $[0,t]$,*

$$\sum_{i:\, t_i \leq t} H_{t_{i-1}}(X_{t_i} - X_{t_{i-1}}) \xrightarrow[|\Delta_n|\downarrow 0]{\mathbb{P}} \int_0^t H_s dX_s.$$

Proof. It suffices to use the proposition with $H_t^n = \sum_i H_{t_{i-1}} \mathbf{1}_{]t_{i-1},t_i]}(t)$.

Indeed one has $\int_0^t H_s^n dX_s = \sum_{i:\, t_i \leq t} H_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})$ and H^n converges to H. □

Proposition 3.2.2. *In the preceding context one has*

$$\sum_{i:\, t_i \leq t} H_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})^2 \xrightarrow[|\Delta_n|\downarrow 0]{\mathbb{P}} \int_0^t H_s d\langle X\rangle_s.$$

Proof. See [4] pp 151-152. $\square$

Main ideas of the proof of Theorem 3.2.1: We deal only with the case $p = 1$, where the Itô formula simply writes

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle X\rangle_s. \tag{3.2.2}$$

A subdivision Δ_n of $[0,t]$ is given. We have

$$f(X_t) - f(X_0) = \sum_{i=1}^n (f(X_{t_i}) - f(X_{t_{i-1}}))$$

and for any $1 \le i \le n$, we use a Taylor expansion between $X_{t_{i-1}}$ and X_{t_i}. This gives

$$f(X_{t_i}) - f(X_{t_{i-1}}) = f'(X_{t_{i-1}})\,(X_{t_i} - X_{t_{i-1}}) + \frac{1}{2}f''(\xi_i)(X_{t_i} - X_{t_{i-1}})^2,$$

where ξ_i is some real number between $X_{t_{i-1}}$ and X_{t_i}. By summation we get

$$\begin{aligned} f(X_t) - f(X_0) &= \sum_{i=1}^n f'(X_{t_{i-1}})\,(X_{t_i} - X_{t_{i-1}}) + \frac{1}{2}\sum_{i=1}^n f''(X_{t_{i-1}})\,(X_{t_i} - X_{t_{i-1}})^2 \\ &\quad + \frac{1}{2}\sum_{i=1}^n (f''(\xi_i) - f''(X_{t_{i-1}}))\,(X_{t_i} - X_{t_{i-1}})^2 \end{aligned}$$

The term $\sum_{i=1}^n f'(X_{t_{i-1}})\,(X_{t_i} - X_{t_{i-1}})$ converges to $\int_0^t f'(X_s)dX_s$ as $|\Delta_n| \downarrow 0$ by Corollary 3.2.1 (in fact a version of this corollary for locally bounded integrands; cf Proposition IV.2.13 in [6]).

The term $\sum_{i=1}^n f''(X_{t_{i-1}})\,(X_{t_i} - X_{t_{i-1}})^2$ converges to $\int_0^t f''(X_s)d\langle X\rangle_s$ as $|\Delta_n| \downarrow 0$, by Proposition 3.2.2.

To finish with the term $\sum_{i=1}^n (f''(\xi_i) - f''(X_{t_{i-1}}))\,(X_{t_i} - X_{t_{i-1}})^2$ is bounded by

$$\sup_i |f''(\xi_i) - f''(X_{t_{i-1}})|\, Q_t^{\Delta_n}(X)$$

but $\sup_i |f''(\xi_i) - f''(X_{t_{i-1}})|$ tends to zero as $|\Delta_n| \downarrow 0$, by continuity of f'', and $Q_t^{\Delta_n}(X)$ tends to $\langle X\rangle_t < \infty$. Thus $\sum_{i=1}^n (f''(\xi_i) - f''(X_{t_{i-1}}))\,(X_{t_i} - X_{t_{i-1}})^2$ tends to zero. Therefore the result.

Remark 3.2.1. Note that if X were of null quadratic variation (for example if X is continuous and of FV), the second order term in (3.2.2) would be zero. In other words we would have $f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s$, which corresponds in some sense to the classical differential calculus. Here the second order term comes from the martingale part of X, which is of non null bracket, and which makes the paths of X non smooth (they have in fact the same kind of smoothness than the Brownian paths: continuous but not differentiable).

Remark 3.2.2. The Itô formula is often written in its differential (and shorter) form: in 1D it gives

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle X\rangle_t. \tag{3.2.3}$$

But note that this differential writing has just an integral meaning: writing (3.2.3) just means that we have (3.2.2).

Example 3.2.1. The Black-Scholes differential stochastic equation writes:

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad S_0 = x \tag{3.2.4}$$

(which means that $S_t = x + \int_0^t \mu S_s ds + \int_0^t \sigma S_s dB_s$ for any $t \ge 0$, see Chapter 5).

Is there a process S solving (3.2.4) ? Let us consider

$$S_t = x\exp\big((\mu - \frac{\sigma^2}{2})t + \sigma B_t\big), \quad t \ge 0. \tag{3.2.5}$$

We denote $X_t = (\mu - \frac{\sigma^2}{2})t + \sigma B_t$, and apply the Itô formula, this gives,

$$\begin{aligned} dS_t &= x\exp(X_t)dX_t + \tfrac{1}{2}x\exp(X_t)d\langle X\rangle_t \\ &= S_t(\mu - \tfrac{\sigma^2}{2})dt + S_t\sigma dB_t + \tfrac{1}{2}S_t\sigma^2 dt \\ &= \mu S_t dt + \sigma S_t dB_t. \end{aligned}$$

Thus the process S defined by (3.2.5) solves (3.2.4).

Remark 3.2.3. The assumption that $\mathbb{E}\int_0^T |\partial_{x_i} f(X_s)|^2 d\langle X^i\rangle_s < +\infty$, $1 \le i \le p$, is required to have the Itô integrals $\int_0^t \partial_{x_i} f(X_s)dM_s^i$ (M^i is the martingale part of X^i) well defined in the Itô formula.

In fact this condition can be relaxed and we get a more general Itô formula under the weaker assumption

$$\int_0^t |\partial_{x_i} f(X_s)|^2 d\langle X^i\rangle_s < +\infty, \text{ a.s. } \quad \forall t \ge 0, \quad \forall 1 \le i \le p$$

(we can even deal with infinite time horizon).

But this requires to use the theory of "local martingales", which is more involved (see [4] and [6]). In this course we will only deal with functions having the right integrability condition.

Chapter 4

Lévy and Girsanov theorems

In this chapter we state and prove the theorem of Lévy and the theorem of Girsanov. The theorem of Girsanov is of crucial importance for the forthcoming Chapter 5 (it will allow to perform a risk-neutral change of probability measure).

4.1 Exponential martingale and theorem of Lévy

In this section some time horizon $0 < T < \infty$ is fixed. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ are given. We start with a lemma.

