Geophysical imaging: Mathematics for imaging the subsurface

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Using Helmoltz-Hodge decomposition of the displacement field $u(\mathrm{x}, t)$ in a scalar potential field $\psi(\mathrm{x}, t)$ and a vectorial potential field $\Psi(\mathrm{x}, t)$

$$\mathrm{u}(\mathrm{x}, t) = \nabla \psi(\mathrm{x}, t) + \mathrm{curl}\ \Psi(\mathrm{x}, t), \quad \mathrm{div}\ \Psi = 0. \tag{1}$$

we can show in the homogenenous isotropic case

$$\nabla \left(\rho \frac{\partial^2 \psi}{\partial t^2} - (\lambda + 2\mu)\, \Delta \psi \right) = -\mathrm{curl}\ \left(\rho \frac{\partial^2 \Psi}{\partial t^2} - \mu \Delta \Psi \right) \tag{2}$$

One solution to this equation is to cancel simultaneously the left and right hand side, which yields one equation on the scalar potential ψ

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\lambda + 2\mu}{\rho} \Delta \psi = 0, \tag{3}$$

and one equation on the vector potential $\mathbf{\Psi}$

$$\frac{\partial^2 \mathbf{\Psi}}{\partial t^2} - \frac{\mu}{\rho} \Delta \mathbf{\Psi}. \tag{4}$$

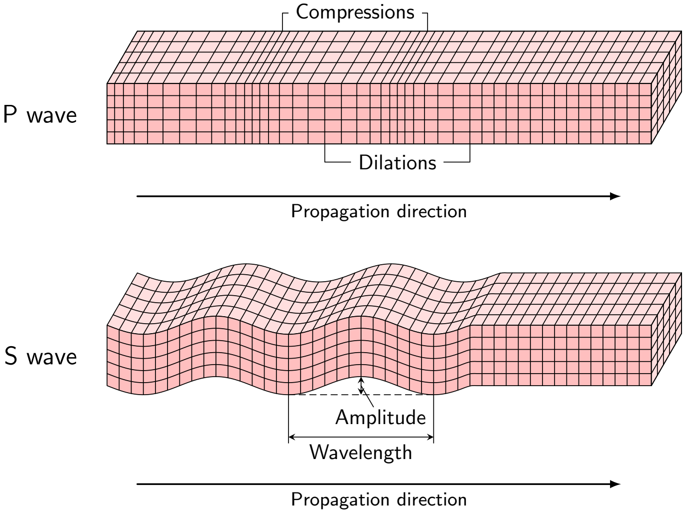
Both are **wave equations**. The first describes the propagation of pressure waves at speed V_P such that

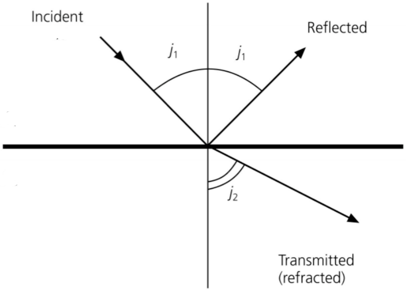
$$V_P = \sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{5}$$

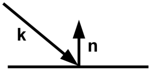
The second describes the propagation of shear-waves at speed V_S such that

$$V_S = \sqrt{\frac{\mu}{\rho}} \tag{6}$$

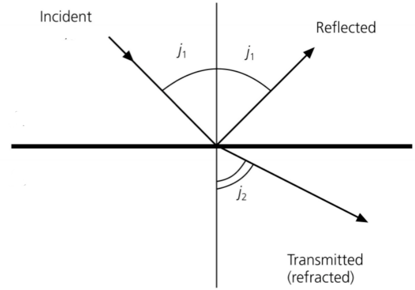
Two-modes of propagation: P-waves and S-waves

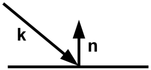






$$\frac{\sin j_1}{v_1} = \frac{\sin j_2}{v_2} \quad j_1 = j_1' \tag{7}$$





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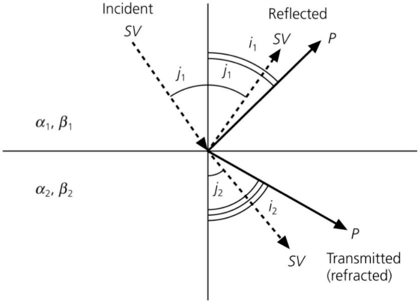
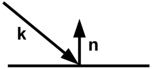
- this setup is valid for single wave type
  - P waves in acoustic media
  - SH waves (no coupling with P waves)





$$\frac{\sin j_1}{v_{S1}} = \frac{\sin j_2}{v_{S2}} = \frac{\sin i_1}{v_{P1}} = \frac{\sin i_2}{v_{P2}} \quad j_1 = j_1' \tag{8}$$

- this setup is generic
    - P waves in solids
    - SV waves

Surface waves

- the free surface is a particular interface
  - on one side a solid : v_P, v_S and ρ_S
  - on the other side, air : $v_{P_{air}} < v_P$, $v_S = 0$ and $\rho_{air} <<< \rho_S$

