Geophysical imaging: Mathematics for imaging the subsurface

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Thursday 25th September, 2025

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C010, ENSIMAG, Grenoble

Outline

Objectives of the course: present some key concepts about the mathematics used for subsurface imaging

- Why and how to image the subsurface
- Physics of propagation of mechanical waves in an elastic medium
- Notions of analysis of hyperbolic systems of partial differential equations
- Absorbing boundary conditions for wave propagation problems
- Discretization and numerical solution of wave propagation problems
- Inverse problems
- Inverse data fitting problems and local optimization techniques
- Physical interpretation of imaging, imaging conditions
- Implementation on high performance computing devices

Main outline: 12 sessions

- Introduction/Context/Motivation (1 session: 25/09/2025)
- Full Waveform Modeling (5 sessions: 02/10/2025, 09/10/2025, **13/10/2025**, 16/10/2025, 06/11/2025)
- Full Waveform Inversion (6 sessions: 13/11/2025, 20/11/2025, 27/11/2025, 04/12/2025, 11/12/2025, 08/01/2026)

Room CO10, 2:00pm-3:30pm, **except 13/10/2025, Room H102**

Contact: **ludovic.metivier@univ-grenoble-alpes.fr**

Lecture notes and (some of the) slides: are available on my webpage.

https://membres-ljk.imag.fr/Ludovic.Metivier/webpage_LMetivier.html

Outline

Introduction

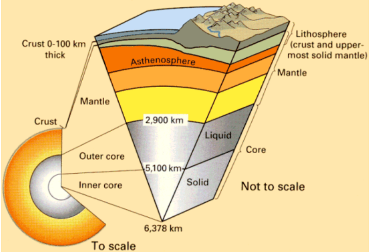
Geophysical imaging: to do what?

Seismic data

A first glance at seismic inversion methods

Understanding Earth's geodynamic

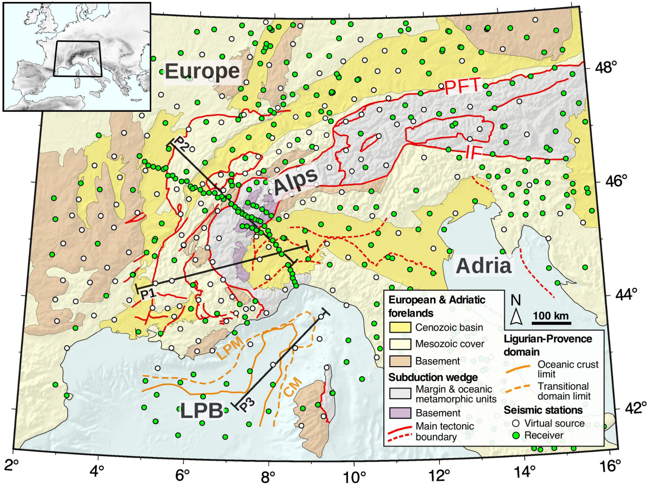
- mantle convection, mantle/core boundary
- planet formation
- geomagnetic field generation through convection in the outer core
- plate tectonic



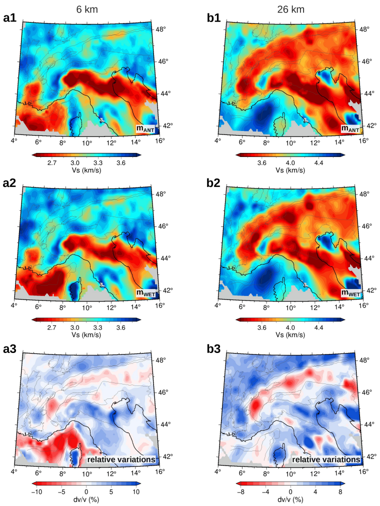
"The layer at the core mantle boundary may serve a the source of material for mantle plumes that give rise to hot spots, which are important in plate tectonics. The thermal properties of this layer might also influence the outward transport of heat from the Earth's core; in turn this could affect the intricate processes that generate the Earth's magnetic field."

Lowrie and Ficthner (2020)

Regional imaging at the Alps scale

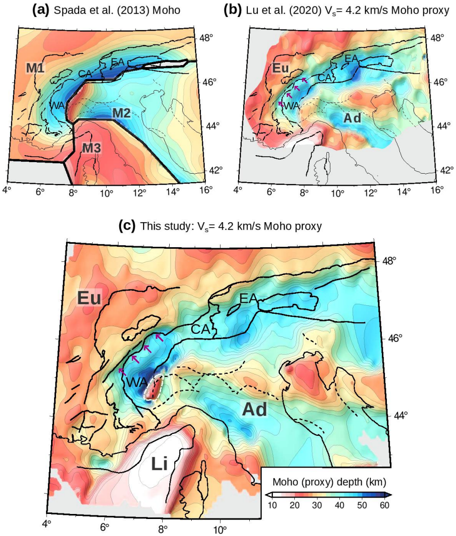


Geological and tectonic setting, locations of seismic stations (white circles: virtual sources; green circles: receivers). LPB: Ligurian-Provence basin, LPM: Ligurian-Provence margin, CM: Corsican margin (Nouibat et al., 2023).

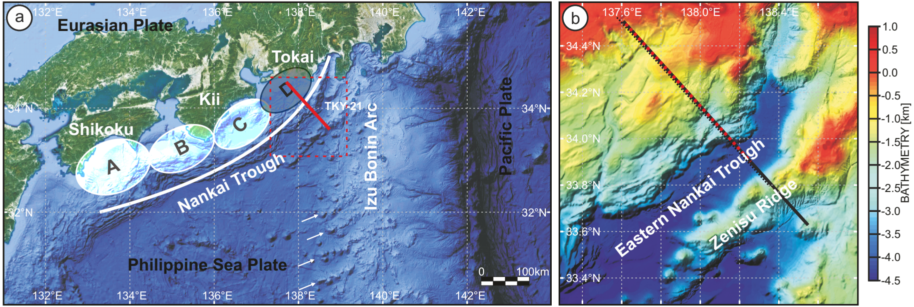


Depth slices of the initial V_S (a1-b1), final V_S(a2-b2) and perturbation (a3-b3)

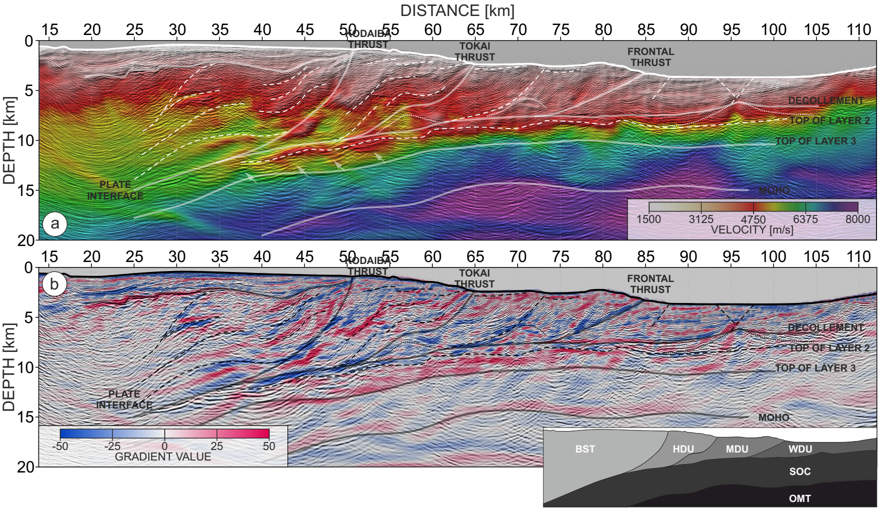
Regional imaging at the Alps scale



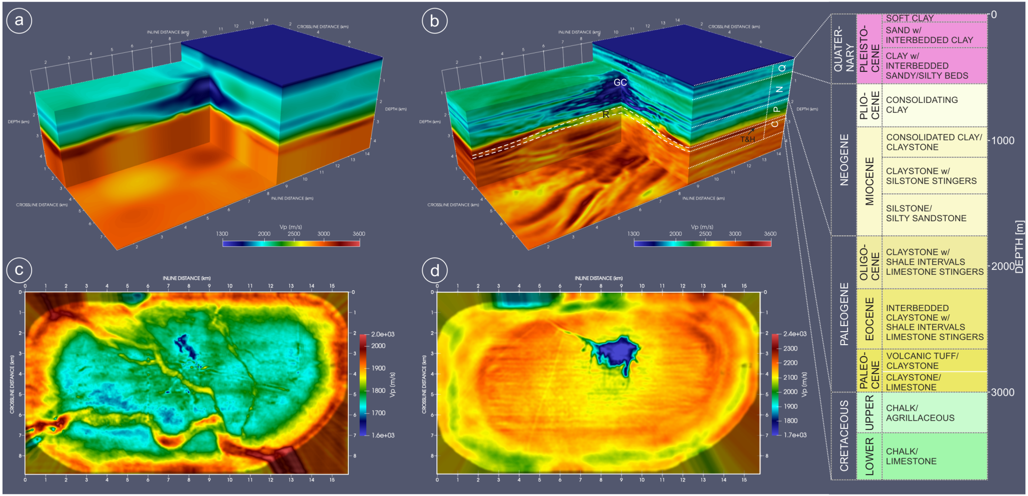
- active zones = active faults, subduction zones, volcanic area
- example: Nankai Trough

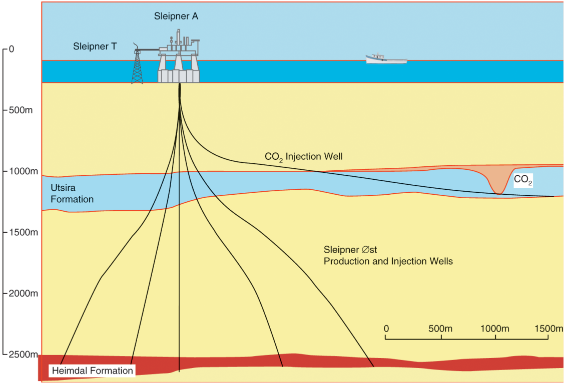


Partitioning of the Nankai Trough into four segments as described by Ando (1975). Region D was left unruptured during the most recent sequence of two large earthquakes (1944 Tonankai and 1946 Nankaido). Figure taken from Gorszczyk et al. (2019).


DISTANCE [km]
15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95 100 105 110
DEPTH [km]
0 5 10 15 20
KODAIBA THRUST
TOKAI THRUST
FRONTAL THRUST
DECOLLEMENT
TOP OF LAYER 2
TOP OF LAYER 3
MOHO
PLATE INTERFACE
a
1500 3125 4750 6375 8000
VELOCITY [m/s]
b
-50 -25 0 25 50
GRADIENT VALUE
BST
HDU
MDU
WDU
SOC
OMT

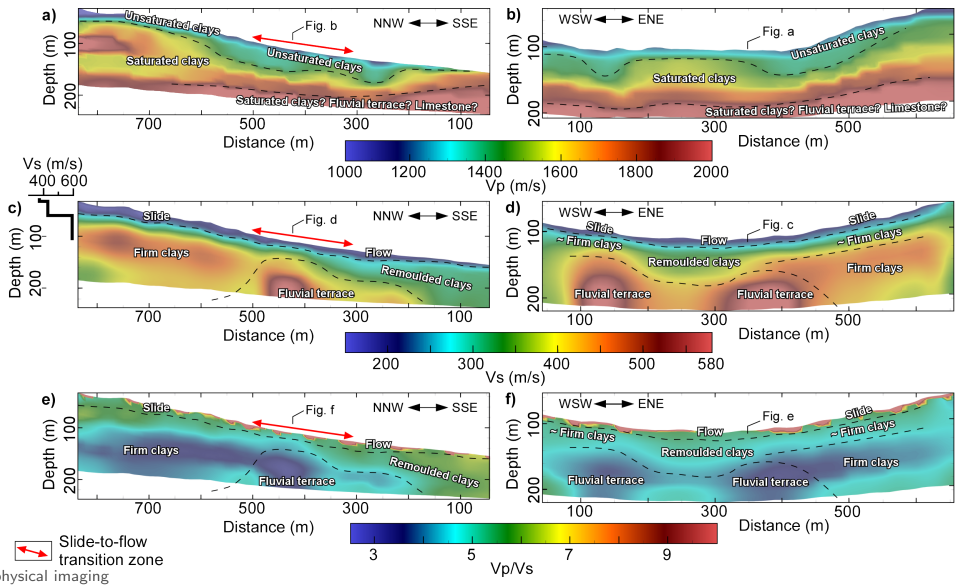
Exploring the crust





Schematic view of the CO2 storage injection in Sleipner (Norway) (Eiken, 2019). Sleipner is a pilot site, which has made possible to test and evaluate the interest of CO2 storage technology since 1996.