Lemma 4.1.1. *Let $X = (X_t)_{0\leq t\leq T}$ be a martingale, satisfying $\mathbb{E}(\exp(\frac{1}{2}\langle X\rangle_T)) < +\infty$. Then the process $\mathcal{E}xp(X) = \big(\exp(X_t - \frac{1}{2}\langle X\rangle_t)\big)_{0\leq t\leq T}$ is a martingale, called the exponential martingale (associated to X).*

Proof. By the Itô rule we get

$$d(\exp(X_t - \frac{1}{2}\langle X\rangle_t) = \big(\mathcal{E}xp(X)\big)_t dX_t - \frac{1}{2}\big(\mathcal{E}xp(X)\big)_t d\langle X\rangle_t + \frac{1}{2}\big(\mathcal{E}xp(X)\big)_t d\langle X\rangle_t = \big(\mathcal{E}xp(X)\big)_t dX_t,$$

meaning that $\mathcal{E}xp(X)$ is the stochastic integral of itself against a martingale. The only point that is not clear is that $\mathcal{E}xp(X)$ has the required integrability (could we say for example that $\mathcal{E}xp(X) \in \Pi_2(X)$?), that we were allowed to use the Itô formula and that the obtained stochastic integral is a martingale. But the assumption $\mathbb{E}(\exp(\frac{1}{2}\langle X\rangle_T)) < +\infty$ is here to ensure this is the case (see Proposition 3.5.12 in [4] for details). □

In the sequel we will deal with multidimensional Brownian motion (B.m.). We revisit Definition 1.4.1 for dimension $d \geq 1$ in the definition below.

Definition 4.1.1. A $\mathbb{R}^d$-valued process $B = (B_t)_{0\leq t\leq T}$ is called a d-dimensional $(\mathcal{F}_t)$-standard Brownian motion if it is adapted and satisfies

i) $B_0 = 0$, $\mathbb{P}$-a.s.
ii) For any $0 \leq s < t$ we have $B_t - B_s \sim \mathcal{N}_d(0, (t-s)I_d)$.
iii) For any $0 \leq s < t$ the increment $B_t - B_s$ is independent from $\mathcal{F}_s$.
iv) B is a.s. continuous.

Remark 4.1.1. If we have d independent one-dimensional $(\mathcal{F}_t)$-standard Brownian motions $B^1, \ldots, B^d$, then $B = (B^1, \ldots, B^d)^T = ((B^1_t, \ldots, B^d_t)^T)_{0\leq t\leq T}$ is a d-dimensional $(\mathcal{F}_t)$-standard Brownian motion.
The converse is true (by projection).

The following theorem allows to identify a d-dimensional standard Brownian motion. It is originally due to Paul Lévy (1948). In 1967 Kunita and Watanabe gave a more modern proof, using Itô rule.

Theorem 4.1.1. *Let $X = (X^1, \ldots, X^d)^T$ be a continuous and adapted $\mathbb{R}^d$-valued process, with $X_0 = 0$. The following statements are equivalent:*
i) X is a d-dimensional $(\mathcal{F}_t)$-standard Brownian motion.
ii) Each X^i is a $(\mathcal{F}_t)$-martingale and we have

$$\langle X^i, X^j\rangle_t = \mathbf{1}_{i=j}\, t, \quad \forall 0 \leq t \leq T, \quad \forall 1 \leq i, j \leq d.$$

Proof. "$i) \Rightarrow ii)$": Each X^i is a martingale with bracket $\langle X^i, X^i\rangle_t = t$ (by Remark 4.1.1, Proposition 1.4.2 and Example 2.3.1).

Let us check that $\langle X^i, X^j\rangle \equiv 0$ for $i \neq j$.

Let $i \neq j$, and let us first establish that $\frac{X^i+X^j}{\sqrt{2}}$ is a one-dimensional B.m. Indeed for any $s < t$ the quantity

$$\Big(\frac{X^i + X^j}{\sqrt{2}}\Big)(t) - \Big(\frac{X^i + X^j}{\sqrt{2}}\Big)(s) = \frac{X^i_t - X^i_s + X^j_t - X^j_s}{\sqrt{2}}$$

is independent from $\mathcal{F}_s$ and is distributed as $\mathcal{N}(0, \sigma^2)$ with

$$\sigma^2 = \mathbb{V}\mathrm{ar}\big(\frac{X^i_t - X^i_s}{\sqrt{2}}\big) + \mathbb{V}\mathrm{ar}\big(\frac{X^j_t - X^j_s}{\sqrt{2}}\big) = \frac{t-s}{2} + \frac{t-s}{2} = t - s$$

(using the fact that $X^i_t - X^i_s$ and $X^j_t - X^j_s$ are independent between themselves, independent from $\mathcal{F}_s$ and distributed as $\mathcal{N}(0, t-s)$).

Thus using Definition 2.3.1 we get for any $0 \le t \le T$,

$$\langle X^i, X^j\rangle_t = \frac{1}{2}\big[\langle X^i + X^j\rangle_t - \langle X^i\rangle_t - \langle X^j\rangle_t\big] = \langle \frac{X^i + X^j}{\sqrt{2}}\rangle_t - \frac{t}{2} - \frac{t}{2} = t - t = 0.$$

"$ii) \Rightarrow i)$": The idea is to identify the law of $X_t - X_s$ through its characteristic function (c.f.). We recall that if $Y \sim \mathcal{N}_d(0, \theta\, I_d)$ its c.f. is given by

$$\varphi_Y(\xi) = \mathbb{E}[e^{i\xi.Y}] = e^{-\frac{1}{2}|\xi|^2\theta}, \;\; \xi \in \mathbb{R}^d$$

(we use the notations $x.y = \sum_{j=1}^d x_j y_j$ and $|x| = \sqrt{\sum_{j=1}^d x_j^2}$ for any $x, y \in \mathbb{R}^d$).

Let $\xi \in \mathbb{R}^d$ and let us consider the (complex valued) martingale $i\xi.X = (i\sum_{j=1}^d \xi^j X^j_t)_t$, with bracket

$$\langle i\xi.X\rangle_t = (i)^2 \langle \sum_{j=1}^d \xi^j X^j, \sum_{l=1}^d \xi^l X^l\rangle_t = -\sum_{j=1}^d \sum_{l=1}^d \xi^j \xi^l \langle X^j, X^l\rangle_t = -|\xi|^2 t.$$

Here we have used the bilinearity of the bracket and the assumption $\langle X^i, X^j\rangle_t = \mathbf{1}_{i=j}\, t$. In fact the result of Lemma 4.1.1 remains true for a complex valued martingale so that we can say that $Z = (Z_t)_{0\le t\le T}$ defined by

$$Z_t = \exp\big(i\xi.X_t + \frac{1}{2}|\xi|^2 t\big), \quad 0 \le t \le T,$$

is a martingale. Let now $0 \le s < t \le T$ and $A \in \mathcal{F}_s$. We have

$$\mathbb{E}[\,\mathbf{1}_A \exp(i\xi.(X_t - X_s))\,] = \mathbb{E}[\,\mathbf{1}_A Z_t Z_s^{-1} e^{-\frac{1}{2}|\xi|^2(t-s)}\,] = e^{-\frac{1}{2}|\xi|^2(t-s)}\mathbb{E}[\,\mathbf{1}_A Z_s^{-1} Z_t\,].$$

