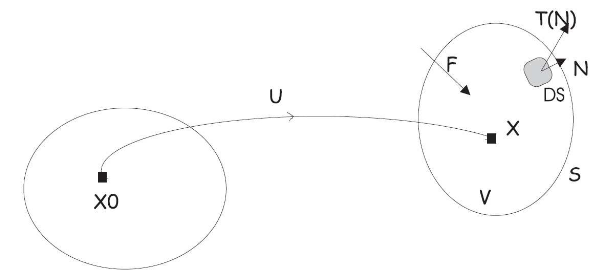
Full waveform modeling

Outline

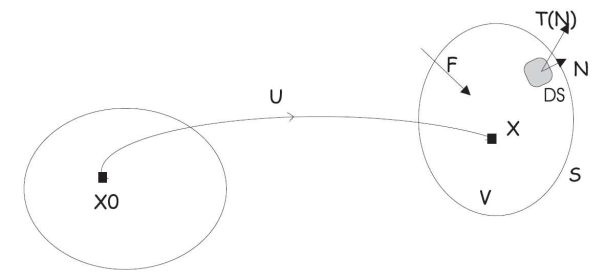
Displacement of a point

- reference system fixed in space and time
- a small element material is considered
- a point P at $\mathbf{x}_0 = (x_0, y_0, z_0)^t$ is considered at time t_0
- the point moves to $\mathbf{x}$ at time t



Displacement of a point

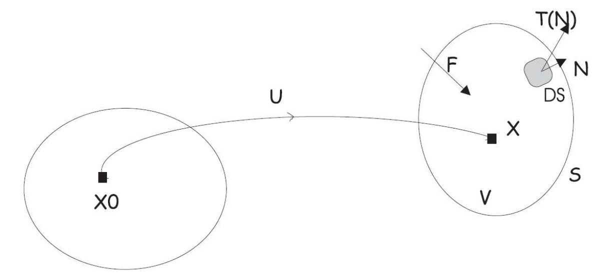
- displacement vector : $\mathbf{u} = \mathbf{x} - \mathbf{x}_0$
- velocity vector : $\frac{\partial \mathbf{u}}{\partial t}$
- acceleration vector $\frac{\partial^2 \mathbf{u}}{\partial t^2}$



Conservation of mass

- we consider a close volume V, with mass conservation

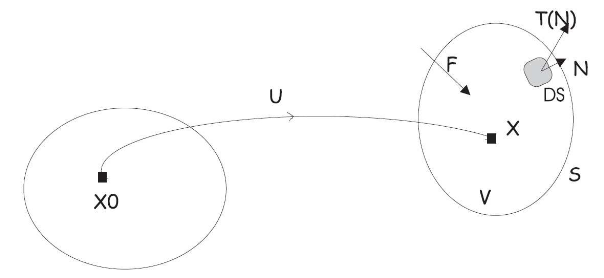
$$\frac{dM(t)}{dt} = 0. \tag{8}$$



Second Newton law

$$\iiint_V \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} dV = \iiint_V \mathbf{F} dV + \iint_S \mathbf{T} dS. \tag{9}$$

- ρ: density
- **F**: density of volumetric forces
- **T**: surface forces

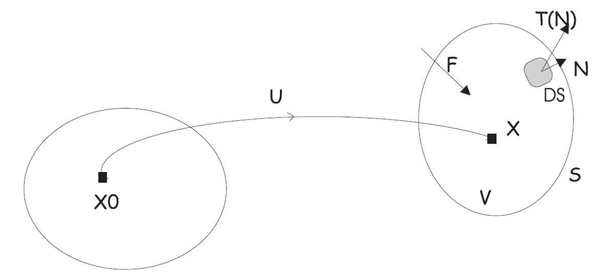


Simplification: surface forces → stress

- We can write

$$T_i = \sum_j \sigma_{ij} n_j, \tag{10}$$

  - σ_{ij} : component ij of the stress tensor
  - n_j : j component of the normal vector to S

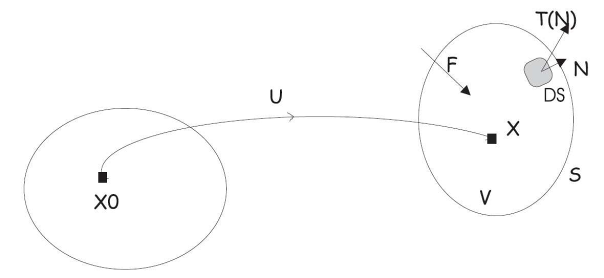


Simplification : divergence theorem

- divergence theorem

$$\iint_S \sigma_{ij} n_j dS = \iiint_V \sum_j \partial_j \sigma_{ij} dV \tag{11}$$

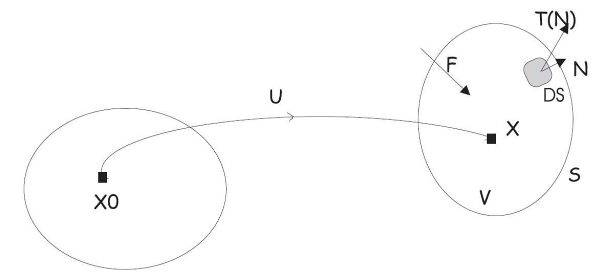
  - ∂_j : spatial partial derivative along j



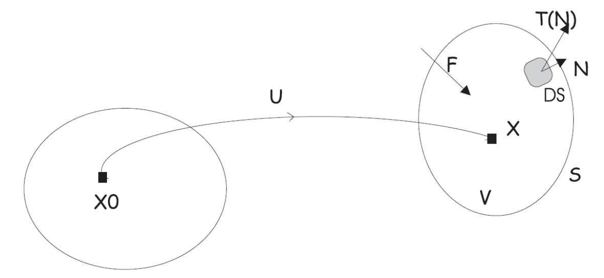
Second Newton law : integration on an elementary volume

$$\rho \frac{\partial^2 u_i}{\partial t^2} = F_i + \sum_j \partial_j \sigma_{ij} \tag{12}$$