SEISCOPE


a)
Depth (m)
100
200
Unsaturated clays
Saturated clays
Unsaturated clays
Saturated clays? Fluvial terrace? Limestone?
Fig. b
NNW SSE
700
500
300
100
Distance (m)
b)
WSW ENE
Fig. a
Unsaturated clays
Saturated clays
Saturated clays? Fluvial terrace? Limestone?
100
300
500
Distance (m)
1000
1200
1400
1600
1800
2000
Vp (m/s)
Vs (m/s)
400 600
c)
Slide
Firm clays
Fig. d
Flow
Remoulded clays
Fluvial terrace
d)
Slide
~ Firm clays
Flow
Fig. c
Remoulded clays
Firm clays
Fluvial terrace
Fluvial terrace
200
300
400
500
580
Vs (m/s)
e)
Fig. f
f)
Fig. e
3
5
7
9
Vp/Vs
Slide-to-flow transition zone

and so many other purposes

- mining
- storage monitoring
- geothermal energy
- geotechnical engineering
- archaeology
- medical imaging with ultra-sound
- medical imaging with elastography
- ...

How to access the subsurface structure: drilling is not (really) an option

Drawback of drilling:

- direct measurement can destroy the target: archeology and geotechnical engineering belong to this category of applications.
- local information only: the subsurface, especially the crust, can not be accurately represented as a layered medium.
- highly technical and complex operation: thus expensive and risky
- for regional scale and global scale imaging, the depth of investigation of a drilling operation is far from being sufficient.

- Deepest drilling in the world: 12, 2 km $\simeq$ **0.2 % of the Earth's radius**
- Location: Kola Peninsula, Russia
- Drilling duration: 1970-1989 *i.e.* 19 years

Localization of the Kola peninsula on Google Earth (left). Picture of the Kola Superdeep Borehole drilling site(right).

Only feasible approach: remote sensing

Starting point: **subsurface rheology has an impact on the propagation of waves**.

- Electromagnetic waves. In this case, the subsurface structure/rheology variations affect the mean **permittivity and conductivity** of the subsurface, which have an effect on the propagation of electromagnetic waves.
- Mechanical (elastic) waves. In this case, the subsurface structure/rheology variations affect the mean **velocity, density, anisotropy, and attenuation** of the subsurface, which have an effect on the propagation of mechanical (elastic) waves.

Principle: from the observation of electromagnetic or elastic waves, infer the mean electromagnetic or mechanical properties of the subsurface

Outline

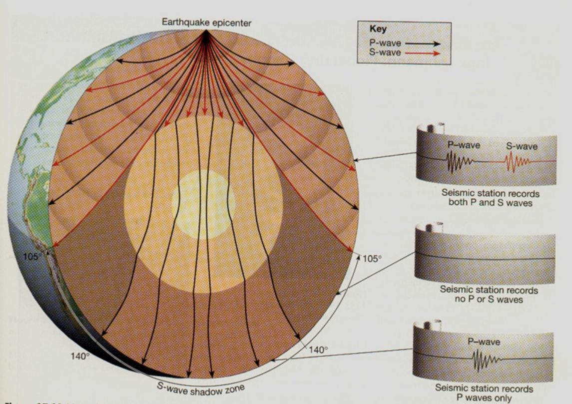
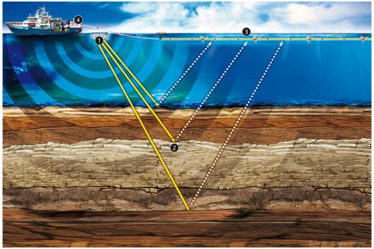
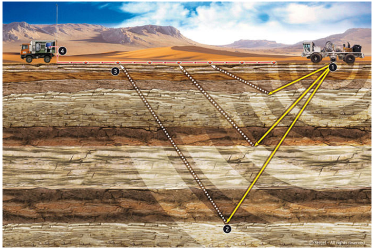


Figure 17.10 View of Earth's interior showing P and S wave paths. Any location more than 105 degrees from the earthquake epicenter will not receive direct S waves since the outer core will not transmit them. Although P waves are also absent beyond 105 degrees, they are recorded beyond 140 degrees, as shown in Figure 17.9.

Global tomography sketch: an earthquake acts as a source which propagates elastic waves. These waves are recorded by seismic stations spread at different locations at the surface.

or controlled sources





Controlled source acquisition sketch, in a marine environment (left) in a land environment (right)

Notations for data

In terms of mathematics, the seismic data is thus a collection of time functions $d(t)$ associated with a source s at position x_s and a receiver r at position x_r. We will denote it as

$$d_{r,s}(t), \tag{1}$$

in the following, or equivalently

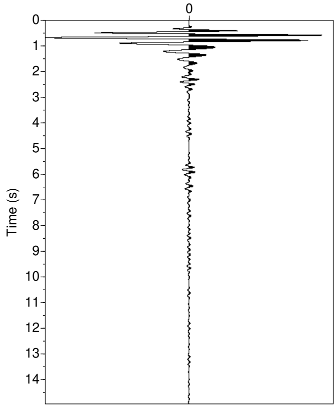
$$d(x_s, x_r, t), \tag{2}$$

or

$$d_s(x_r, t), \tag{3}$$

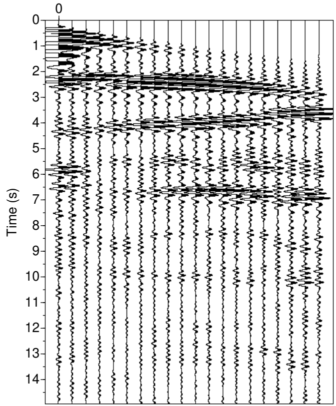
depending on the context. A single function $d_{r,s}(t)$ will be referred to as a *seismic trace* in the following.

A typical example of a seismic trace



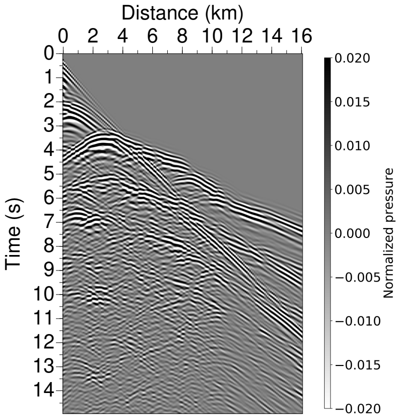
Seismic trace $d(t)$ as a function of time. We can identify a first wave packet of larger amplitude and later wave packets with smaller amplitude.

Instead of analyzing the data trace by trace: look simultaneously at several traces.



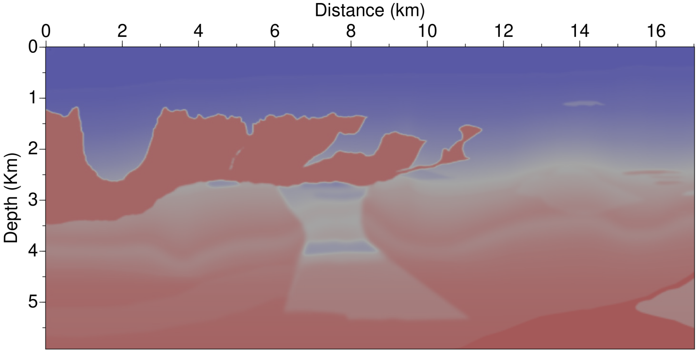
20 seismic traces $d_r(t)$ as a function of time, depending on the receiver/source distance, also referred to as *offset* in the following.

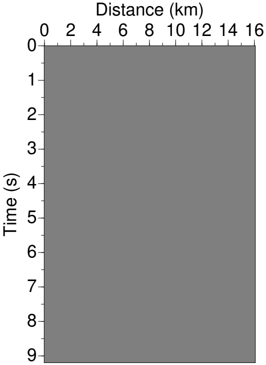
When the number of traces is even larger: use a 2D plot with a black & white chart



A typical seismogram in black and white representation. 161 traces spanning 16 km are used here. White correspond to negative values, black to positive values, while gray corresponds to 0. This yields the typical seismogram representation, widely used in exploration geophysics

Where does this seismogram come from?





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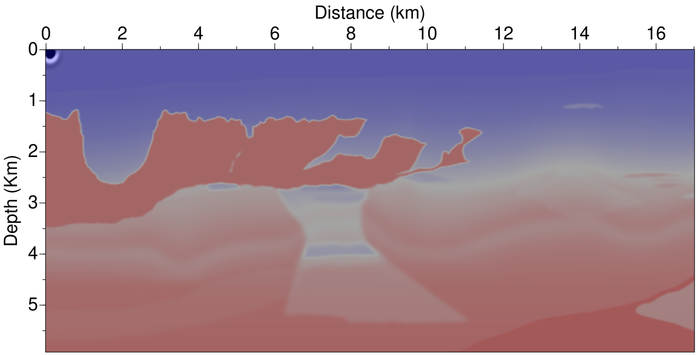
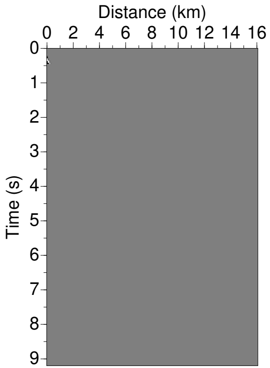
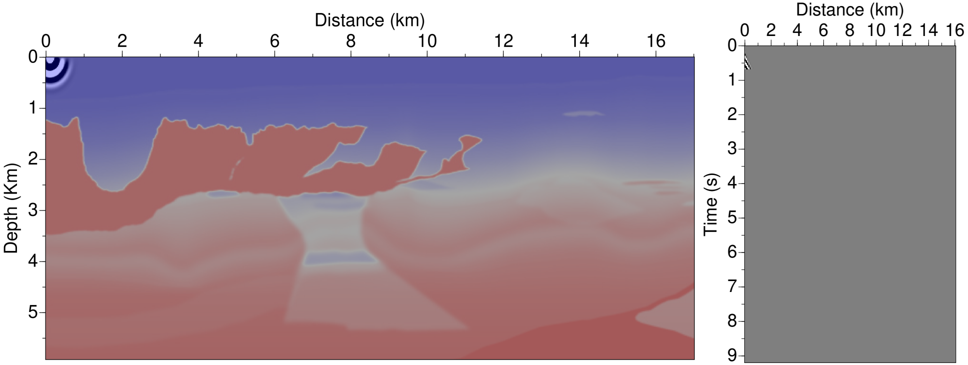
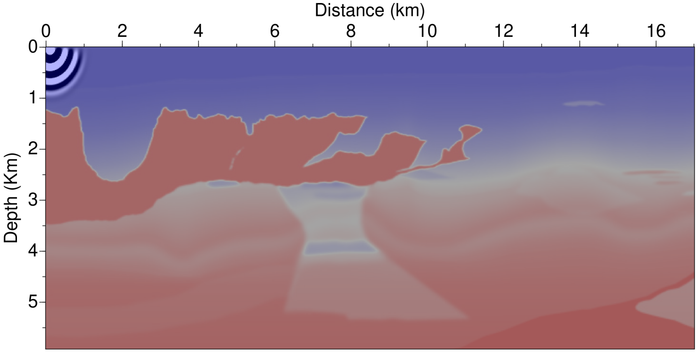
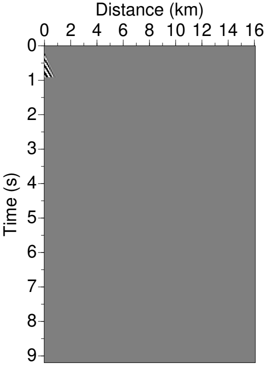