But $\mathbf{1}_A Z_s^{-1}$ is $\mathcal{F}_s$-measurable so that by definition of the conditional expectation we have $\mathbb{E}[\,\mathbf{1}_A Z_s^{-1} Z_t\,] = \mathbb{E}[\,\mathbf{1}_A Z_s^{-1}\mathbb{E}(Z_t|\mathcal{F}_s)\,] = \mathbb{E}[\,\mathbf{1}_A Z_s^{-1} Z_s\,] = \mathbb{E}[\mathbf{1}_A]$ and finally

$$\mathbb{E}[\,\mathbf{1}_A \exp(i\xi.(X_t - X_s))\,] = e^{-\frac{1}{2}|\xi|^2(t-s)}\mathbb{E}[\mathbf{1}_A].$$

Taking $A = \Omega$ this shows that $X_t - X_s \sim \mathcal{N}_d(0, (t-s)I_d)$. Denoting $\mathbb{E}_A(Y) = \mathbb{E}[\mathbf{1}_A Y]/\mathbb{P}(A)$ the expectation of a random variable Y knowing the event $A \in \mathcal{F}_s$, this also shows that

$$\mathbb{E}_A[e^{i\xi.(X_t - X_s)}] = e^{-\frac{1}{2}|\xi|^2(t-s)}, \quad \forall A \in \mathcal{F}_s.$$

This shows that $X_t - X_s$ is independent from $\mathcal{F}_s$. □

4.2 Girsanov theorem

We stay under the assumptions of Section 4.1.

Definition 4.2.1. A probability measure $\widetilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ is said to be absolutely continuous with respect to $\mathbb{P}$ if for any $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$ we have $\widetilde{\mathbb{P}}(A) = 0$.

This is denoted by $\widetilde{\mathbb{P}} << \mathbb{P}$.

We say that $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent if $\widetilde{\mathbb{P}} << \mathbb{P}$ and $\mathbb{P} << \widetilde{\mathbb{P}}$.

Remark 4.2.1. If an event is true $\mathbb{P}$-a.s. and $\widetilde{\mathbb{P}} << \mathbb{P}$, then it is true $\widetilde{\mathbb{P}}$-a.s.

Exercise 4.2.1. Show that if $X = \lim_{\mathbb{P}} X_n$ (i.e. X_n tends to X in probability, as $n \to \infty$, for the probability measure $\mathbb{P}$) and $\widetilde{\mathbb{P}} << \mathbb{P}$ then $X = \lim_{\widetilde{\mathbb{P}}} X_n$.

Theorem 4.2.1 (Radon-Nykodim)**.** *There is equivalence between:*

i) We have $\widetilde{\mathbb{P}} << \mathbb{P}$.

ii) There exists $Z \geq 0$, *with* $\mathbb{E}(Z) < +\infty$ *such that* $\widetilde{\mathbb{P}}(A) = \mathbb{E}_{\mathbb{P}}(Z\mathbf{1}_A)$ *for any* $A \in \mathcal{F}$.

We note $Z =: \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}$, *this is the density of* $\widetilde{\mathbb{P}}$ *with respect to* $\mathbb{P}$.

Proof. Cf [1], Theorem 32.2. □

Definition 4.2.2. Let $\widetilde{\mathbb{P}}$ be a probability measure on $(\Omega, \mathcal{F})$. We say that $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are locally equivalent if for all $0 \leq t \leq T$, their restriction to $\mathcal{F}_t$, denoted $\mathbb{P}_t$ and $\widetilde{\mathbb{P}}_t$ are equivalent. We denote

$$Z_t := \frac{d\widetilde{\mathbb{P}}_t}{d\mathbb{P}_t} =: \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t}, \quad 0 \leq t \leq T,$$

the local density of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$.

For any $0 \leq t \leq T$, and any $\mathcal{F}_t$-measurable random variable Y we have

$$\mathbb{E}_{\widetilde{\mathbb{P}}}(Y) = \int_\Omega Y d\widetilde{\mathbb{P}}_t = \int_\Omega Y \frac{d\widetilde{\mathbb{P}}_t}{d\mathbb{P}_t} d\mathbb{P}_t = \mathbb{E}_{\mathbb{P}}[Y Z_t]. \tag{4.2.1}$$

With all these definitions we can now state the Girsanov theorem.

Theorem 4.2.2 (Girsanov)**.** *Let* $B = ((B_t^1, \ldots, B_t^d)^T)_{0 \leq t \leq T}$ *be a* d*-dimensional* $(\mathcal{F}_t)$*-standard Brownian motion (in particular* $B_0 = 0$*) defined on* $(\Omega, \mathcal{F}, \mathbb{P})$. *Let* $X = (X^1, \ldots, X^d)^T$ *be an adapted process, with* $X^i \in \Pi_2(B^i)$, $1 \leq i \leq d$ *and*

$$\mathbb{E}_{\mathbb{P}}\big[\exp\big(\frac{1}{2}\int_0^T |X_s|^2 ds\big)\big] < +\infty.$$

Let $Z = \mathcal{E}xp\big(\int_0^{\cdot} X_s.dB_s\big)$ *be defined by*

$$Z_t = \exp\Big(\int_0^t X_s.dB_s - \frac{1}{2}\int_0^t |X_s|^2 ds\Big), \quad 0 \leq t \leq T,$$

(we denote $\int_0^t X_s.dB_s = \sum_{i=1}^d \int_0^t X_s^i dB_s^i$ *for any* t*).*

Define $\widetilde{\mathbb{P}}$ *locally equivalent to* $\mathbb{P}$ *by* $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t} = Z_t$.

Then $\widetilde{B} = (\widetilde{B}^1, \ldots, \widetilde{B}^d)^T$ *defined* $\widetilde{B}_t^i = B_t^i - \int_0^t X_s^i ds$, $0 \leq t \leq T$, $1 \leq i \leq d$ *is a* d*-dimensional* $(\mathcal{F}_t)$*-standard Brownian motion under* $\widetilde{\mathbb{P}}$.

The remainder of the chapter is devoted to the proof of Theorem 4.2.2. We need two lemmas and a proposition. We denote $\mathbb{E} = \mathbb{E}_{\mathbb{P}}$ and $\widetilde{\mathbb{E}} = \mathbb{E}_{\widetilde{\mathbb{P}}}$.