- i : component of the vector
- j : spatial direction



$$
\begin{aligned}
\rho\frac{\partial^2 u_x}{\partial t^2} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + F_x \\
\rho\frac{\partial^2 u_y}{\partial t^2} &= \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + F_y \\
\rho\frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z \qquad (13)
\end{aligned}
$$



displacement/strain linearity

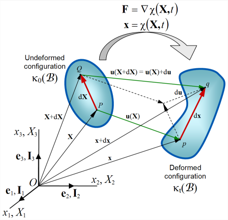
- strain tensor ϵ_{ij}

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k \partial u_k}{\partial x_i \partial x_j}\right) \tag{14}$$

- linear part

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{15}$$

$\rightarrow$ linear relation between strain and displacement



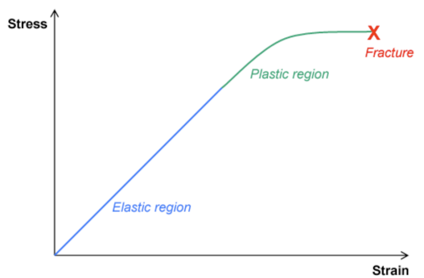
stress/strain linearity

- rheology law between stress and strain

$$\sigma_{ij} = \sigma_{ij}^0 + C_{ijkl}\epsilon_{kl} + D_{ijklmn}\epsilon_{kl}\epsilon_{mn} + \mathcal{O}(\epsilon^3) \quad (16)$$

  - σ_{ij}^0 : pre-stress
  - C_{ijkl} et D_{ijklmn} : order 4 and 6 tensors
- linear part

$$\sigma_{ij} = \sigma_{ij}^0 + C_{ijkl}\epsilon_{kl} \quad (17)$$



We can gather everything

- dynamic equation

$$\rho \frac{\partial^2 u_i}{\partial t^2} = F_i + \sum_j \partial_j \sigma_{ij} \tag{18}$$

- strain/displacement relation

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{19}$$

- rheology relation

$$\sigma_{ij} = \sigma_{ij}^0 + C_{ijkl}\epsilon_{kl} \tag{20}$$

Matrix formulation

Voigt notations

$$11 \to 1,\ 22 \to 2,\ 33 \to 3,\ 23 \to 4,\ 13 \to 5,\ 12 \to 6. \tag{21}$$

This makes

$$\sigma = \left[\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{23}\ \sigma_{13}\ \sigma_{12}\right]^T = \left[\sigma_1\ \sigma_2\ \sigma_3\ \sigma_4\ \sigma_5\ \sigma_6\right]^T, \tag{22}$$

$$\epsilon = \left[\epsilon_{11}\ \epsilon_{22}\ \epsilon_{33}\ 2\epsilon_{23}\ 2\epsilon_{13}\ 2\epsilon_{12}\right]^T = \left[\epsilon_1\ \epsilon_2\ \epsilon_3\ \epsilon_4\ \epsilon_5\ \epsilon_6\right]^T. \tag{23}$$

We introduce the matrix operator D as

$$D = \begin{pmatrix} \partial x_1 & 0 & 0 & 0 & \partial x_2 & \partial x_3 \\ 0 & \partial x_2 & 0 & \partial x_3 & 0 & \partial x_1 \\ 0 & 0 & \partial x_3 & \partial x_2 & \partial x_1 & 0 \end{pmatrix}. \tag{24}$$

We can gather everything

Gathering equations we obtain

$$\begin{aligned}\rho\frac{\partial^2 u}{\partial t^2} &= D\sigma + F \\ \sigma &= \sigma^0 + CD^T u.\end{aligned} \tag{25}$$

Isotropic medium example

- In elastic isotropic media, C_{ijkl} depends on 2 independent coefficients (λ and μ, or E and ν), giving

$$\begin{aligned}
\sigma_{xx} &= (\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda\frac{\partial u_y}{\partial y} + \lambda\frac{\partial u_z}{\partial z} + \sigma_{xx_0} \\
\sigma_{yy} &= \lambda\frac{\partial u_x}{\partial x} + (\lambda + 2\mu)\frac{\partial u_y}{\partial y} + \lambda\frac{\partial u_z}{\partial z} + \sigma_{yy_0} \\
\sigma_{zz} &= \lambda\frac{\partial u_x}{\partial x} + \lambda\frac{\partial u_y}{\partial y} + (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \sigma_{zz_0} \\
\sigma_{xy} &= \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \sigma_{xy_0} \\
\sigma_{yz} &= \mu\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) + \sigma_{yz_0} \\
\sigma_{xz} &= \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) + \sigma_{xz_0}
\end{aligned} \tag{26}$$

Isotropic medium example

- we combine the sets of equations

$$
\begin{aligned}
\rho \frac{\partial^2 u_x}{\partial t^2} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + F_x \\
\rho \frac{\partial^2 u_y}{\partial t^2} &= \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + F_y \\
\rho \frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z \\
\sigma_{xx} &= (\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_y}{\partial y} + \lambda \frac{\partial u_z}{\partial z} + \sigma_{xx_0} \\
\sigma_{yy} &= \lambda \frac{\partial u_x}{\partial x} + (\lambda + 2\mu)\frac{\partial u_y}{\partial y} + \lambda \frac{\partial u_z}{\partial z} + \sigma_{yy_0} \\
\sigma_{zz} &= \lambda \frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_y}{\partial y} + (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \sigma_{zz_0} \\
\sigma_{xy} &= \mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) + \sigma_{xy_0} \\
\sigma_{yz} &= \mu \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) + \sigma_{yz_0} \\
\sigma_{xz} &= \mu \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) + \sigma_{xz_0}
\end{aligned} \tag{27}
$$