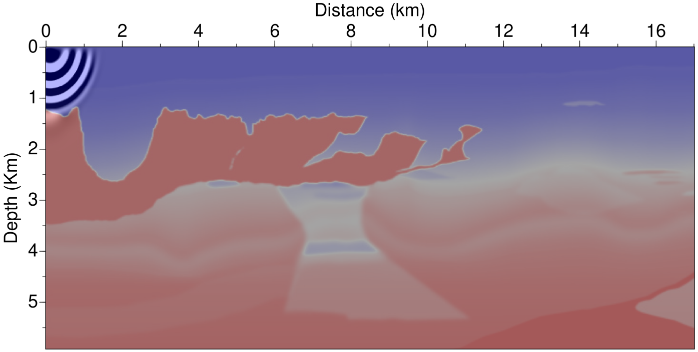


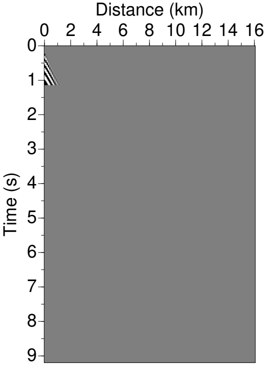

Distance (km)
0
2
4
6
8
10
12
14
16
Depth (Km)
0
1
2
3
4
5
Distance (km)
0
2
4
6
8
10
12
14
16
Time (s)
0
1
2
3
4
5
6
7
8
9

Where does this seismogram come from?






Distance (km)
0 2 4 6 8 10 12 14 16
0 1 2 3 4 5
Depth (Km)


Distance (km)
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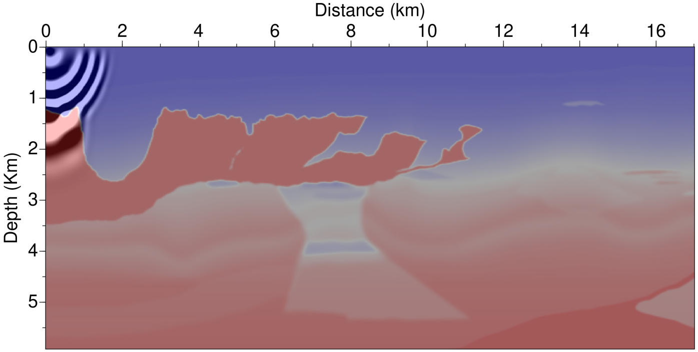
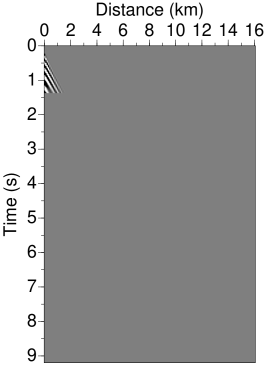
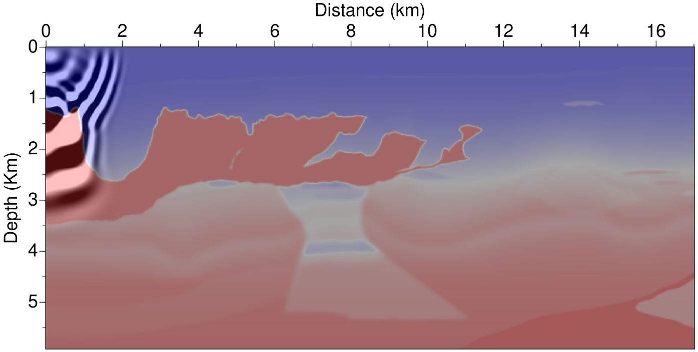
0 1 2 3 4 5 6 7 8 9
Time (s)

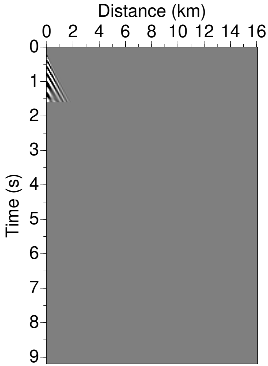
Where does this seismogram come from?





Where does this seismogram come from?





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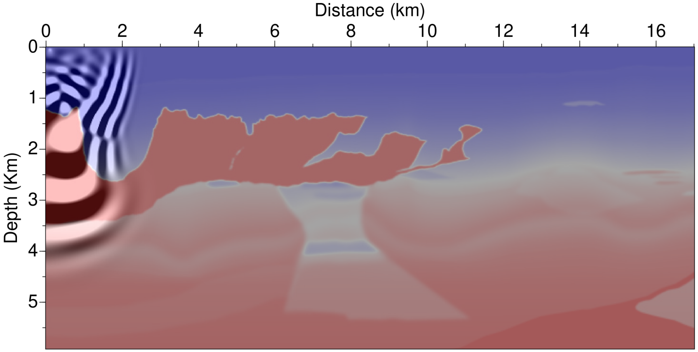

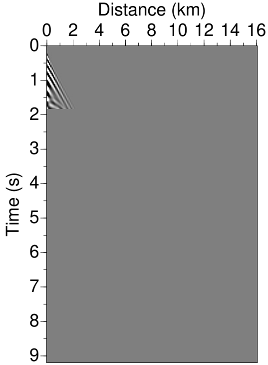

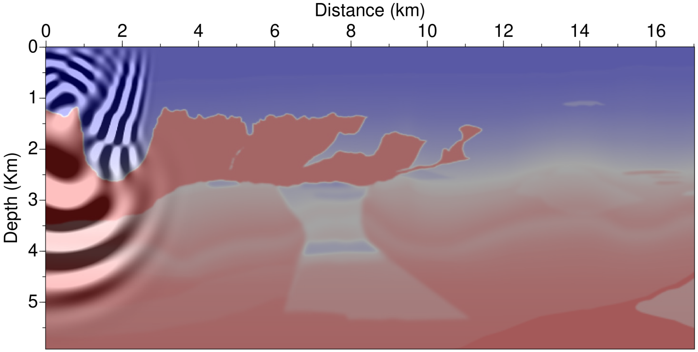
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Depth (Km)
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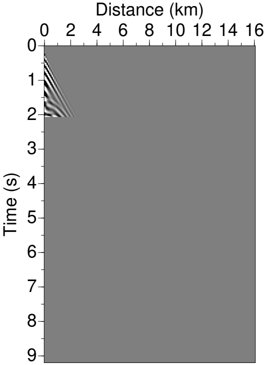
Distance (km)
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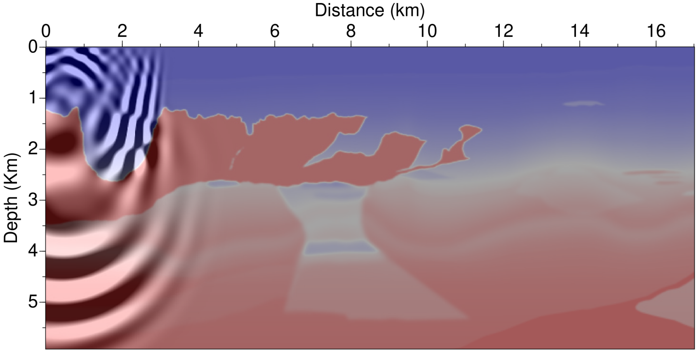
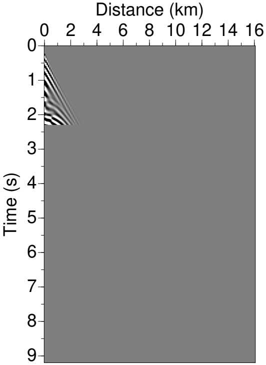
Where does this seismogram come from?





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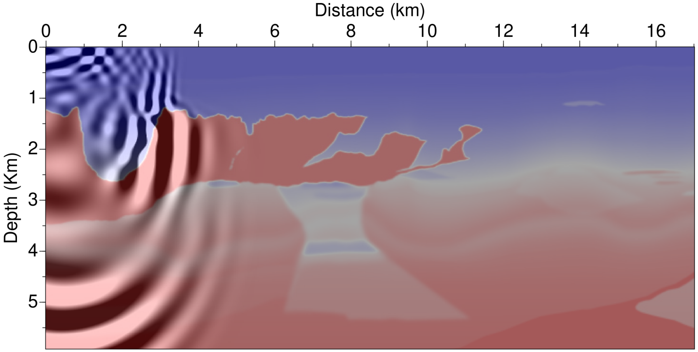

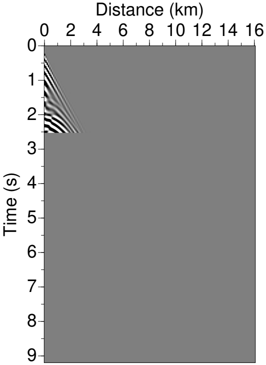

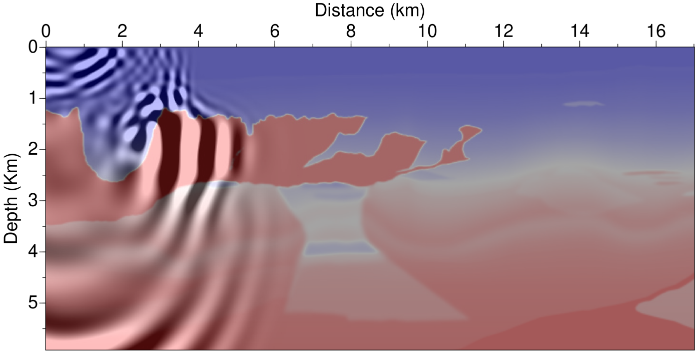
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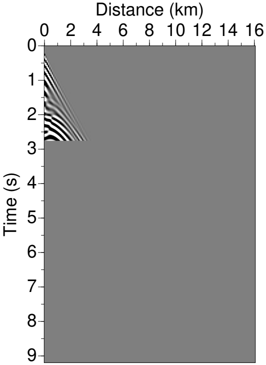
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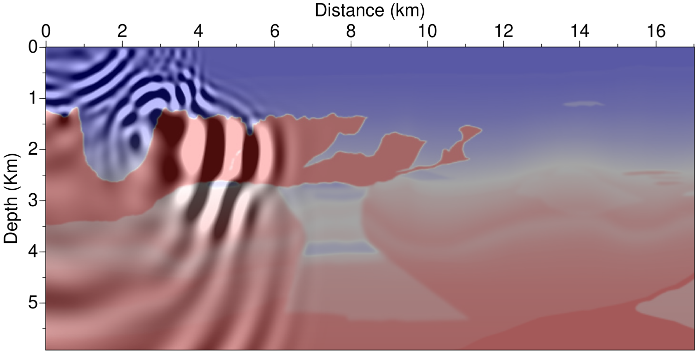
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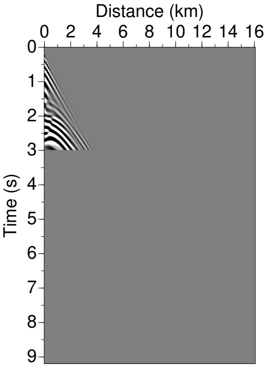
Distance (km)
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Time (s)
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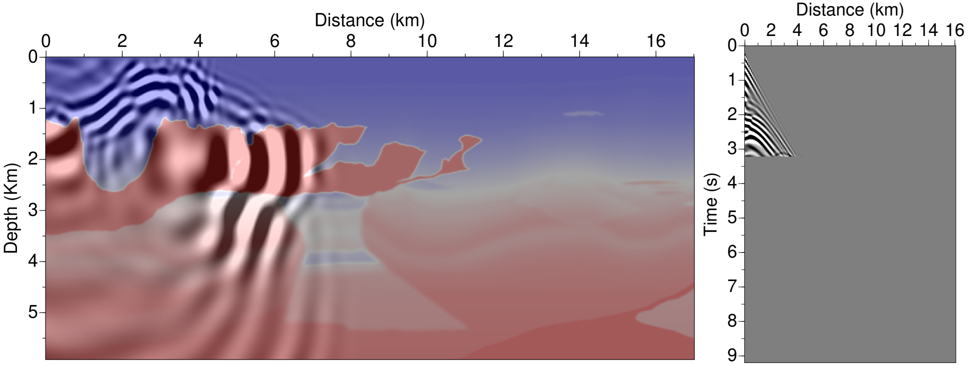
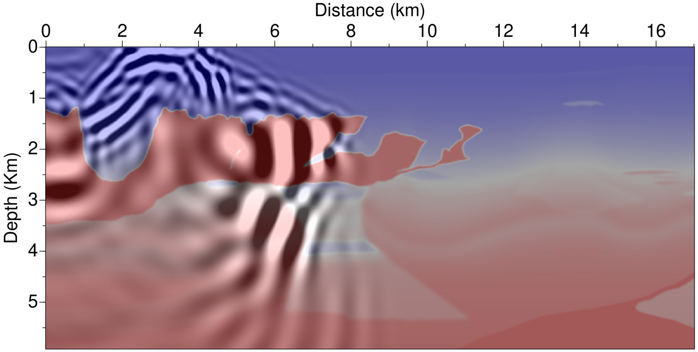
Where does this seismogram come from?