Lemma 4.2.1. *In the context of Theorem 4.2.2, for any* $0 \leq s < t \leq T$ *and any* $\mathcal{F}_t$*-measurable random variable* Y *with* $\widetilde{\mathbb{E}}|Y| < +\infty$, *we have*

$$Z_s \widetilde{\mathbb{E}}(Y|\mathcal{F}_s) = \mathbb{E}[Y Z_t \,|\, \mathcal{F}_s] \quad (a.s.)$$

In other words Lemma 4.2.1 says that we have for any $\mathcal{F}_t$-measurable Y

$$\widetilde{\mathbb{E}}(Y|\mathcal{F}_s) = \mathbb{E}\big[Y\frac{Z_t}{Z_s}\,|\,\mathcal{F}_s\big].$$

This gives in some sense the density that allows to compute the conditional expectation $\widetilde{\mathbb{E}}(\cdot|\mathcal{F}_s)$ of a $\mathcal{F}_t$-measurable random variable (we have to be very cautious when saying that, because remember that $\widetilde{\mathbb{E}}(Y|\mathcal{F}_s)$ is a random variable, not always an expectation).

Proof of Lemma 4.2.1. Let $0 \leq s < t \leq T$. For $A \in \mathcal{F}_s$ we have

$$\mathbb{E}[\mathbf{1}_A Z_s \widetilde{\mathbb{E}}(Y|\mathcal{F}_s)] = \widetilde{\mathbb{E}}[\mathbf{1}_A \widetilde{\mathbb{E}}(Y|\mathcal{F}_s)] = \widetilde{\mathbb{E}}[\mathbf{1}_A Y] = \mathbb{E}[\mathbf{1}_A Y Z_t].$$

Here we have used at the first equality (4.2.1) and the fact that $\mathbf{1}_A\widetilde{\mathbb{E}}(Y|\mathcal{F}_s)$ is $\mathcal{F}_s$-measurable. At the second equality we have used the definition of the conditional expectation. At the third inequality we have used (4.2.1) and the fact that $\mathbf{1}_A Y$ is $\mathcal{F}_t$-measurable.

As $Z_s\widetilde{\mathbb{E}}(Y|\mathcal{F}_s)$ is $\mathcal{F}_s$-measurable the above equality shows that $\mathbb{E}[YZ_t\,|\,\mathcal{F}_s] = Z_s\widetilde{\mathbb{E}}(Y|\mathcal{F}_s)$. □

Now remember two possible ways to define $\langle X, Y\rangle$ for $X = M + A$ and $Y = N + C$ two continuous semimartingales. One is

$$\langle X, Y\rangle_t = \lim_{|\Delta_n|\downarrow 0} |_{\mathbb{P}}\, Q_t^{\Delta_n}(X, Y),$$

the other one is $\langle X, Y\rangle = \langle M, N\rangle$ where $\langle M, N\rangle$ is the unique continuous adapted process of FV starting from zero such that $MN - \langle M, N\rangle$ is a martingale under $\mathbb{P}$ (as M and N are; see Remark 1.3.3).

So one has the feeling that the bracket $\langle X, Y\rangle_{(\mathbb{P})}$ depends on the underlying probability measure $\mathbb{P}$ (i.e. could be altered by a change of measure) ...

Lemma 4.2.2. *Let $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ equivalent (or locally equivalent). Let X and Y be two semimartingales under $\mathbb{P}$. Then $\langle X, Y\rangle_{(\mathbb{P})} = \langle X, Y\rangle_{(\tilde{\mathbb{P}})}$, $\mathbb{P}$ and $\widetilde{\mathbb{P}}$-a.s.*

... this lemma says that in fact not, as long as the change of measure is equivalent.

Proof of Lemma 4.2.2. By Exercise 4.2.1 we have $\lim_{\mathbb{P}} Q_t^{\Delta_n}(X, Y) = \lim_{\widetilde{\mathbb{P}}} Q_t^{\Delta_n}(X, Y)$, as $\widetilde{\mathbb{P}} << \mathbb{P}$. This equality in understood in the $\mathbb{P}$ and $\widetilde{\mathbb{P}}$-almost sure sense and we have also $\lim_{\mathbb{P}} Q_t^{\Delta_n}(X, Y) = \langle X, Y\rangle_{t\ (\mathbb{P})}$ $\mathbb{P}$-a.s. and $\lim_{\widetilde{\mathbb{P}}} Q_t^{\Delta_n}(X, Y) = \langle X, Y\rangle_{t\ (\widetilde{\mathbb{P}})}$ $\widetilde{\mathbb{P}}$-a.s. Therefore the result. □

Proposition 4.2.1. *In the context of Theorem 4.2.2 let M be a martingale under $\mathbb{P}$ (with $M_0 = 0$). Then $\widetilde{M}$ defined by*

$$\widetilde{M}_t = M_t - \int_0^t X_s.d\langle M, B\rangle_s$$

(we denote $\int_0^t X_s.d\langle M, B\rangle_s = \sum_{i=1}^d \int_0^t X_s^i d\langle M, B^i\rangle_s$) is a martingale under $\widetilde{\mathbb{P}}$.

Proof. We have, remembering that $dZ_t = Z_t X_t.dB_t = \sum_{i=1}^d Z_t X_t^i dB_t^i$ (see the proof of Lemma 4.1.1),

$$\begin{aligned}
Z_t\widetilde{M}_t &= \int_0^t Z_s d\widetilde{M}_s + \int_0^t \widetilde{M}_s dZ_s + \langle Z, \widetilde{M}\rangle_t \\
&= \int_0^t Z_s dM_s - \sum_{i=1}^d \int_0^t Z_s X_s^i d\langle M, B^i\rangle_s + \sum_{i=1}^d \int_0^t \widetilde{M}_s Z_s X_s^i dB_s^i + \sum_{i=1}^d \int_0^t Z_s X_s^i d\langle M, B^i\rangle_s \\
&= \int_0^t Z_s dM_s + \sum_{i=1}^d \int_0^t \widetilde{M}_s Z_s X_s^i dB_s^i.
\end{aligned}$$

Thanks to the assumption $\mathbb{E}_{\mathbb{P}}\big[\exp\big(\frac{1}{2}\int_0^T |X_s|^2 ds\big)\big] < +\infty$ it is possible to proceed as if the stochastic integrals are martingales under $\mathbb{P}$ (see [4] for details).

Thus $(Z_t \widetilde{M}_t)$ is a martingale under $\mathbb{P}$. Thus, using Lemma 4.2.1,

$$\widetilde{\mathbb{E}}[\widetilde{M}_t | \mathcal{F}_s] = \frac{1}{Z_s} \mathbb{E}[\widetilde{M}_t Z_t \,|\, \mathcal{F}_s] = \frac{1}{Z_s} Z_s \widetilde{M}_s = \widetilde{M}_s.$$

Thus $\widetilde{M}$ is a martingale under $\widetilde{\mathbb{P}}$. □

Proof of Theorem 4.2.2. For each $1 \leq i \leq d$ we apply Proposition 4.2.1 with $M = B^i$. This gives

$$B_t^i - \int_0^t X_s . d\langle B^i, B \rangle_s = B_t^i - \sum_{j=1}^d \int_0^t X_s^j d\langle B^i, B^j \rangle_s = B_t^i - \int_0^t X_s^i ds = \widetilde{B}_t^i,$$

and $\widetilde{B}^i$ is a martingale under $\widetilde{\mathbb{P}}$ by the proposition.