Isotropic medium example

- we combine the sets of equations
- PDE with 9 unknowns : 3 displacement + 9 stresses

$$
\begin{aligned}
\rho \frac{\partial^2 u_x}{\partial t^2} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + F_x \\
\rho \frac{\partial^2 u_y}{\partial t^2} &= \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + F_y \\
\rho \frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z \\
\sigma_{xx} &= (\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_y}{\partial y} + \lambda \frac{\partial u_z}{\partial z} + \sigma_{xx_0} \\
\sigma_{yy} &= \lambda \frac{\partial u_x}{\partial x} + (\lambda + 2\mu)\frac{\partial u_y}{\partial y} + \lambda \frac{\partial u_z}{\partial z} + \sigma_{yy_0} \\
\sigma_{zz} &= \lambda \frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_y}{\partial y} + (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \sigma_{zz_0} \\
\sigma_{xy} &= \mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) + \sigma_{xy_0} \\
\sigma_{yz} &= \mu \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) + \sigma_{yz_0} \\
\sigma_{xz} &= \mu \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) + \sigma_{xz_0}
\end{aligned}
\tag{27}
$$

Isotropic medium example

- we replace
  $v_i = \partial_t u_i$

$$
\begin{aligned}
\frac{\partial v_x}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}\right) + f_x \\
\frac{\partial v_y}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}\right) + f_y \\
\frac{\partial v_z}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) + f_z
\end{aligned} \tag{28}
$$

Isotropic medium example

- we replace $v_i = \partial_t u_i$
- we derive over time the rheology relation

$$
\begin{aligned}
\frac{\partial v_x}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}\right) + f_x \\
\frac{\partial v_y}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}\right) + f_y \\
\frac{\partial v_z}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) + f_z \\
\frac{\partial \sigma_{xx}}{\partial t} &= (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \lambda\frac{\partial v_y}{\partial y} + \lambda\frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{xx_0}}{\partial t} \\
\frac{\partial \sigma_{yy}}{\partial t} &= \lambda\frac{\partial v_x}{\partial x} + (\lambda + 2\mu)\frac{\partial v_y}{\partial y} + \lambda\frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{yy_0}}{\partial t} \\
\frac{\partial \sigma_{zz}}{\partial t} &= \lambda\frac{\partial v_x}{\partial x} + \lambda\frac{\partial v_y}{\partial y} + (\lambda + 2\mu)\frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{zz_0}}{\partial t} \\
\frac{\partial \sigma_{xy}}{\partial t} &= \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) + \frac{\partial \sigma_{xy_0}}{\partial t} \\
\frac{\partial \sigma_{yz}}{\partial t} &= \mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) + \frac{\partial \sigma_{yz_0}}{\partial t} \\
\frac{\partial \sigma_{xz}}{\partial t} &= \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) + \frac{\partial \sigma_{xz_0}}{\partial t}
\end{aligned}
\tag{28}
$$

Isotropic medium example

- we replace $v_i = \partial_t u_i$
- we derive over time the rheology relation

we obtain an homogeneous system of order 1 in time

$$
\begin{aligned}
\frac{\partial v_x}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}\right) + f_x \\
\frac{\partial v_y}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}\right) + f_y \\
\frac{\partial v_z}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) + f_z \\
\frac{\partial \sigma_{xx}}{\partial t} &= (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \lambda\frac{\partial v_y}{\partial y} + \lambda\frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{xx_0}}{\partial t} \\
\frac{\partial \sigma_{yy}}{\partial t} &= \lambda\frac{\partial v_x}{\partial x} + (\lambda + 2\mu)\frac{\partial v_y}{\partial y} + \lambda\frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{yy_0}}{\partial t} \\
\frac{\partial \sigma_{zz}}{\partial t} &= \lambda\frac{\partial v_x}{\partial x} + \lambda\frac{\partial v_y}{\partial y} + (\lambda + 2\mu)\frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{zz_0}}{\partial t} \\
\frac{\partial \sigma_{xy}}{\partial t} &= \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) + \frac{\partial \sigma_{xy_0}}{\partial t} \\
\frac{\partial \sigma_{yz}}{\partial t} &= \mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) + \frac{\partial \sigma_{yz_0}}{\partial t} \\
\frac{\partial \sigma_{xz}}{\partial t} &= \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) + \frac{\partial \sigma_{xz_0}}{\partial t}
\end{aligned}
\tag{28}
$$

Acoustic medium

- in acoustic medium, shear effects vanish

Acoustic medium

- in acoustic medium, shear effects vanish

$$
\begin{aligned}
\frac{\partial v_x}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x} + \cancel{\frac{\partial \sigma_{xy}}{\partial y}} + \cancel{\frac{\partial \sigma_{xz}}{\partial z}}\right) + f_x \\
\frac{\partial v_y}{\partial t} &= \frac{1}{\rho}\left(\cancel{\frac{\partial \sigma_{xy}}{\partial x}} + \frac{\partial \sigma_{yy}}{\partial y} + \cancel{\frac{\partial \sigma_{yz}}{\partial z}}\right) + f_y \\
\frac{\partial v_z}{\partial t} &= \frac{1}{\rho}\left(\cancel{\frac{\partial \sigma_{xz}}{\partial x}} + \cancel{\frac{\partial \sigma_{yz}}{\partial y}} + \frac{\partial \sigma_{zz}}{\partial z}\right) + f_z \\
\frac{\partial \sigma_{xx}}{\partial t} &= (\lambda + \cancel{2\mu})\frac{\partial v_x}{\partial x} + \lambda\frac{\partial v_y}{\partial y} + \lambda\frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{xx_0}}{\partial t} \\
\frac{\partial \sigma_{yy}}{\partial t} &= \lambda\frac{\partial v_x}{\partial x} + (\lambda + \cancel{2\mu})\frac{\partial v_y}{\partial y} + \lambda\frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{yy_0}}{\partial t} \\
\frac{\partial \sigma_{zz}}{\partial t} &= \lambda\frac{\partial v_x}{\partial x} + \lambda\frac{\partial v_y}{\partial y} + (\lambda + \cancel{2\mu})\frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{zz_0}}{\partial t} \\
\cancel{\frac{\partial \sigma_{xy}}{\partial t}} &= \cancel{\mu}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) + \frac{\partial \sigma_{xy_0}}{\partial t} \\
\cancel{\frac{\partial \sigma_{yz}}{\partial t}} &= \cancel{\mu}\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) + \frac{\partial \sigma_{yz_0}}{\partial t} \\
\cancel{\frac{\partial \sigma_{xz}}{\partial t}} &= \cancel{\mu}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) + \frac{\partial \sigma_{xz_0}}{\partial t}
\end{aligned}
\tag{29}
$$