Distance (km)
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Depth (Km)
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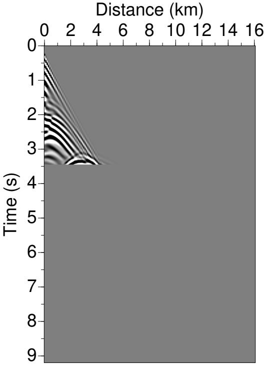
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Time (s)
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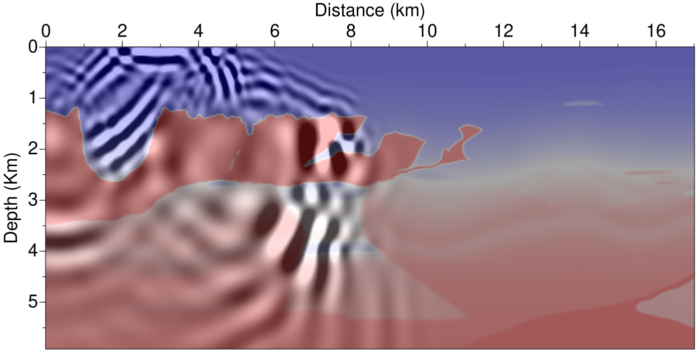

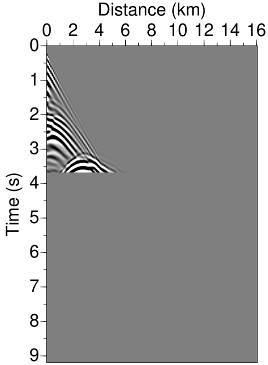

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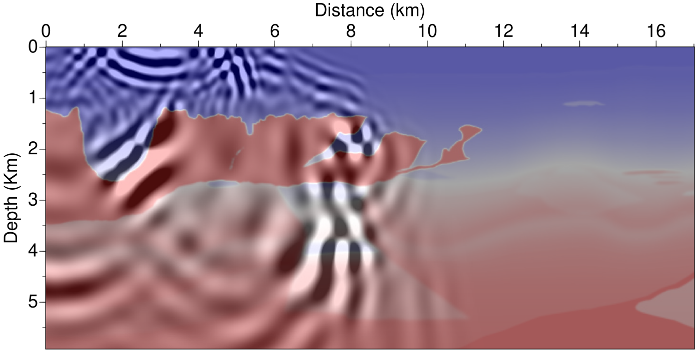

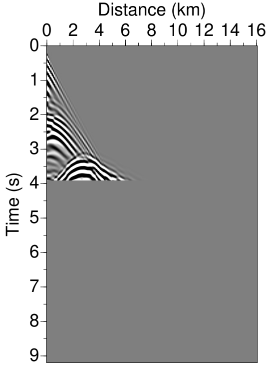

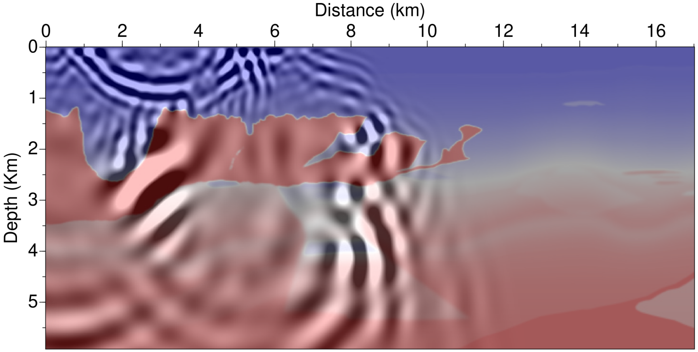
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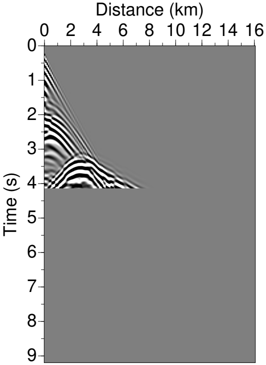
Distance (km)
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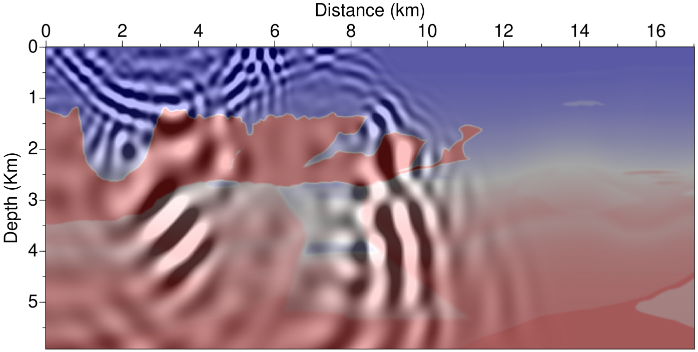
Distance (km)
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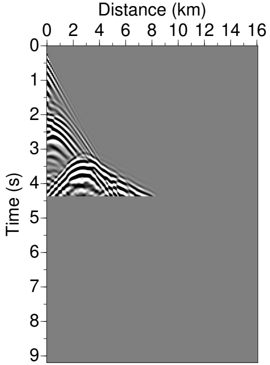
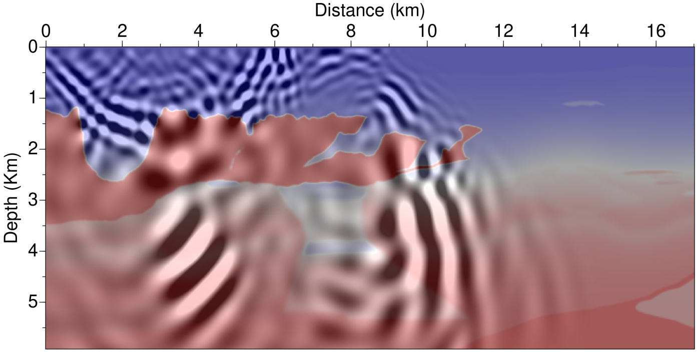
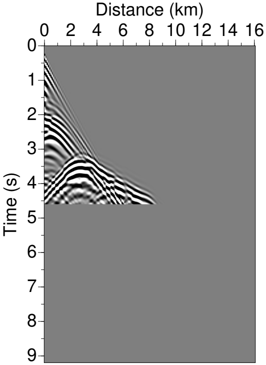
Depth (Km)
0 1 2 3 4 5

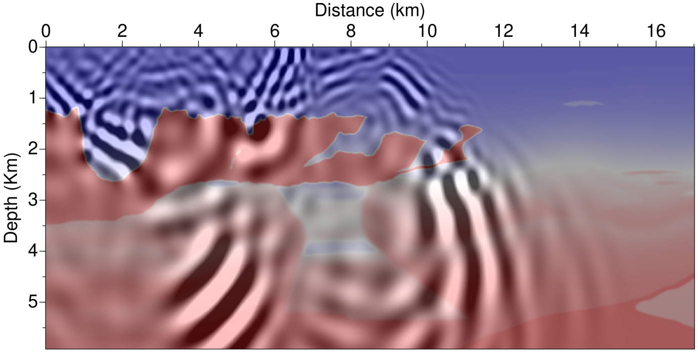

Distance (km)
0 2 4 6 8 10 12 14 16
Time (s)
0 1 2 3 4 5 6 7 8 9

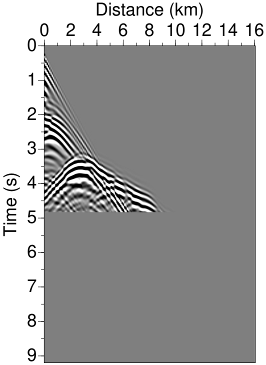
Where does this seismogram come from?



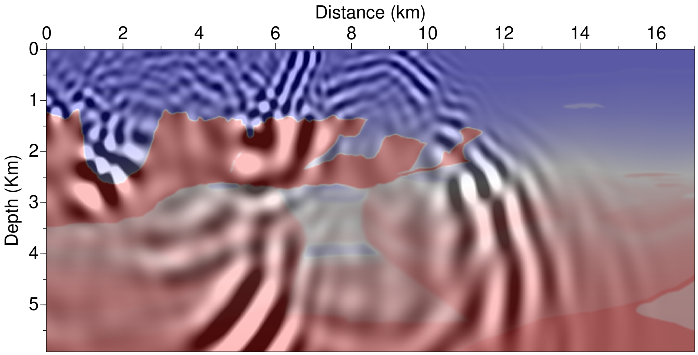


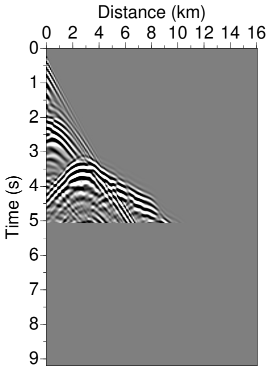
Where does this seismogram come from?



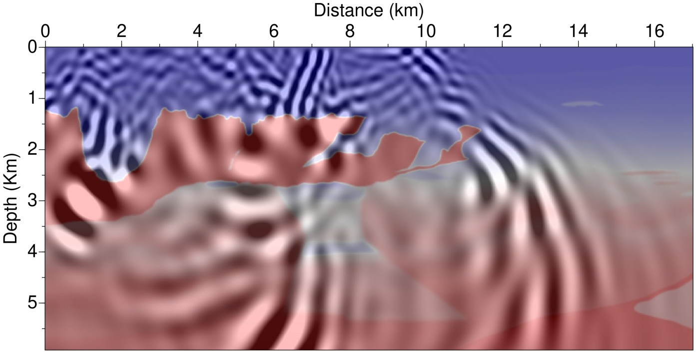


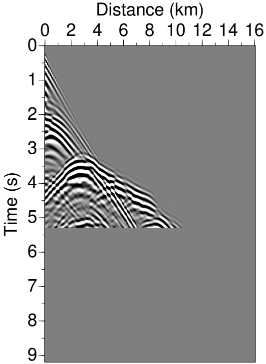
Where does this seismogram come from?