It remains to check that $\widetilde{B}$ is a d-dimensional $(\mathcal{F}_t)$-standard Brownian motion under $\widetilde{\mathbb{P}}$, using the theorem of Lévy. We have

$$\langle \widetilde{B}^i, \widetilde{B}^j \rangle_{(\widetilde{\mathbb{P}})} = \langle \widetilde{B}^i, \widetilde{B}^j \rangle_{(\mathbb{P})} = \langle B^i, B^j \rangle_{(\mathbb{P})},$$

using Lemma 4.2.2 at the first equality. But B is a B.m. under $\mathbb{P}$ thus $\langle B^i, B^j \rangle_{t\,(\mathbb{P})} = \mathbf{1}_{i=j} t$ for any $0 \leq t \leq T$. □

Chapter 5

Applications to Finance, Stochastic Differential Equations and link with Partial Differential Equations

In this chapter we present the applications of stochastic calculus to continuous time financial models. We also illustrate the link between Stochastic Differential Equations (SDE) and Partial Differential Equations (PDE). We are inspired mostly by [7].

5.1 Introduction and motivations, one-dimensional Black and Scholes model

Some time horizon $0 < T < \infty$ is fixed. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ are given.

In the sequel we will have one risky asset (or stock) of price $S(t)$ at time $0 \leq t \leq T$ (the risky asset can be an action, a baril of petrol....). One assumes that the process S follows the Black-Scholes SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dW_t, \quad S_0 = x \tag{5.1.1}$$

that we have already encountered in Example 3.2.1. In (5.1.1) the process W is some 1D $(\mathcal{F}_t)$-Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$, $\mu \in \mathbb{R}$ is the trend of the model and $\sigma > 0$ is its volatility. At the end of the forthcoming Section 5.2 we will have more informations about (5.1.1): in particular its existing solution is unique and remains strictly positive as long as $x > 0$.

In addition to the risky asset S we will have a non-risky asset (or bond) of price $S_0(t)$ at time $0 \leq t \leq T$. Its dynamic is given by

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = 1,$$

where $r > 0$ is the short interest rate. Note that for simplicity r is constant so that the dynamic of S_0 is deterministic. Note that $S_0(t) = e^{rt}$, $0 \leq t \leq T$ (it is immediate to solve the involved ordinary differential equation). The non-risky asset will help us to modelize the money we put at or borrow from the bank (for example if one borrows 1 euro at time $t = 0$ one has to give back $S_0(T) = e^{rT}$ euros at time $t = T$).

Our object of interest is a derivative product (or "derivative security", or "option") on the risky asset S.

Example 5.1.1. We start with the classical example of the European Call option. This option has a maturity T (for simplicity this maturity is our time horizon T). It has a strike $K > 0$.

It has a buyer (owner) and a seller: at time $t = 0$ the buyer gives C_0 euros to the seller in exchange of the option.

This option gives the right to its owner to buy the risky asset at price K at time $t = T$.

At time T there are only two possible situations: if $S(T) > K$, it is interesting for the owner of the option to use this right; he may indeed buy the asset at price K, and immediately sell it at its market price $S(T)$; he gets then a benefit of $S(T) - K$ euros.

If $S(T) \leq K$, it is not interesting to use the option. We say that the option is dead. The owner gets nothing.

If we sum up both situations in one formula the European Call option pays $(S(T) - K)_+$ euros to its owner at maturity $t = T$.

Now if I am the seller of the option two questions arise:

i) At which "fair price" C_0 do I sell the option ? (at time $t = 0$). In other words what money do I claim to the buyer at time $t = 0$ in exchange to the fact that I promise to give him $(S(T) - K)_+$ euros at $t = T$? (**Question of Pricing**)

ii) Once I have received the C_0 euros at time $t = 0$, what do I do with this money to be (almost) sure to have $(S(T) - K)_+$ at my disposal at time $t = T$? (in oder to be able to provide this to the owner) (**Question of Hedging**)

Concerning Point i) we will see that

$$C_0 = \mathbb{E}_{\mathbb{Q}}[e^{-rT}(S(T) - K)_+] \tag{5.1.2}$$

where $\mathbb{Q}$ is some "risk-neutral probability measure" that we will exhibit in the sequel (Section 5.4).

But in fact Point i) is not separable from Point ii): we will solve both issues at the same time.

5.2 A digression on SDE

We have $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ (the "drift term") and $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ (the "diffusion term").

Let B a $(\mathcal{F}_t)$-Brownian motion of dimension m, defined on $(\Omega, \mathcal{F}, \mathbb{P})$. A strong solution of the SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = x \tag{5.2.1}$$

is an a.s. continuous process X satisfying

i) X is $(\mathcal{F}_t)$-adapted.

ii) $\mathbb{P}(X_0 = x) = 1$.

iii) For any $1 \leq i \leq d$, any $1 \leq j \leq m$, $\mathbb{E} \int_0^T \{|b_i(s, X_s) + \sigma_{ij}^2(s, X_s)\}ds < \infty$ (this is to ensure that the integrals in Point iv) are correctly defined).

iv) For any $1 \leq i \leq d$ and any $0 \leq t \leq T$ we have

$$X_t^i = x^i + \int_0^t b_i(s, X_s)ds + \sum_{j=1}^m \int_0^t \sigma_{ij}(s, X_s)dB_s^j.$$

Concerning SDEs we have this kind of result, due again to Itô.

Theorem 5.2.1. *Assume we have*

$$|b(t, x) - b(t, y)| + \Big(\sum_{i=1}^d \sum_{j=1}^m (\sigma_{ij}(t, x) - \sigma_{ij}(t, y))^2\Big)^{1/2} \leq K|x - y|, \quad \forall x, y \in \mathbb{R}^d,\ \forall 0 \leq t \leq T$$

(this means that b and σ are globally Lipschitz) and

$$|b(t, x)|^2 + \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}^2(t, x) \leq K^2(1 + |x|^2), \quad \forall x \in \mathbb{R}^d,\ \forall 0 \leq t \leq T$$

(this is some "linear growth condition"), for some constant $0 < K < \infty$.

Then (5.2.1) *has a unique strong solution.*

Proof. See Theorems 5.2.5 and 5.2.9 in [4]. □

Example 5.2.1. The Black-Scholes SDE (5.1.1) writes

$$dS(t) = \bar{\sigma}(S(t))dW_t + b(S(t))dt$$

with the functions $\bar{\sigma}(x) = \sigma x$ and $b(x) = \mu x$.

We immediately see that the assumptions of Theorem 5.2.1 are satisfied. Thus we see that a solution to (5.1.1) exists and is unique. In fact we had already seen the existence in Example 3.2.1. Now we know that the sole solution to (5.1.1) is given by

$$S(t) = x \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W_t\right)$$

(this is formula (3.2.5) in Example 3.2.1). Note that this implies that this solution is strictlty positive at any time, as long as $x > 0$.

5.3 Self-financing portfolio

Let $(H(t))$ and $(H_0(t))$ be adapted processes.