Acoustic medium

- in acoustic medium, shear effects vanish

$$
\begin{aligned}
\frac{\partial v_x}{\partial t} &= \frac{1}{\rho}\frac{\partial \sigma_{xx}}{\partial x} + f_x \\
\frac{\partial v_y}{\partial t} &= \frac{1}{\rho}\frac{\partial \sigma_{yy}}{\partial y} + f_y \\
\frac{\partial v_z}{\partial t} &= \frac{1}{\rho}\frac{\partial \sigma_{zz}}{\partial y} + f_z \\
\frac{\partial \sigma_{xx}}{\partial t} &= \lambda\frac{\partial v_x}{\partial x} + \lambda\frac{\partial v_y}{\partial y} + \lambda\frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{xx_0}}{\partial t} \\
\frac{\partial \sigma_{yy}}{\partial t} &= \lambda\frac{\partial v_x}{\partial x} + \lambda\frac{\partial v_y}{\partial y} + \lambda\frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{yy_0}}{\partial t} \\
\frac{\partial \sigma_{zz}}{\partial t} &= \lambda\frac{\partial v_x}{\partial x} + \lambda\frac{\partial v_y}{\partial y} + \lambda\frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{zz_0}}{\partial t}
\end{aligned} \tag{30}
$$

Acoustic medium

- in acoustic medium, shear effects vanish

$$
\begin{aligned}
\frac{\partial v_x}{\partial t} &= \frac{1}{\rho}\frac{\partial P}{\partial x} + f_x \\
\frac{\partial v_y}{\partial t} &= \frac{1}{\rho}\frac{\partial P}{\partial y} + f_y \\
\frac{\partial v_z}{\partial t} &= \frac{1}{\rho}\frac{\partial P}{\partial y} + f_z \\
\frac{\partial P}{\partial t} &= \lambda\frac{\partial v_x}{\partial x} + \lambda\frac{\partial v_y}{\partial y} + \lambda\frac{\partial v_z}{\partial z} + \frac{\partial P_0}{\partial t} \qquad (31)
\end{aligned}
$$

2D isotropic medium

hypothesis

- invariant properties along one direction
- source and receivers invariant along one direction : line source and receivers

2D isotropic medium

- invariance along y (for example)

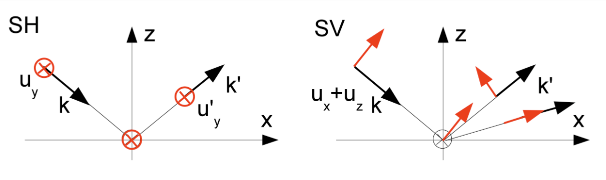
$$
\begin{aligned}
\frac{\partial v_x}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x} + \cancel{\frac{\partial \sigma_{xy}}{\partial y}} + \frac{\partial \sigma_{xz}}{\partial z}\right) + f_x \\
\frac{\partial v_y}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xy}}{\partial x} + \cancel{\frac{\partial \sigma_{yy}}{\partial y}} + \frac{\partial \sigma_{yz}}{\partial z}\right) + f_y \\
\frac{\partial v_z}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xz}}{\partial x} + \cancel{\frac{\partial \sigma_{yz}}{\partial y}} + \frac{\partial \sigma_{zz}}{\partial z}\right) + f_z \\
\frac{\partial \sigma_{xx}}{\partial t} &= (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \lambda \cancel{\frac{\partial v_y}{\partial y}} + \lambda \frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{xx_0}}{\partial t} \\
\cancel{\frac{\partial \sigma_{yy}}{\partial t}} &= \lambda \frac{\partial v_x}{\partial x} + (\lambda + 2\mu)\cancel{\frac{\partial v_y}{\partial y}} + \lambda \frac{\partial v_z}{\partial z} + \cancel{\frac{\partial \sigma_{yy_0}}{\partial t}} \\
\frac{\partial \sigma_{zz}}{\partial t} &= \lambda \frac{\partial v_x}{\partial x} + \lambda \cancel{\frac{\partial v_y}{\partial y}} + (\lambda + 2\mu)\frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{zz_0}}{\partial t} \\
\frac{\partial \sigma_{xy}}{\partial t} &= \mu\left(\cancel{\frac{\partial v_x}{\partial y}} + \frac{\partial v_y}{\partial x}\right) + \frac{\partial \sigma_{xy_0}}{\partial t} \\
\frac{\partial \sigma_{yz}}{\partial t} &= \mu\left(\frac{\partial v_y}{\partial z} + \cancel{\frac{\partial v_z}{\partial y}}\right) + \frac{\partial \sigma_{yz_0}}{\partial t} \\
\frac{\partial \sigma_{xz}}{\partial t} &= \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) + \frac{\partial \sigma_{xz_0}}{\partial t}
\end{aligned}
\tag{32}
$$