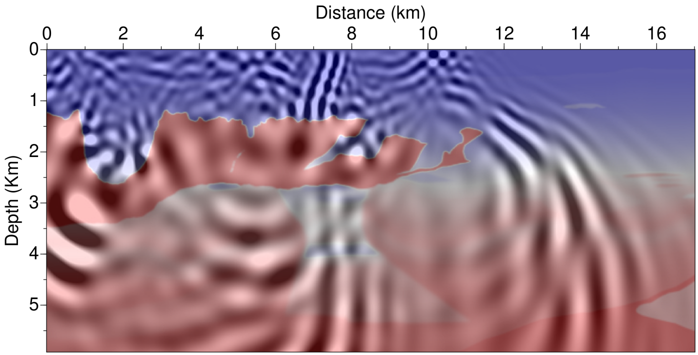


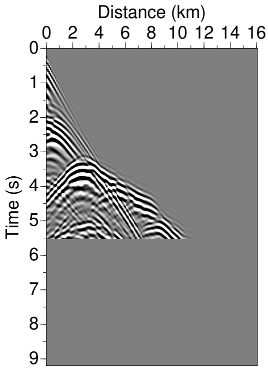


Where does this seismogram come from?

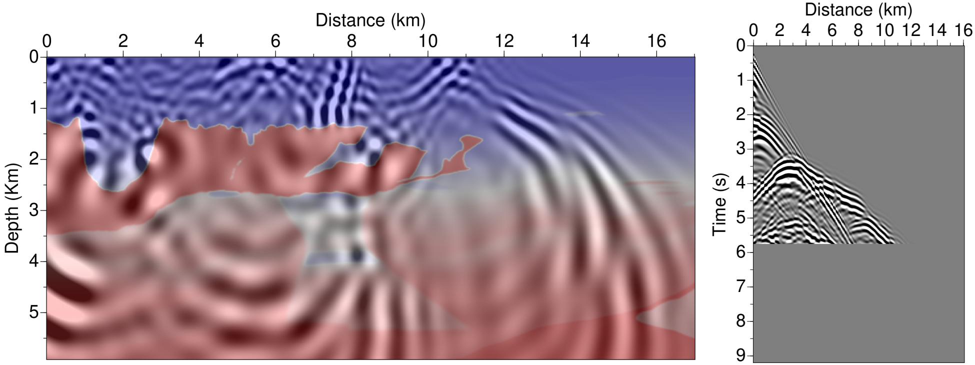




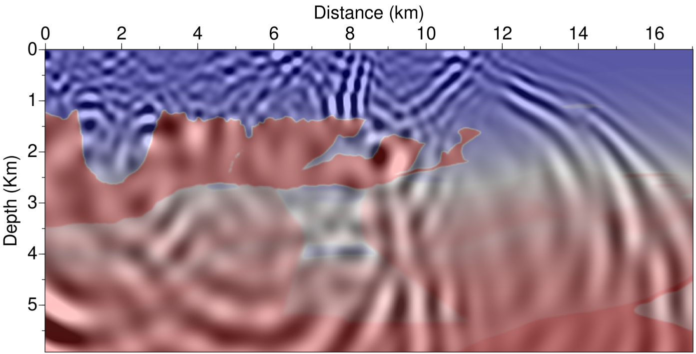

Distance (km)
0 2 4 6 8 10 12 14 16
Depth (Km)
0 1 2 3 4 5

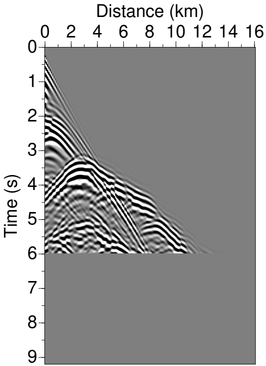

Distance (km)
0 2 4 6 8 10 12 14 16
Time (s)
0 1 2 3 4 5 6 7 8 9

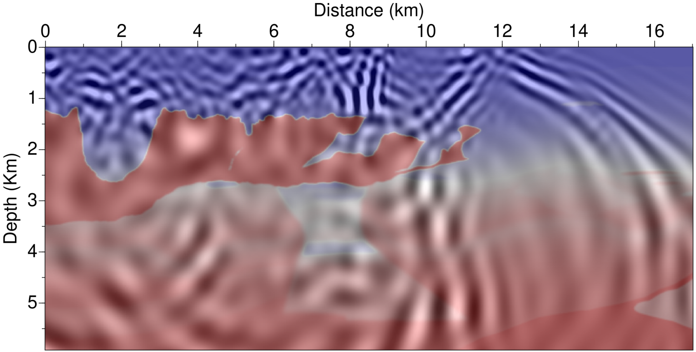
Where does this seismogram come from?

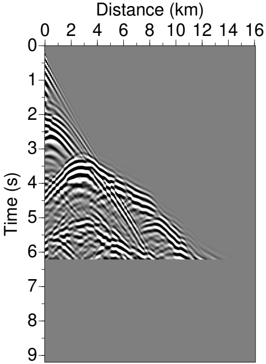


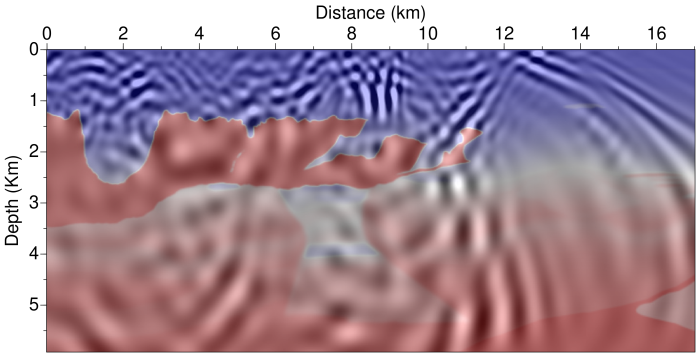
Where does this seismogram come from?

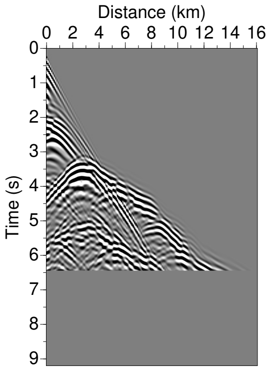




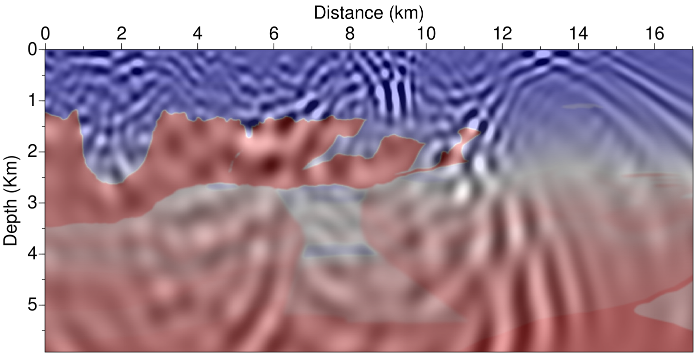

Distance (km)
0 2 4 6 8 10 12 14 16
Depth (Km)
0 1 2 3 4 5

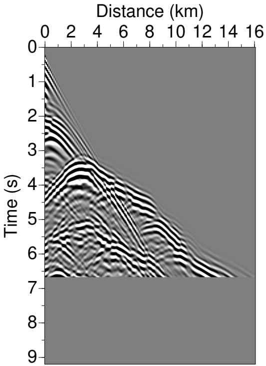

Distance (km)
0 2 4 6 8 10 12 14 16
Time (s)
0 1 2 3 4 5 6 7 8 9


Distance (km)
0 2 4 6 8 10 12 14 16
Depth (Km)
0 1 2 3 4 5

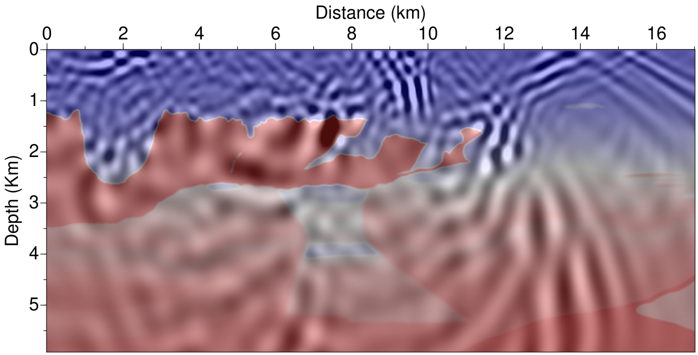

Distance (km)
0 2 4 6 8 10 12 14 16
Time (s)
0 1 2 3 4 5 6 7 8 9

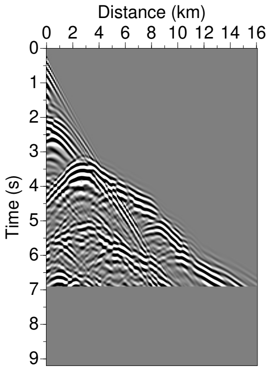
Where does this seismogram come from?

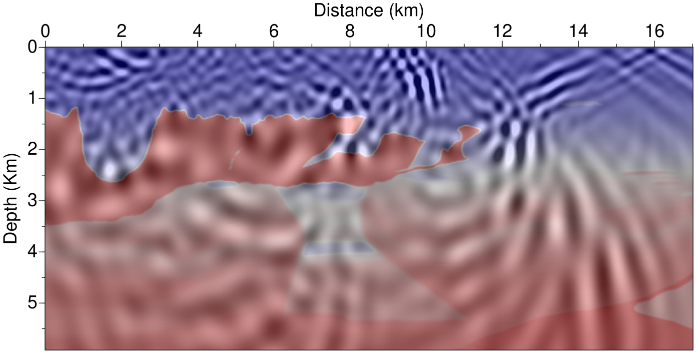


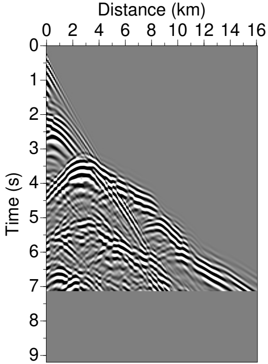


Where does this seismogram come from?

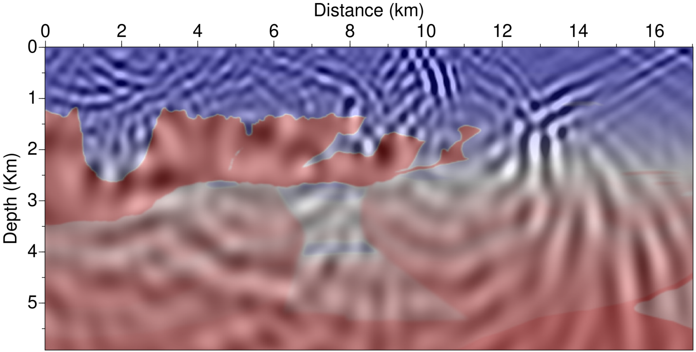


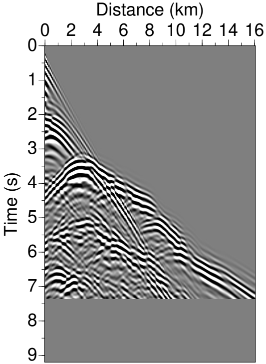



Distance (km)
0 2 4 6 8 10 12 14 16
Depth (Km)
0 1 2 3 4 5

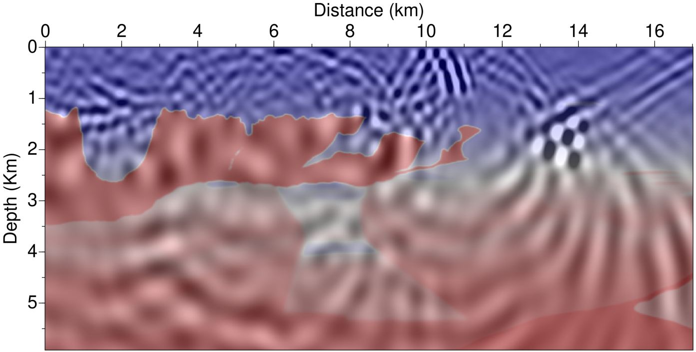

Distance (km)
0 2 4 6 8 10 12 14 16
Time (s)
0 1 2 3 4 5 6 7 8 9

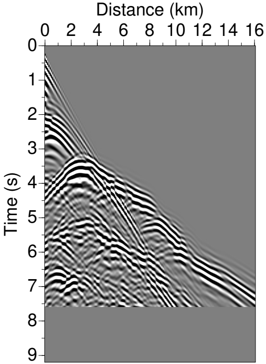
Where does this seismogram come from?