- the free surface is a particular interface
  - on one side a solid : v_P, v_S and ρ_S
  - on the other side, air : $v_{P_{air}} < v_P$, $v_S = 0$ and $\rho_{air} <<< \rho_S$
    $\rightarrow$ we generally assume air as void
- we model the free surface by imposing no traction force normal to the surface

$$\mathrm{t}(\mathrm{n}) = \sigma \mathrm{n} = 0, \tag{9}$$

  where n is the unit vector normal to the surface

Assuming the topography is flat, and the normal points upward, we have

$$n = [0 \; 0 \; 1]^T \tag{10}$$

and thus the free surface condition translates into

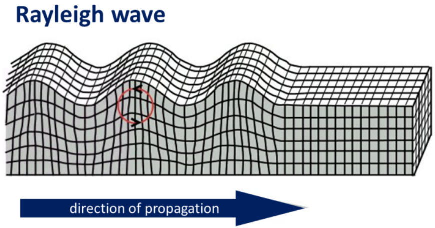
$$\sigma_{13} = 0, \;\; \sigma_{23} = 0, \;\; \sigma_{33} = 0. \tag{11}$$

- particles move in the P-SV plane
- result from interferences between P and SV waves
- homogeneous medium and no-free surface: non-dispersive waves, otherwise dispersive waves
- ellptical retrograde motion in the vertical plane parallel to propagation

$$v_R < v_S < v_P \tag{12}$$

$$v_R^6 - 8v_S^2 v_R^4 + (24 - 16v_S^2/v_P^2)v_S^4 v_R^2 + 16(v_S^2/v_P^2 - 1)v_S^6 = 0 \tag{13}$$

where v_R is the Rayleigh wave velocity, for perfect solid media, $\nu \approx 0.25$ giving $v_R = 0.919 v_S$

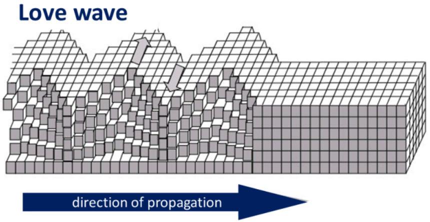


- particle motion in the SH plane
- result from interferences between incident, reflected and refracted SH in an heterogenous zone close to the surface (layered media) → does not exist in homogeneous media
- we have

$$v_{S1} < v_L < v_{S2} \tag{14}$$

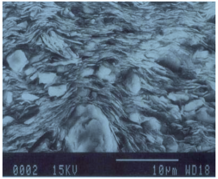
where V_{S1} and V_{S2} are the S-wave velocities in the two layers close from the surface

- Love waves are always dispersive
- Particle motion transverse and confined to the plane perpendicular to the direction of propagation waves

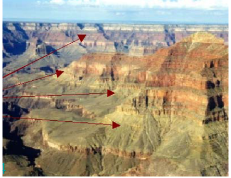


- Anisotropy: changing of the behavior depending on the direction
- In geologic media, wave anisotropy can have two origins: internal (mineral composition → static anisotropy) and external (layer structure for example → dynamic anisotropy)

Micro-scale anisotropy: mineral anisotropy

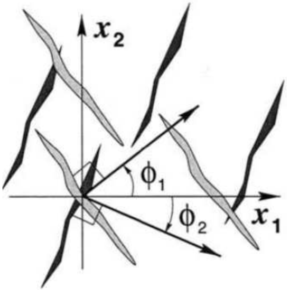


Macro-scale anisotropy: VTI/TTI anisotropy



- in all cases, anisotropy is a scale problem : a medium is seen as anisotropic when it contains "small scale" heterogeneities or structures
  → "small scale" is related to wavelength $\lambda = V/f$ for wave propagation

- In anisotropic media, the reologic relation is more complex
  - **triclinic** : 21 independent coefficients in C_{ijkl}
  - **monoclinic** : 13 independent coefficients in C_{ijkl}



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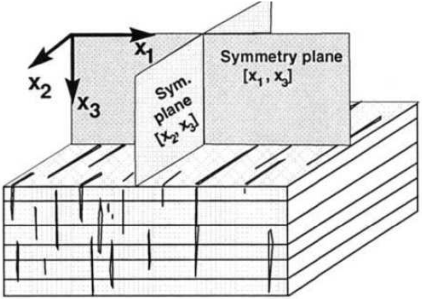
  - **orthorombic** : 9 independent coefficients in C_{ijkl}



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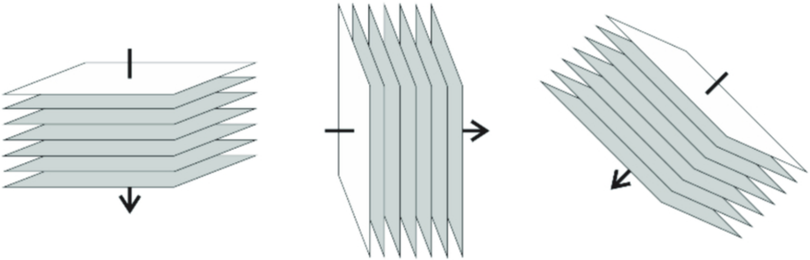
  - **monoclinic** : 13 independent coefficients in C_{ijkl}
  - **orthorombic** : 9 independent coefficients in C_{ijkl}
  - **transverse isotropic** :5 independent coefficients in C_{ijkl}



(a) VTI (b) HTI (c) TTI

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