2D isotropic medium

- invariance along y (for example)
- 2 independent systems appear

$$
\begin{aligned}
\frac{\partial v_x}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x} + \cancel{\frac{\partial \sigma_{xy}}{\partial y}} + \frac{\partial \sigma_{xz}}{\partial z}\right) + f_x \\
\frac{\partial v_y}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xy}}{\partial x} + \cancel{\frac{\partial \sigma_{yy}}{\partial y}} + \frac{\partial \sigma_{yz}}{\partial z}\right) + f_y \\
\frac{\partial v_z}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xz}}{\partial x} + \cancel{\frac{\partial \sigma_{yz}}{\partial y}} + \frac{\partial \sigma_{zz}}{\partial z}\right) + f_z \\
\frac{\partial \sigma_{xx}}{\partial t} &= (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \lambda\cancel{\frac{\partial v_y}{\partial y}} + \lambda\frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{xx_0}}{\partial t} \\
\cancel{\frac{\partial \sigma_{yy}}{\partial t}} &= \lambda\frac{\partial v_x}{\partial x} + (\lambda + 2\mu)\cancel{\frac{\partial v_y}{\partial y}} + \lambda\frac{\partial v_z}{\partial z} + \cancel{\frac{\partial \sigma_{yy_0}}{\partial t}} \\
\frac{\partial \sigma_{zz}}{\partial t} &= \lambda\frac{\partial v_x}{\partial x} + \lambda\cancel{\frac{\partial v_y}{\partial y}} + (\lambda + 2\mu)\frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{zz_0}}{\partial t} \\
\frac{\partial \sigma_{xy}}{\partial t} &= \mu\left(\cancel{\frac{\partial v_x}{\partial y}} + \frac{\partial v_y}{\partial x}\right) + \frac{\partial \sigma_{xy_0}}{\partial t} \\
\frac{\partial \sigma_{yz}}{\partial t} &= \mu\left(\frac{\partial v_y}{\partial z} + \cancel{\frac{\partial v_z}{\partial y}}\right) + \frac{\partial \sigma_{yz_0}}{\partial t} \\
\frac{\partial \sigma_{xz}}{\partial t} &= \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) + \frac{\partial \sigma_{xz_0}}{\partial t}
\end{aligned} \tag{32}
$$

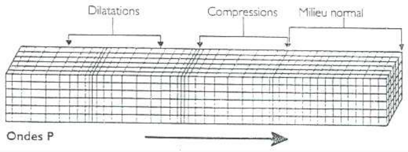
SV and SH waves



2D isotropic medium : P-SV system

- P-SV system
  - P waves
  - S waves polarized in plane xz

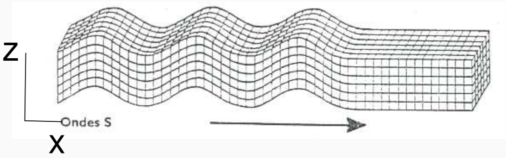
$$
\begin{aligned}
\frac{\partial v_x}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z}\right) + f_x \\
\frac{\partial v_z}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z}\right) + f_z \\
\frac{\partial \sigma_{xx}}{\partial t} &= (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \lambda\frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{xx_0}}{\partial t} \\
\frac{\partial \sigma_{zz}}{\partial t} &= \lambda\frac{\partial v_x}{\partial x} + (\lambda + 2\mu)\frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{zz_0}}{\partial t} \\
\frac{\partial \sigma_{xz}}{\partial t} &= \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) + \frac{\partial \sigma_{xz_0}}{\partial t}.
\end{aligned}
\tag{33}
$$



2D isotropic medium : P-SV system

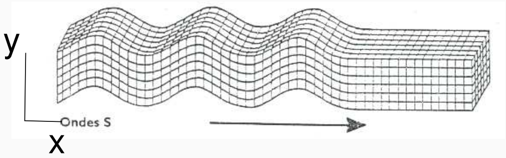
- P-SV system
    - P waves
    - S waves polarized in plane *xz*

$$
\begin{aligned}
\frac{\partial v_x}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z}\right) + f_x \\
\frac{\partial v_z}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z}\right) + f_z \\
\frac{\partial \sigma_{xx}}{\partial t} &= (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \lambda\frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{xx_0}}{\partial t} \\
\frac{\partial \sigma_{zz}}{\partial t} &= \lambda\frac{\partial v_x}{\partial x} + (\lambda + 2\mu)\frac{\partial v_z}{\partial z} + \frac{\partial \sigma_{zz_0}}{\partial t} \\
\frac{\partial \sigma_{xz}}{\partial t} &= \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) + \frac{\partial \sigma_{xz_0}}{\partial t}.
\end{aligned}
\tag{33}
$$



- SH system
  - S waves only in plane *xy*

$$
\begin{aligned}
\frac{\partial v_y}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z}\right) + f_y \\
\frac{\partial \sigma_{xy}}{\partial t} &= \mu\left(\frac{\partial v_y}{\partial x}\right) + \frac{\partial \sigma_{xy_0}}{\partial t} \\
\frac{\partial \sigma_{yz}}{\partial t} &= \mu\left(\frac{\partial v_y}{\partial z}\right) + \frac{\partial \sigma_{yz_0}}{\partial t}.
\end{aligned} \tag{34}
$$



1D isotropic medium

Hypothesis

- invariant properties in 2 directions
- invariant source and receiver in 2 directions : plane wave

1D isotropic medium

- y and z directions for invariance (for exemple)

$$
\begin{aligned}
\frac{\partial v_x}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x} + \cancel{\frac{\partial \sigma_{xy}}{\partial y}} + \cancel{\frac{\partial \sigma_{xz}}{\partial z}}\right) + f_x \\
\frac{\partial v_y}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xy}}{\partial x} + \cancel{\frac{\partial \sigma_{yy}}{\partial y}} + \cancel{\frac{\partial \sigma_{yz}}{\partial z}}\right) + f_y \\
\frac{\partial v_z}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xz}}{\partial x} + \cancel{\frac{\partial \sigma_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_{zz}}{\partial z}}\right) + f_z \\
\frac{\partial \sigma_{xx}}{\partial t} &= (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \lambda\cancel{\frac{\partial v_y}{\partial y}} + \lambda\cancel{\frac{\partial v_z}{\partial z}} + \frac{\partial \sigma_{xx_0}}{\partial t} \\
\cancel{\frac{\partial \sigma_{yy}}{\partial t}} &= \lambda\frac{\partial v_x}{\partial x} + (\lambda + 2\mu)\cancel{\frac{\partial v_y}{\partial y}} + \lambda\cancel{\frac{\partial v_z}{\partial z}} + \cancel{\frac{\partial \sigma_{yy_0}}{\partial t}} \\
\cancel{\frac{\partial \sigma_{zz}}{\partial t}} &= \lambda\frac{\partial v_x}{\partial x} + \lambda\cancel{\frac{\partial v_y}{\partial y}} + (\lambda + 2\mu)\cancel{\frac{\partial v_z}{\partial z}} + \cancel{\frac{\partial \sigma_{zz_0}}{\partial t}} \\
\frac{\partial \sigma_{xy}}{\partial t} &= \mu\left(\cancel{\frac{\partial v_x}{\partial y}} + \frac{\partial v_y}{\partial x}\right) + \frac{\partial \sigma_{xy_0}}{\partial t} \\
\frac{\partial \sigma_{yz}}{\partial t} &= \mu\left(\cancel{\frac{\partial v_y}{\partial z}} + \cancel{\frac{\partial v_z}{\partial y}}\right) + \frac{\partial \sigma_{yz_0}}{\partial t} \\
\frac{\partial \sigma_{xz}}{\partial t} &= \mu\left(\cancel{\frac{\partial v_x}{\partial z}} + \frac{\partial v_z}{\partial x}\right) + \frac{\partial \sigma_{xz_0}}{\partial t}
\end{aligned}
\tag{35}
$$