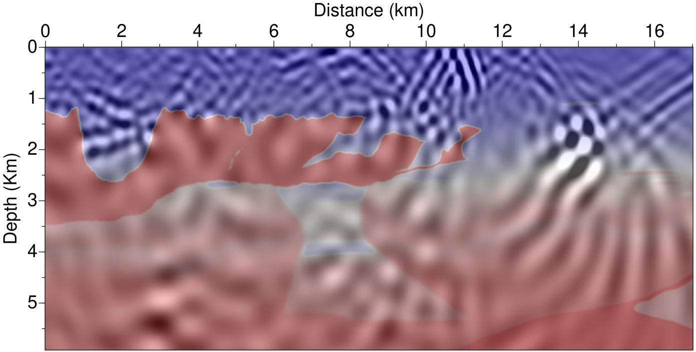


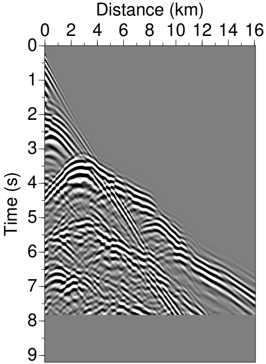


Where does this seismogram come from?

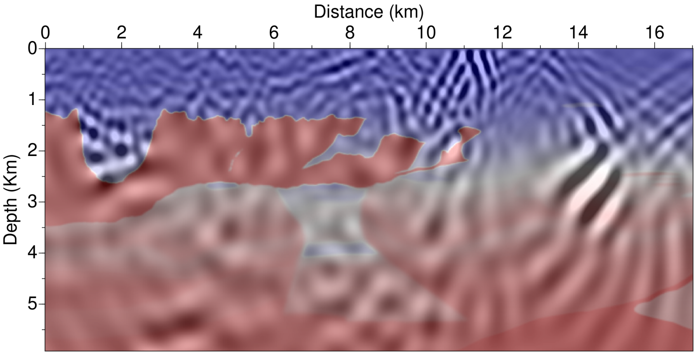


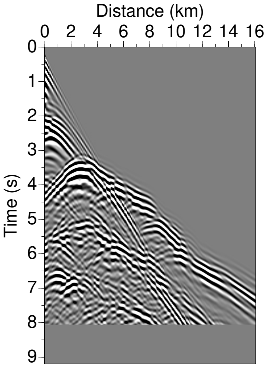


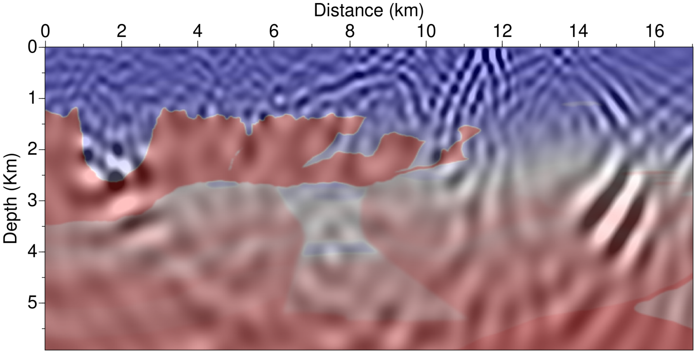

Distance (km)
0 2 4 6 8 10 12 14 16
Depth (Km)
0 1 2 3 4 5

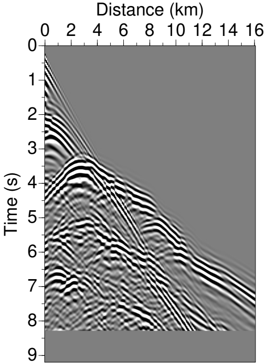

Distance (km)
0 2 4 6 8 10 12 14 16
Time (s)
0 1 2 3 4 5 6 7 8 9

Where does this seismogram come from?

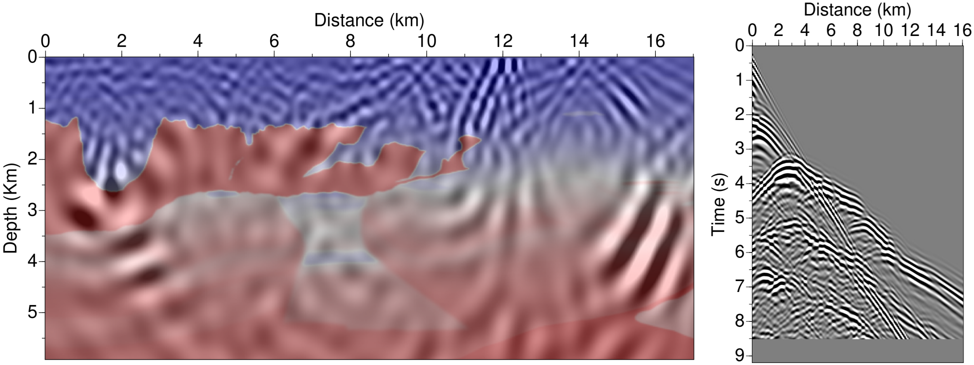




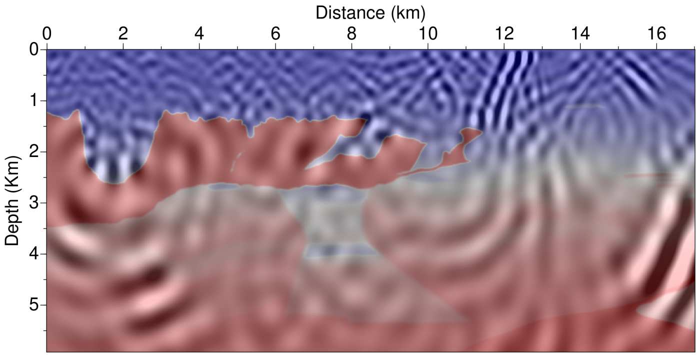

Distance (km)
0 2 4 6 8 10 12 14 16
Depth (Km)
0 1 2 3 4 5

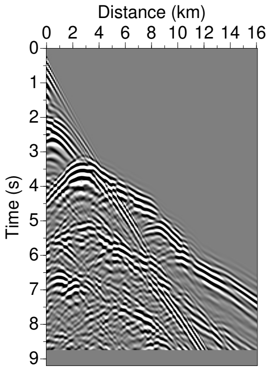

Distance (km)
0 2 4 6 8 10 12 14 16
Time (s)
0 1 2 3 4 5 6 7 8 9

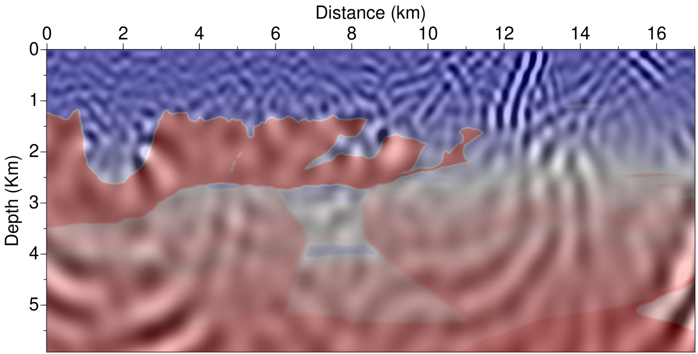
Where does this seismogram come from?

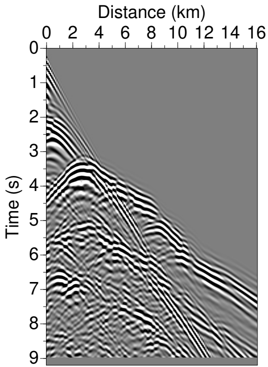


Where does this seismogram come from?

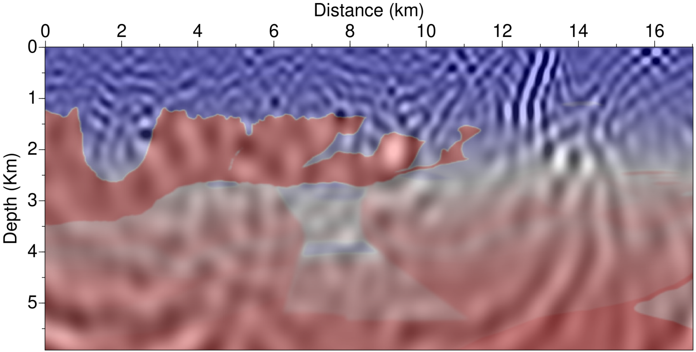


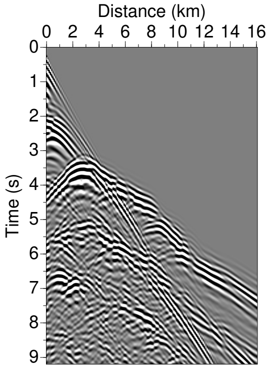


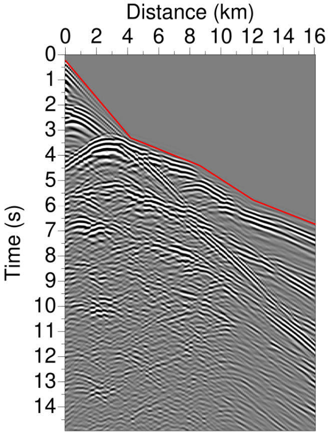

Distance (km)
0 2 4 6 8 10 12 14 16
Depth (Km)
0 1 2 3 4 5


Distance (km)
0 2 4 6 8 10 12 14 16
Time (s)
0 1 2 3 4 5 6 7 8 9

Where does this seismogram come from?







Same seismogram as in previous Figure with first-arrival travel time denoted by the red line.

Inverse tomography problem

$$d_{obs} = t_{obs}(x_s, x_r), \quad m = v_P \tag{4}$$

where $t_{obs}(x_s, x_r)$ denotes the picked travel times from source s to receiver r, and v_P is the pressure wave velocity.