We consider a strategy (or portfolio) constituted at time $0 \leq t \leq T$ with $H(t)$ shares of risky asset and $H_0(t)$ shares of non-risky asset.

The value at time $0 \leq t \leq T$ of such a portfolio is

$$V_t(H) = H(t)S(t) + H_0(t)S_0(t).$$

The process $(V_t(H))_{0 \leq t \leq T}$ is sometimes called the wealth process.

Definition 5.3.1. We say that $(H, H_0)^T$ is a self-financing strategy if

$$dV_t(H) = H(t)dS(t) + H_0(t)dS_0(t).$$

Where does this definition come from, and what does "self-financing" mean ?

Imagine that H and H_0 do not evolve permanently (continuously) but are piecewise constant, for a fixed randomness ω. In fact imagine that they are simple processes (like in Chapter 3).

That is we have a time grid $0 = t_0 < t_1 < \ldots < t_n = T$. At time t_i we decide the quantity $H(t_{i+1})$ of risky asset and the quantity $H_0(t_{i+1})$ of non-risky asset that will be held in the portfolio on time interval $(t_i, t_{i+1}]$, i.e. $H(t) = H(t_{i+1})$ and $H_0(t) = H_0(t_{i+1})$ for any $t \in (t_i, t_{i+1}]$. Note that $H(t_{i+1})$ and $H_0(t_{i+1})$ are $\mathcal{F}_{t_i}$-measurable.

We want that when we take a new position in the portfolio (that is we pass from $(H, H_0)^T(t_i)$ to $(H, H_0)^T(t_{i+1})$; we say that we rebalance the portfolio) its global value remains the same. That is to say:

$$H(t_i)S(t_i) + H_0(t_i)S_0(t_i) = H(t_{i+1})S(t_i) + H_0(t_{i+1})S_0(t_i) \tag{5.3.1}$$

(in the above expression the left hand side is the value of the portfolio just before rebalancing and time t_i, and the right hand side is its value just after rebalancing and time t_i).

This traducts the fact that the portfolio value evolves just because of the value of S, the capitalization at rate r (contained in S_0) and our choice of H and H_0: we get no money from outside and do not drop any; the portfolio is self-financing.

From (5.3.1) we have

$$\begin{aligned}
V_{t_{i+1}}(H) - V_{t_i}(H) &= H(t_{i+1})S(t_{i+1}) + H_0(t_{i+1})S_0(t_{i+1}) - H(t_i)S(t_i) - H_0(t_i)S_0(t_i) \\
&= H(t_{i+1})S(t_{i+1}) + H_0(t_{i+1})S_0(t_{i+1}) - H(t_{i+1})S(t_i) + H_0(t_{i+1})S_0(t_i) \\
&= H(t_{i+1})(S(t_{i+1}) - S(t_i)) + H_0(t_{i+1})(S_0(t_{i+1}) - S_0(t_{i+1})) \\
&= H(t_{i+1})\Delta S(t_{i+1}) + H_0(t_{i+1})\Delta S_0(t_{i+1})
\end{aligned}$$

Therefore Definition 5.3.1 in infinitesimal time.

Definition 5.3.2. Let $f : \mathbb{R}_+^* \to \mathbb{R}$ such that $f(S_T)$ is square integrable.

The price at time $0 \le t \le T$ of a derivative product paying $f(S(T))$ at maturity T is the value $V_t(H)$ of a self-financing strategy that replicates the pay-off $f(S(T))$, i.e. such that $V_T(H) = f(S(T))$ $\mathbb{P}$-a.s.

Remark 5.3.1. This give an answer to Question i) and ii) at the end of Section 5.1: if I am given $V_0(H)$ euros at time $t = 0$, by investing this money in a self-financing replicating portfolio containing shares of S, and shares of S_0 (that is borrowing if necessary from the bank, or putting money at the bank), I will be sure to have $f(S(T))$ euros at time $t = T$.

Remark 5.3.2. The quantities $H(t)$ and $H_0(t)$ are signed (if the sign is negative that means that we have a debt, in the risky asset, or the bond).

Consider $(H, H^0)^T$ a self-financing strategy. We have

$$\begin{aligned} dV_t(H) &= H(t)dS(t) + H_0(t)dS_0(t) \\ &= H(t)dS(t) + H_0(t)rS_0(t)dt \\ &= r(V_t(H) - H(t)S(t))dt + H(t)dS(t) \\ &= rV_t(H)dt + H(t)(dS(t) - rS(t)dt). \end{aligned} \tag{5.3.2}$$

Examine now the discounted wealth process $(e^{-rt}V_t)$ (here $e^{-rt} = 1/S_0(t)$ is the discount factor and we have written $V_t \equiv V_t(H)$).

We have, using Itô rule, the fact that (e^{-rt}) is of finite variation, and (5.3.2),

$$\begin{aligned} d(e^{-rt}V_t) &= -re^{-rt}V_t dt + e^{-rt}dV_t + 0 \\ &= -re^{-rt}V_t dt + re^{-rt}V_t dt + H(t)e^{-rt}(dS(t) - rS(t)dt). \\ &= H(t)d\big(e^{-rt}S(t)\big), \end{aligned} \tag{5.3.3}$$

by noticing that

$$d\big(e^{-rt}S(t)\big) = -re^{-rt}S(t)dt + e^{-rt}dS(t) = e^{-rt}(dS(t) - rS(t)dt).$$

Imagine now that we find a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ under which $(e^{-rt}S(t))$ is a martingale.

Assuming $H \in \Pi_2(e^{r\cdot}S(\cdot))$ we will have that $(e^{-rt}V_t)$ is a martingale under $\mathbb{Q}$. Thus if the strategy is replicating we have

$$e^{-rt}V_t = \mathbb{E}_{\mathbb{Q}}[e^{-rT}V_T \,|\, \mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[e^{-rT}f(S(T)), |\, \mathcal{F}_t], \quad 0 \le t \le T.$$

Thus

$$V_t = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}f(S(T))|\, \mathcal{F}_t] \tag{5.3.4}$$

for any time $0 \le t \le T$. In particular at time $t = 0$ we have

$$V_0 = \mathbb{E}_{\mathbb{Q}}[e^{-rT}f(S(T))],$$

this is the result announced in (5.1.2).

We are now seeking for $\mathbb{Q}$ (Section 5.4) and will aim at constructing a self-financing replicating portfolio (Section 5.5).

5.4 Risk -neutral probability measure

Definition 5.4.1. We say that a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is risk-neutral if it is locally equivalent to $\mathbb{P}$, and if the discounted risky asset price process $\big(e^{-rt}S(t)\big)_{0 \le t \le T}$ is a martingale under $\mathbb{Q}$.