1D isotropic medium

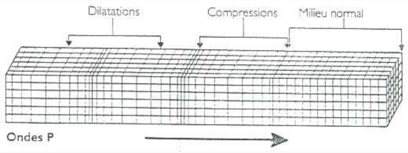
- y and z directions for invariance (for exemple)
- 3 independent systems appear

$$
\begin{aligned}
\frac{\partial v_x}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x} + \cancel{\frac{\partial \sigma_{xy}}{\partial y}} + \cancel{\frac{\partial \sigma_{xz}}{\partial z}}\right) + f_x \\
\frac{\partial v_y}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xy}}{\partial x} + \cancel{\frac{\partial \sigma_{yy}}{\partial y}} + \cancel{\frac{\partial \sigma_{yz}}{\partial z}}\right) + f_y \\
\frac{\partial v_z}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \sigma_{xz}}{\partial x} + \cancel{\frac{\partial \sigma_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_{zz}}{\partial z}}\right) + f_z \\
\frac{\partial \sigma_{xx}}{\partial t} &= (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \lambda \cancel{\frac{\partial v_y}{\partial y}} + \lambda \cancel{\frac{\partial v_z}{\partial z}} + \frac{\partial \sigma_{xx_0}}{\partial t} \\
\cancel{\frac{\partial \sigma_{yy}}{\partial t}} &= \lambda \frac{\partial v_x}{\partial x} + (\lambda + 2\mu)\cancel{\frac{\partial v_y}{\partial y}} + \lambda \cancel{\frac{\partial v_z}{\partial z}} + \cancel{\frac{\partial \sigma_{yy_0}}{\partial t}} \\
\cancel{\frac{\partial \sigma_{zz}}{\partial t}} &= \lambda \frac{\partial v_x}{\partial x} + \lambda \cancel{\frac{\partial v_y}{\partial y}} + (\lambda + 2\mu)\cancel{\frac{\partial v_z}{\partial z}} + \cancel{\frac{\partial \sigma_{zz_0}}{\partial t}} \\
\frac{\partial \sigma_{xy}}{\partial t} &= \mu\left(\cancel{\frac{\partial v_x}{\partial y}} + \frac{\partial v_y}{\partial x}\right) + \frac{\partial \sigma_{xy_0}}{\partial t} \\
\frac{\partial \sigma_{yz}}{\partial t} &= \mu\left(\cancel{\frac{\partial v_y}{\partial z}} + \cancel{\frac{\partial v_z}{\partial y}}\right) + \frac{\partial \sigma_{yz_0}}{\partial t} \\
\frac{\partial \sigma_{xz}}{\partial t} &= \mu\left(\cancel{\frac{\partial v_x}{\partial z}} + \frac{\partial v_z}{\partial x}\right) + \frac{\partial \sigma_{xz_0}}{\partial t}
\end{aligned}
\tag{35}
$$

1D isotropic medium: P wave

- P wave along x direction

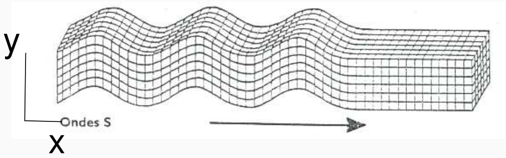
$$
\begin{aligned}
\frac{\partial v_x}{\partial t} &= \frac{1}{\rho}\frac{\partial \sigma_{xx}}{\partial x} + f_x \\
\frac{\partial \sigma_{xx}}{\partial t} &= (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \frac{\partial \sigma_{xx_0}}{\partial t}
\end{aligned}
\tag{36}
$$



1D isotropic medium: S wave (1)

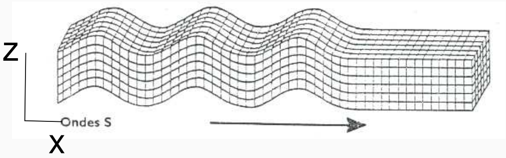
- S wave along direction x, polarized in plane xy

$$
\begin{aligned}
\frac{\partial v_y}{\partial t} &= \frac{1}{\rho}\frac{\partial \sigma_{xy}}{\partial x} + f_y \\
\frac{\partial \sigma_{xy}}{\partial t} &= \mu \frac{\partial v_y}{\partial x} + \frac{\partial \sigma_{xy_0}}{\partial t}
\end{aligned}
\quad (37)
$$



- S wave along direction x, polarized in plane xz

$$
\begin{aligned}
\frac{\partial v_z}{\partial t} &= \frac{1}{\rho}\frac{\partial \sigma_{xz}}{\partial x} + f_y \\
\frac{\partial \sigma_{xz}}{\partial t} &= \mu \frac{\partial v_z}{\partial x} + \frac{\partial \sigma_{xz_0}}{\partial t}
\end{aligned}
\tag{38}
$$



Which velocities for these waves ?

We can write the wave equation as a compact matrix-vector formulation

$$\frac{\partial \mathbf{U}}{\partial t} = A\frac{\partial \mathbf{U}}{\partial x} + B\frac{\partial \mathbf{U}}{\partial y} + C\frac{\partial \mathbf{U}}{\partial z} + \frac{\partial \mathbf{U}_0}{\partial t}, \tag{39}$$

where $\mathbf{U} = (v_x, v_y, v_z, \sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{xz})^t$.