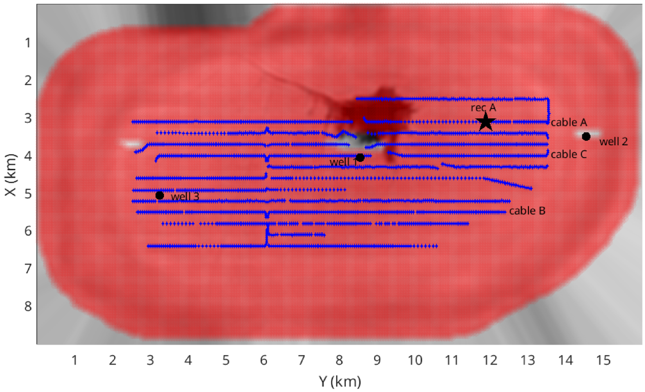
Least-squares first-arrival travel time tomography

$$\min_{v_P} \frac{1}{2} \| t_{cal} - t_{obs} \|^2 + \eta R(v_P), \quad t_{cal} = g(v_P). \tag{5}$$

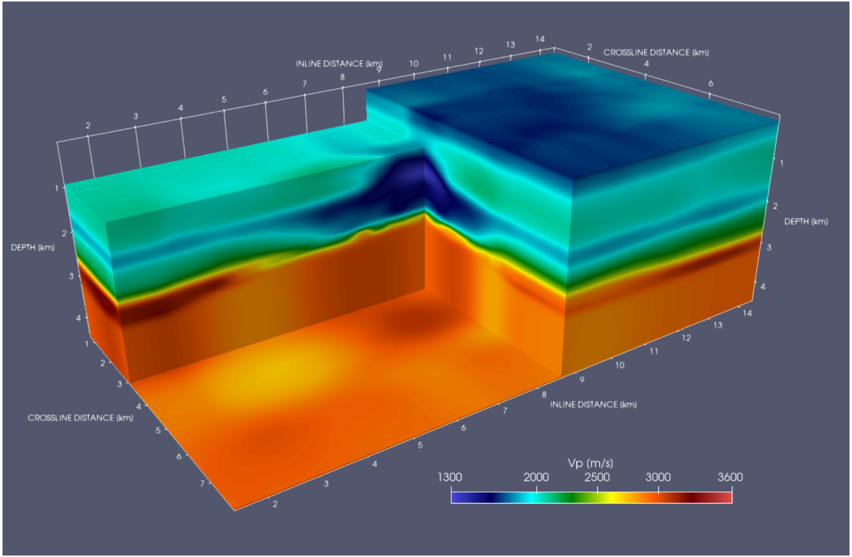
Idea: replace the forward modeling operator $g(m)$ by a full wave modeling solver, and to compare the resulting synthetic data to the full observed data $d_{obs}(x_s, x_r, t)$. The FWI problem is thus formulated as

$$\min_m \frac{1}{2}\|d_{cal} - d_{obs}\|^2 + \eta R(m), \quad d_{cal} = g(m) \tag{6}$$

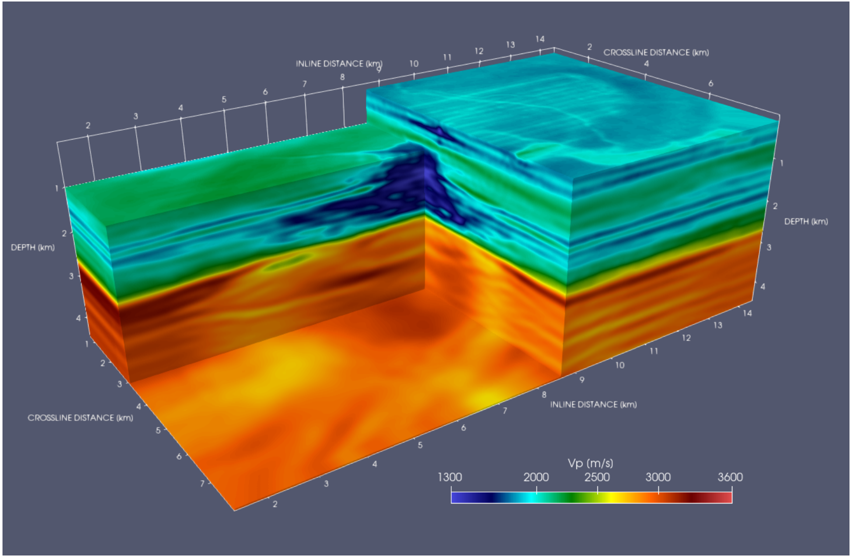
- Reciprocity: 50,000 explosive sources (airguns)+ 2048 hydrophones along the cables $\Longrightarrow$ 2048 explosive source + 50,000 hydrophones



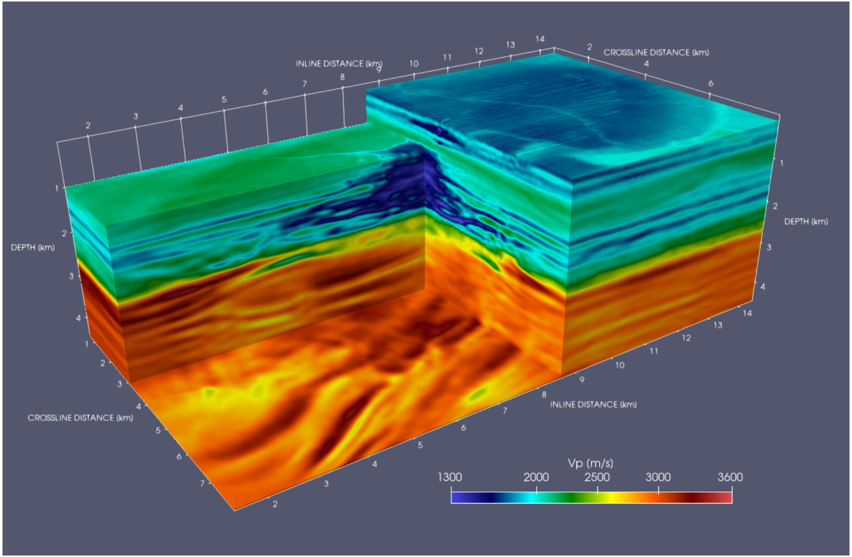
Tomo V_P

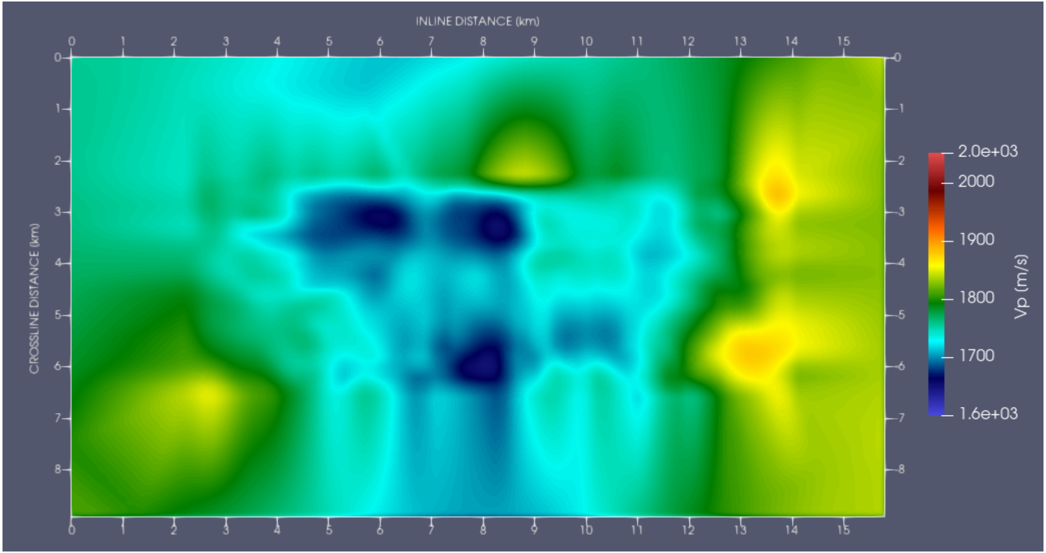


INLINE DISTANCE (km)
CROSSLINE DISTANCE (km)
DEPTH (km)
DEPTH (km)
CROSSLINE DISTANCE (km)
INLINE DISTANCE (km)
Vp (m/s)
1300
2000
2500
3000
3600

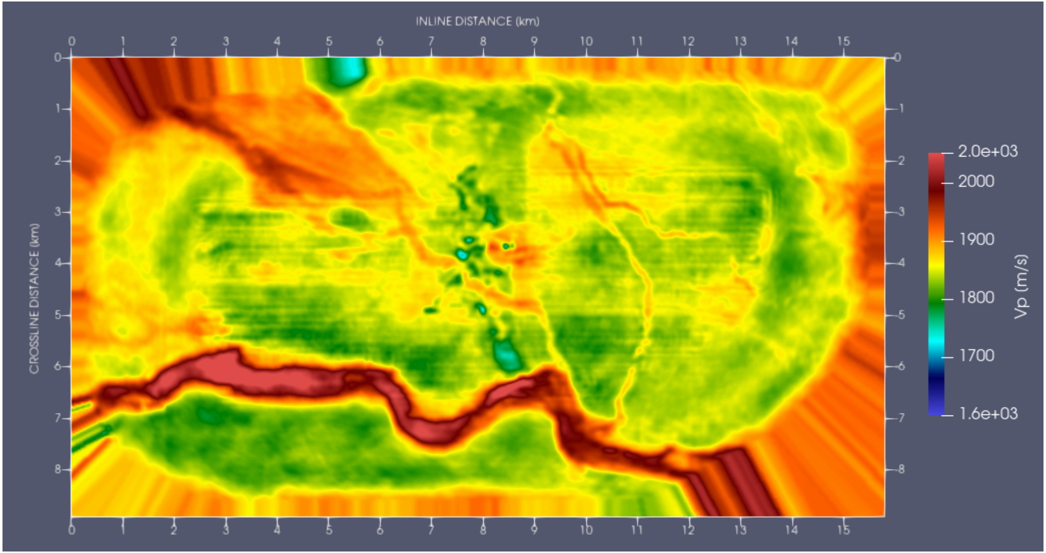

INLINE DISTANCE (km)
CROSSLINE DISTANCE (km)
DEPTH (km)
DEPTH (km)
CROSSLINE DISTANCE (km)
INLINE DISTANCE (km)
Vp (m/s)
1300
2000
2500
3000
3600

22.5 Hz V_P

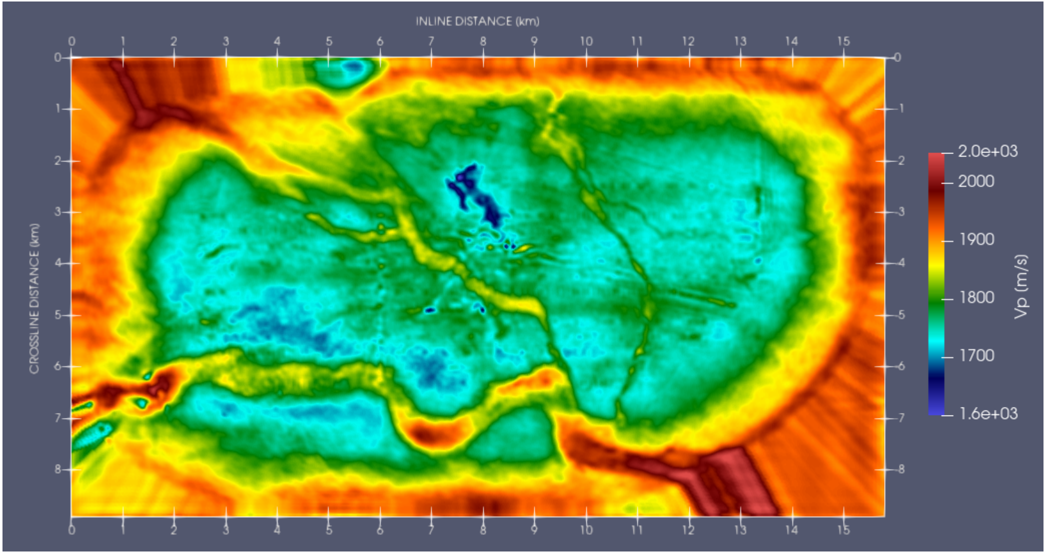

INLINE DISTANCE (km)
CROSSLINE DISTANCE (km)
DEPTH (km)
DEPTH (km)
CROSSLINE DISTANCE (km)
INLINE DISTANCE (km)
Vp (m/s)
1300
2000
2500
3000
3600


INLINE DISTANCE (km)
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
CROSSLINE DISTANCE (km)
0 1 2 3 4 5 6 7 8
2.0e+03
2000
1900
1800
1700
1.6e+03
Vp (m/s)