It is quite easy to construct $\mathbb{Q}$ in the 1D Black-Scholes model. Remember that we have $d\big(e^{-rt}S(t)\big) = e^{-rt}(dS(t) - rS(t)dt)$, thus

$$d\big(e^{-rt}S(t)\big) = e^{-rt}(\mu S(t)dt + \sigma S(t)dW_t - rS(t)dt) = e^{-rt}S(t)\sigma\big(dW_t + \frac{\mu - r}{\sigma}dt\big).$$

We have $\frac{\mu-r}{\sigma} < \infty$ and thus the assumption $\mathbb{E}_{\mathbb{P}}\big[\exp\big(\frac{1}{2}\int_0^T |X_s|^2 ds\big)\big] < +\infty$ of Theorem 4.2.2 is immediately satisfied with $X \equiv -\frac{\mu-r}{\sigma}$.

Thus $\widetilde{W} = \big(W_t + \frac{\mu-r}{\sigma}t\big)_{0\le t\le T}$ is a $(\mathcal{F}_t)$-B.m. under $\mathbb{Q}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp\Big(-\frac{\mu - r}{\sigma}W_t - \frac{1}{2}\big(\frac{\mu - r}{\sigma}\big)^2 t\Big).$$

Thus we have $d\big(e^{-rt}S(t)\big) = e^{-rt}S(t)\sigma d\widetilde{W}_t$ and $(e^{-rt}S(t))$ is a $(\mathcal{F}_t)$ martingale under $\mathbb{Q}$ (one could check that $(e^{-rt}S(t)\sigma)$ is in $\Pi_2(\widetilde{W})$.

5.5 Construction of the self financing replicating portfolio, pricing and hedging formulae, link with PDE

To construct the portfolio of interest, one could use a martingale representation theorem (cf Theorem 6.1.1 in the Appendix).

One can also use PDE arguments: one chooses this path to highlight the link between SDE and PDE.

Proposition 5.5.1. *For any continuous function $f : \mathbb{R}_+^* \to \mathbb{R}$ satisfying $|f(e^x)| \le K'e^{K|x|^2}$, $\forall x \in \mathbb{R}$, with $K < \frac{\sigma^2}{8T}$, there is a unique solution $v \in C([0,T] \times \mathbb{R}_+^*) \cap C^{1,2}([0,T) \times \mathbb{R}_+^*)$ to the PDE*

$$\begin{cases} \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} v(t,x) + rx\partial_x v(t,x) + \partial_t v(t,x) - rv(t,x) &= 0, \qquad \forall (t,x) \in [0,T) \times \mathbb{R}_+^*, \\ v(T,x) &= f(x), \quad \forall x \in \mathbb{R}_+^*. \end{cases} \tag{5.5.1}$$

Proof. See the appendix. We use a log-change of variable trick, to deal with the fact that the coefficient $x \mapsto \sigma^2x^2$ is not elliptic, nor bounded and that $x \mapsto rx$ is not bounded. □

Let us now compute $dv(t, S(t))$ using Itô rule. We get

$$\begin{aligned} dv(t,S(t)) &= \partial_t v(t,S(t))dt + \partial_x v(t,S(t))dS(t) + \frac{1}{2}\partial^2_{xx}v(t,S(t))d\langle S\rangle_t + 0 \\ &= \partial_t v(t,S(t))dt + \frac{1}{2}\partial^2_{xx}v(t,S(t))\sigma^2 S^2(t)dt + \partial_x v(t,S(t))dS(t) \\ &= rv(t,S(t))dt - rS(t)\partial_x v(t,S(t))dt + \partial_x v(t,S(t))dS(t) \\ &= rv(t,S(t))dt + \partial_x v(t,S(t))[dS(t) - rS(t)dt] \end{aligned}$$

(here we have used (5.5.1) at the third line).

That is to say the process $\big(v(t,S(t))\big)$ satisfies (5.3.2) with $H(t) = \partial_x v(t,S(t))$. For clarity let us do again the computations of (5.3.2), but in the reverse sense, setting $H_0(t) = \big(v(t,S(t)) - \partial_x v(t,S(t))\, S(t)\big)/S_0(t)$. We have

$$\begin{aligned} dv(t,S(t)) &= rv(t,S(t))dt + \partial_x v(t,S(t))[dS(t) - rS(t)dt] \\ &= r\big(v(t,S(t)) - \partial_x v(t,S(t))\, S(t)\big)dt + \partial_x v(t,S(t))dS(t) \\ &= \partial_x v(t,S(t))dS(t) + H_0(t)rS_0(t)dt \\ &= \partial_x v(t,S(t))dS(t) + H_0(t)dS_0(t). \end{aligned}$$

Here we have used the definition of $H_0(t)$ at the third line and $dS_0(t) = rS_0(t)dt$ at the fourth line. Then the strategy defined by $H(t) = \partial_x v(t, S(t))$ and $H_0(t) = \big(v(t, S(t)) - \partial_x v(t, S(t))\, S(t)\big)/S_0(t)$, which is such that $V_t(H) = H(t)S(t) + H_0(t)S_0(t) = v(t, S(t))$, $0 \le t \le T$ is self-financing. In addition it satisfies

$$V_T(H) = v(T, S(T)) = f(S(T)),$$

that is to say it is replicating.

Thus the announced program is accomplished and the formulae (5.1.2) and (5.3.4) are valid: the price at time $0 \le t \le T$ of the derivative product of interest is

$$v(t, S(t)) = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)} f(S(T)) | \mathcal{F}_t]. \tag{5.5.2}$$

Remark 5.5.1. Let us check that we have (5.5.2); but by direct martingale computations. We have, using the same computations than in (5.3.3) - and the forthcoming formula (5.5.3),

$$d\big(e^{-rt}v(t, S(t))\big) = \partial_x v(t, S(t))e^{-rt}\big(dS(t) - rS(t)dt\big) = \partial_x v(t, S(t))e^{-rt}\sigma S(t)\, d\widetilde{W}_t.$$

This yields

$$e^{-rs}v(s, S(s)) = e^{-rt}v(t, S(t)) + \int_t^s e^{-ru}\partial_x v(u, S(u))\sigma S(u)\, d\widetilde{W}_u,$$

for any $0 \le t < s < T$. The r.v. $e^{-rt}v(t, S(t))$ is $\mathcal{F}_t$-measurable and $\big(\int_t^s e^{-ru}\partial_x v(u, S(u))\sigma S(u)\, d\widetilde{W}_u\big)_{t\le s<T}$ is a $(\mathcal{F}_t)$ martingale under $\mathbb{Q}$ (we admit $e^{-r\cdot}\partial_x v(\cdot, S(\cdot))\sigma S(\cdot) \in \Pi_2(\widetilde{W})$), with value 0 at time t. Taking then the conditional expectation $\mathbb{E}_{\mathbb{Q}}[\,\cdot\,|\,\mathcal{F}_t]$ of the above expression we get

$$\mathbb{E}_{\mathbb{Q}}[e^{-rs}v(s, S(s))\,|\,\mathcal{F}_t] = e^{-rt}v(t, S(t))$$

Using dominated convergence it is possible to prove that $\mathbb{E}_{\mathbb{Q}}[e^{-rs}v(s, S(s))\,|\,\mathcal{F}_t]$ converges to $\mathbb{E}_{\mathbb{Q}}[e^{-rT}f(S(T))\,|\,\mathcal{F}_t]$ when $s \to T$. This yields (5.5.2).