Which velocities for these waves ?

Example in 1D

$$\frac{\partial \mathbf{U}}{\partial t} = A\frac{\partial \mathbf{U}}{\partial x} + \frac{\partial \mathbf{U}_0}{\partial t}, \tag{40}$$

- P wave

$$\begin{aligned}\frac{\partial v_x}{\partial t} &= \frac{1}{\rho}\frac{\partial \sigma_{xx}}{\partial x} + f_x \\ \frac{\partial \sigma_{xx}}{\partial t} &= (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \frac{\partial \sigma_{xx_0}}{\partial t}\end{aligned}$$

- S wave (in plane xz)

$$\begin{aligned}\frac{\partial v_z}{\partial t} &= \frac{1}{\rho}\frac{\partial \sigma_{xz}}{\partial x} + f_y \\ \frac{\partial \sigma_{xz}}{\partial t} &= \mu\frac{\partial v_z}{\partial x} + \frac{\partial \sigma_{xz_0}}{\partial t}\end{aligned}$$

Which velocities for these waves ?

- For P-wave system we have

$$A = \begin{pmatrix} 0 & \frac{1}{\rho} \\ \lambda + 2\mu & 0 \end{pmatrix} \tag{41}$$

therefore the eigenvalues are solution of

$$X^2 - \frac{\lambda + 2\mu}{\rho} = 0 \tag{42}$$

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- P wave

$$l = \pm\sqrt{\frac{\lambda + 2\mu}{\rho}} \rightarrow v_P = \sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{43}$$

Which velocities for these waves ?

- For P-wave system we have

$$A = \begin{pmatrix} 0 & \frac{1}{\rho} \\ \lambda + 2\mu & 0 \end{pmatrix} \tag{41}$$

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$$X^2 - \frac{\lambda + 2\mu}{\rho} = 0 \tag{42}$$

- P wave

$$l = \pm\sqrt{\frac{\lambda + 2\mu}{\rho}} \rightarrow v_P = \sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{43}$$

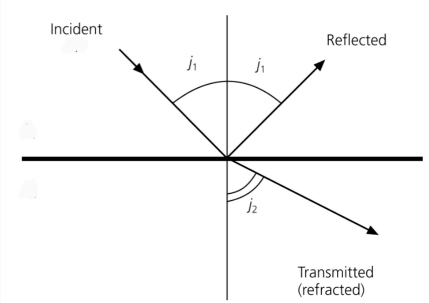
- S wave (in plane xz)

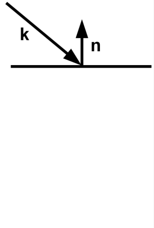
$$l = \pm\sqrt{\frac{\mu}{\rho}} \rightarrow v_S = \sqrt{\frac{\mu}{\rho}} \tag{44}$$

Outline

Reflection and refraction : Snell-Descartes law (1)

As for light...

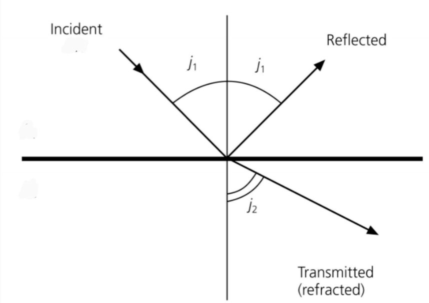


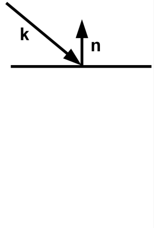


$$\frac{\sin j_1}{v_1} = \frac{\sin j_2}{v_2} \quad j_1 = j_1' \tag{45}$$

Reflection and refraction : Snell-Descartes law (1)

As for light...



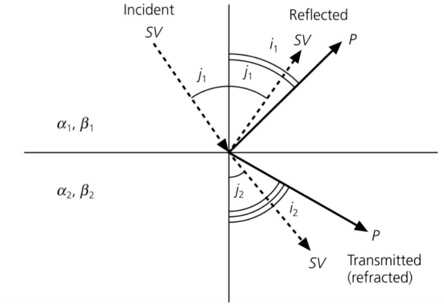


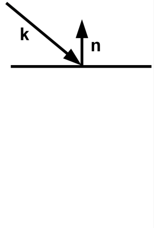
$$\frac{\sin j_1}{v_1} = \frac{\sin j_2}{v_2} \quad j_1 = j_1' \tag{45}$$

- this setup is valid for single wave type
    - P waves in acoustic media
    - SH waves (no coupling with P waves)

Reflection and refraction : Snell-Descartes law (2)

As for light...





$$\frac{\sin j_1}{v_{S1}} = \frac{\sin j_2}{v_{S2}} = \frac{\sin i_1}{v_{P1}} = \frac{\sin i_2}{v_{P2}} \quad j_1 = j_1' \tag{46}$$

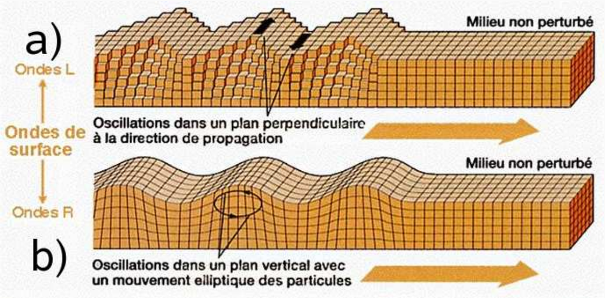
- this setup is generic
    - P waves in solids
    - SV waves

Surface waves

- the free surface is a particular interface
  - on one side a solid : v_{P_S}, v_{S_S} and ρ_S
  - on the other side, air : $v_{P_{air}} < v_{P_S}$, $v_S = 0$ and $\rho_{air} <<< \rho_S$

Surface waves

- the free surface is a particular interface
  - on one side a solid : v_{P_S}, v_{S_S} and ρ_S
  - on the other side, air : $v_{P_{air}} < v_{P_S}$, $v_S = 0$ and $\rho_{air} <<< \rho_S$
    $\rightarrow$ we generally assume air as void