One step further and conclusion of the section. Remember that $\widetilde{W}$ is $(\mathcal{F}_t)$-Markov under $\mathbb{Q}$ (Proposition 1.4.1). A consequence is stated at the end of the following exercise.

Exercise 5.5.1. 1) Show that

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}_t. \tag{5.5.3}$$

2) Show that S is a $(\mathcal{F}_t)$-Markov process under $\mathbb{Q}$ (hint: one may use the explicit solution of (5.5.3)).

Thus the price at time $0 \le t \le T$ of the process of interest is given by

$$v(t, S(t)) = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)} f(S(T)) | \, S(t)].$$

So that the number of shares of risky asset at time $0 \le t \le T$ in the corresponding strategy is

$$H(t) = \partial_x v(t, S(t)) = \partial_x \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)} f(S(T)) | \, S(t) = x]_{|x=S(t)}.$$

For conciseness the RHS is often denoted $\partial_{S(t)}\mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)} f(S(T)) | \, S(t)]$. This is called the delta of the option.

Thus we have answered the questions of pricing and hedging (questions i) and ii) and the end of Section 5.1): they "reduce" to the question of computing $\mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)} f(S(T)) | \, S(t)]$ (for the price) and $\partial_{S(t)}\mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)} f(S(T)) | \, S(t)]$ (for the hedge).

Chapter 6

Appendix

This appendix gathers the proofs of various technical results, that have been used in the previous chapters.

6.1 Functions of finite variation

Proposition 6.1.1. *If a function $f : [a,b] \to \mathbb{R}$ is of finite variation, then $f = f_1 - f_2$ where f_1 and f_2 are two increasing functions.*

Proof. We set $Vf(x) := V_{[a,x]}(f)$ for any $x \in [a,b]$ and consider the function Vf. We have $f = f + Vf - Vf$. We will check that Vf and $f + Vf$ are increasing functions.

Let $a \leq x < y \leq b$. Let $(x_i)_{i=0}^I$ a subdivision of $[a,x]$ (i.e. $a = x_0 < x1 < \ldots < x_I = x$), then $(x_i)_{i=0}^I \cup \{y\}$ is a subdivision of $[a,y]$. Thus by definition of $Vf(y)$ we have

$$\sum_{i=1}^{I} |f(x_i) - f(x_{i-1})| + |f(y) - f(x)| \leq Vf(y),$$

and then

$$Vf(x) + |f(y) - f(x)| \leq Vf(y)$$

(using this time the definition of $Vf(x)$ and the fact that the supremum is the smallest upper bound). We thus get

$$|f(y) - f(x)| \leq Vf(y) - Vf(x). \tag{6.1.1}$$

Note that this show that Vf is increasing. Now from (6.1.1) one gets

$$f(x) - f(y) \leq |f(y) - f(x)| \leq Vf(y) - Vf(x)$$

and then

$$f(x) + Vf(x) \leq f(y) + Vf(y).$$

We are done. □

Proof of Proposition 5.5.1. Consider the PDE

$$\left\{ \begin{array}{rcll} \frac{1}{2}\sigma^2\partial^2_{yy}\tilde{v}(t,y) + (r - \frac{\sigma^2}{2})\partial_y\tilde{v}(t,y) + \partial_t\tilde{v}(t,y) - r\tilde{v}(t,x) & = & 0, & \forall (t,x) \in [0,T) \times \mathbb{R}, \\ \tilde{v}(T,y) & = & \tilde{f}(y), & \forall x \in \mathbb{R}, \end{array} \right. \tag{6.1.2}$$

with $\tilde{f}(y) = f(e^y)$ for all $y \in \mathbb{R}$.

The (constant) coefficients $\frac{1}{2}\sigma^2$, $(r - \frac{\sigma^2}{2})$ and r are obviously bounded. The coefficient $\frac{1}{2}\sigma^2$ is uniformly strictly elliptic (as it is strictly positive !). The terminal condition $\tilde{f}$ satisfies $|\tilde{f}(y)| \leq K' e^{K|y|^2}$, $\forall y \in \mathbb{R}$, with $K \leq \sigma^2/8T$.

Thus there is a unique solution $\tilde{v} \in C([0,T] \times \mathbb{R}) \cap C^{1,2}([0,T) \times \mathbb{R})$ to (6.1.2) (this is a classical result on parabolic PDEs; see [3], in particular Theorem 1.12 and 1.16 therein).

Then it suffices to set $v(t,x) = \tilde{v}(t, \log(x))$ to get a solution of (5.5.1). Indeed, using $\tilde{v}(t,y) = v(t,e^y)$, and then $\partial_y \tilde{v}(t,y) = e^y \partial_x v(t,e^y)$ and $\partial^2_{yy} \tilde{v}(t,y) = e^y \partial_x v(t,e^y) + e^{2y} \partial^2_{xx} v(t,e^y)$ one gets

$$\begin{aligned}
0 &= \tfrac{1}{2}\sigma^2 \partial^2_{yy}\tilde{v}(t,y) + (r - \tfrac{\sigma^2}{2})\partial_y \tilde{v}(t,y) + \partial_t \tilde{v}(t,y) - r\tilde{v}(t,x) \\
&= \tfrac{1}{2}\sigma^2 \Big(e^y \partial_x v(t,e^y) + e^{2y}\partial^2_{xx} v(t,e^y) \Big) + (r - \tfrac{\sigma^2}{2}) e^y \partial_x v(t,e^y) + \partial_t v(t,e^y) - rv(t,e^y) \\
&= \tfrac{1}{2}\sigma^2 (e^y)^2 \partial^2_{xx} v(t,e^y) + re^y \partial_x v(t,e^y) + \partial_t v(t,e^y) - rv(t,e^y)
\end{aligned}$$

for any $y \in \mathbb{R}$, any $t \in [0,T)$. Therefore the first line of (5.5.1), by bijection. The terminal condition $v(T,x) = \tilde{f}(\log(x)) = f(x)$, $x \in \mathbb{R}^*_+$ is easily checked. Once this solution $v(t,x)$ has been constructed it is easy to check it is unique, using the uniqueness of $\tilde{v}$ and bijection arguments. □

Theorem 6.1.1 (Brownian martingale representation theorem)**.** *Let $B(t)$ a r-dimensional Brownian motion defined on some probability space $(E, \mathcal{E}, P)$ and $(\mathcal{G}_t)$ its natural filtration. If $(N(t))$ is a $(\mathcal{G}_t)$-martingale under P, with $\mathbb{E}_P |N(t)|^2 < \infty$ for any t, there exist unique adapted processes $\Gamma_j(t)$, $1 \le j \le r$, satisfying $\Gamma_j \in \Pi_2(B_j)$ for all $1 \le j \le d$, s.t.,*

$$\forall t, \quad N(t) = N(0) + \sum_{j=1}^{r} \int_0^t \Gamma_j(s) dB_j(s).$$

Proof. See Theorem V.3.5 in [6]. □

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