Rayleigh waves

- particles move in the P-SV plane
- result from interferences between P and SV waves

$$v_R < v_S < v_P \tag{47}$$

$$v_R^6 - 8v_S^2 v_R^4 + (24 - 16v_S^2/v_P^2)v_S^4 v_R^2 + 16(v_S^2/v_P^2 - 1)v_S^6 = 0 \tag{48}$$

where v_R is the Rayleigh wave velocity

Rayleigh waves

- particles move in the P-SV plane
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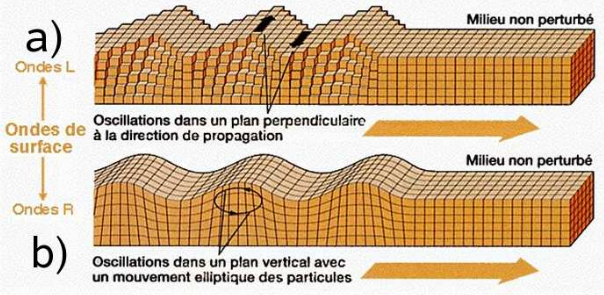
$$v_R^6 - 8v_S^2 v_R^4 + (24 - 16v_S^2/v_P^2)v_S^4 v_R^2 + 16(v_S^2/v_P^2 - 1)v_S^6 = 0 \tag{48}$$

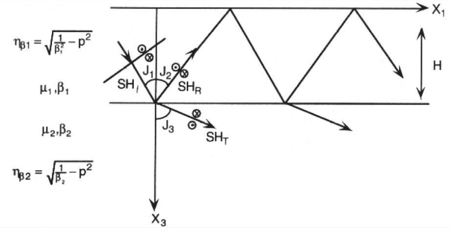
where v_R is the Rayleigh wave velocity
for perfect solid media, $\nu \approx 0.25$ giving $v_R = 0.919 v_S$

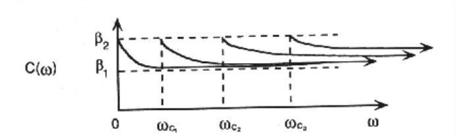


- particle motion in the SH plane
- result from interferences between incident, reflected and refracted SH in an heterogenous near surface
  $\rightarrow$ does not exist in homogeneous media
- we have

$$v_{S1} < v_L < v_{S2} \tag{49}$$

- giving dispersion in all cases

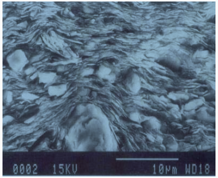




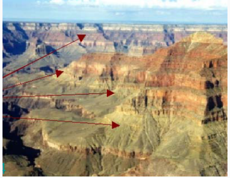
Anisotropy

- Anisotropy : changing of the behavior depending on the direction
- In geologic media, wave anisotropy can have two origins : internal (mineral composition → static anisotropy) and external (layer structure for example → dynamic anisotropy)

Micro-scale anisotropy: mineral anisotropy



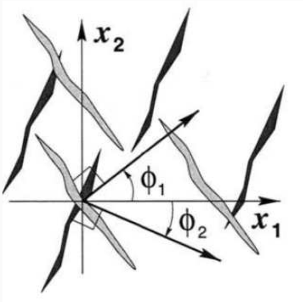
Macro-scale anisotropy: VTI/TTI anisotropy



- in all cases, anisotropy is a scale problem : a medium is seen as anisotropic when it contains "small scale" heterogeneities or structures
  → "small scale" is related to wavelength $\lambda = V/f$ for wave propagation

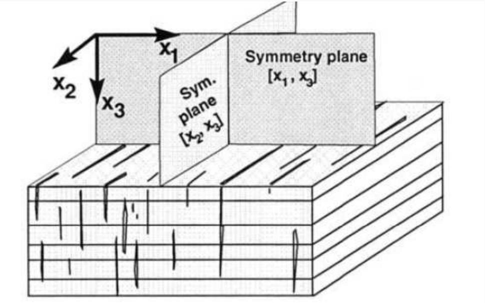
Anisotropy

- In anisotropic media, the reologic relation is more complex
  - **triclinic** : 21 independent coefficients in C_{ijkl}
  - **monoclinic** : 13 independent coefficients in C_{ijkl}



Anisotropy

- In anisotropic media, the reologic relation is more complex
  - **triclinic** : 21 independent coefficients in C_{ijkl}
  - **monoclinic** : 13 independent coefficients in C_{ijkl}
  - **orthorombic** : 9 independent coefficients in C_{ijkl}



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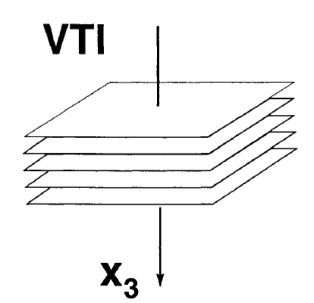

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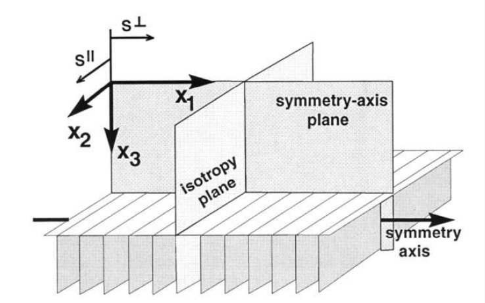

Attenuation: intrinsic, extrinsic and apparent ....

- Quality factor : relate the energy loss per wavelength (effect is frequency-dependent)
- energy loss can be related to intrinsic (fluid motion in porous media, plastic deformation, anelastic reology...) or extrinsic (multi-scattering in heterogeneous media)

Shallow layer
Strong attenuation

Difficult to find equivalent visco-elastic model

Crust and Mantle
Visco-elastic model of anelasticity

Rheological laws (near-elastic & plastic deformation)

Frequency dependence of the Q factor

(Pertinence of the Nearly Constant Q - NCQ- model)