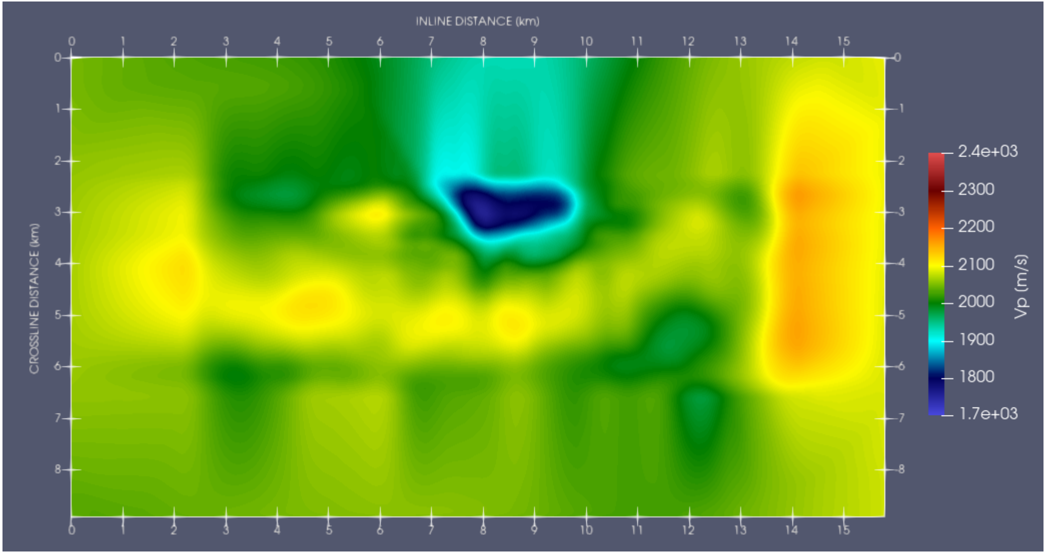
10 Hz V_P, shallow depth slice



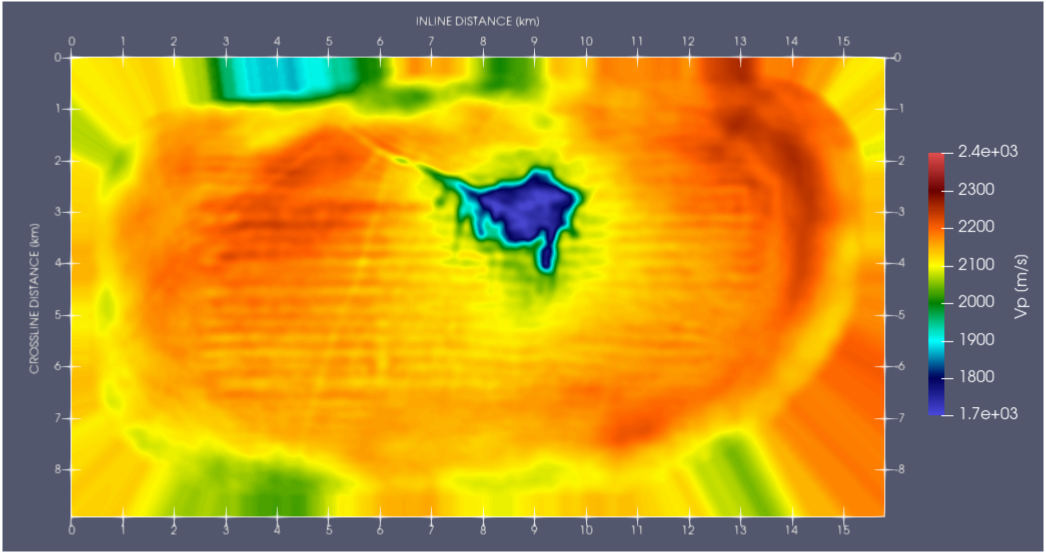
22.5 Hz V_P, shallow depth slice



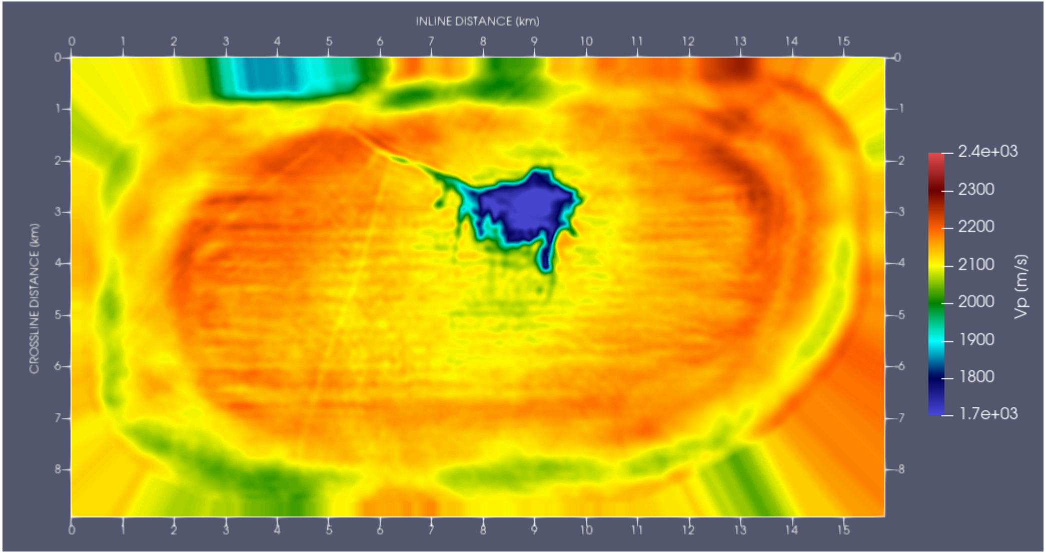
Tomo V_P, deeper depth slice



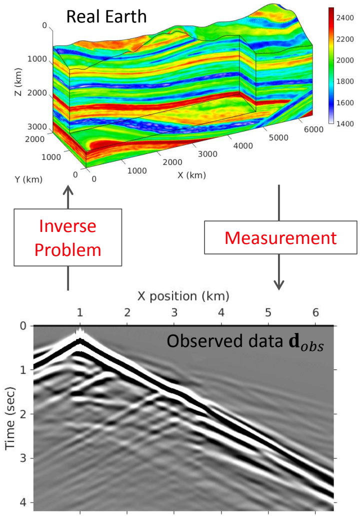
10 Hz V_P, deeper depth slice

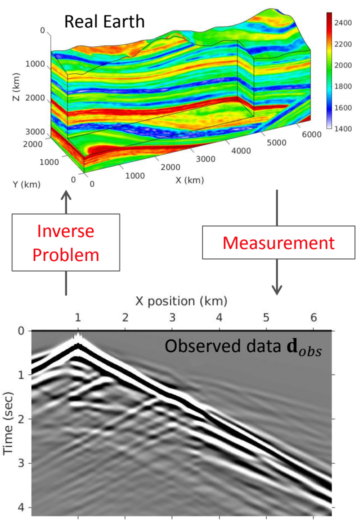


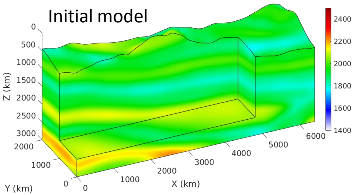
22.5 Hz V_P, deeper depth slice



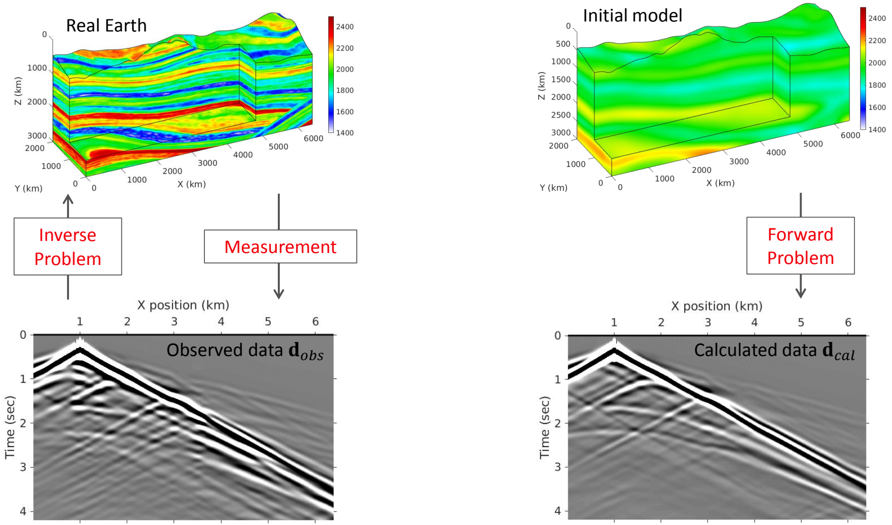
Full waveform inversion principle



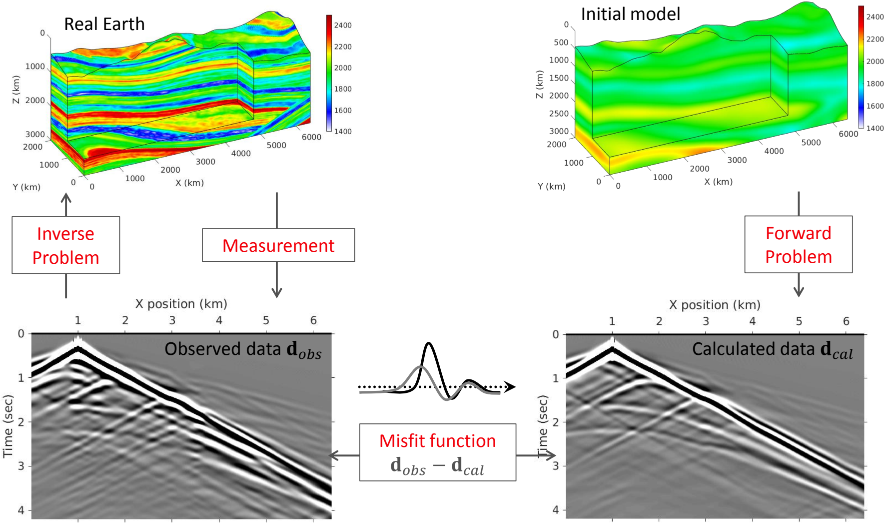

Real Earth
Z (km)
Y (km)
X (km)
Inverse Problem
Measurement
X position (km)
Time (sec)
Observed data $\mathbf{d}_{obs}$


Initial model
Z (km)
Y (km)
X (km)

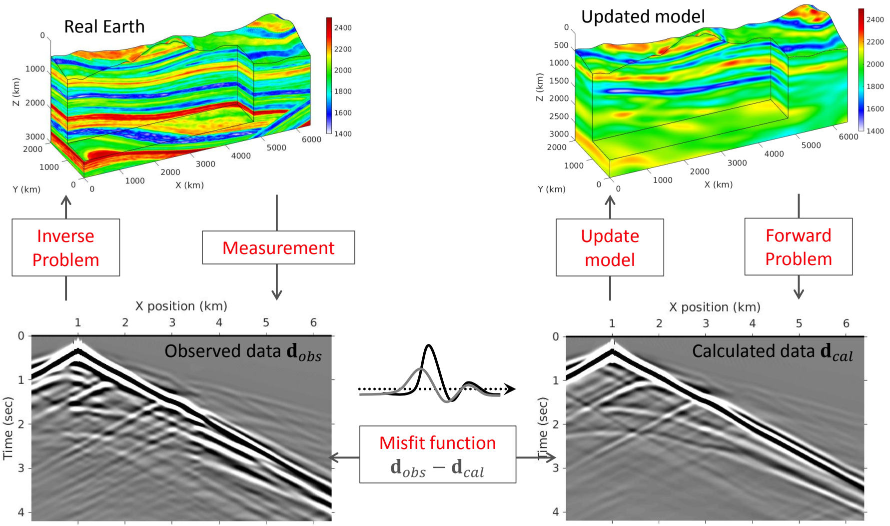
Full waveform inversion principle



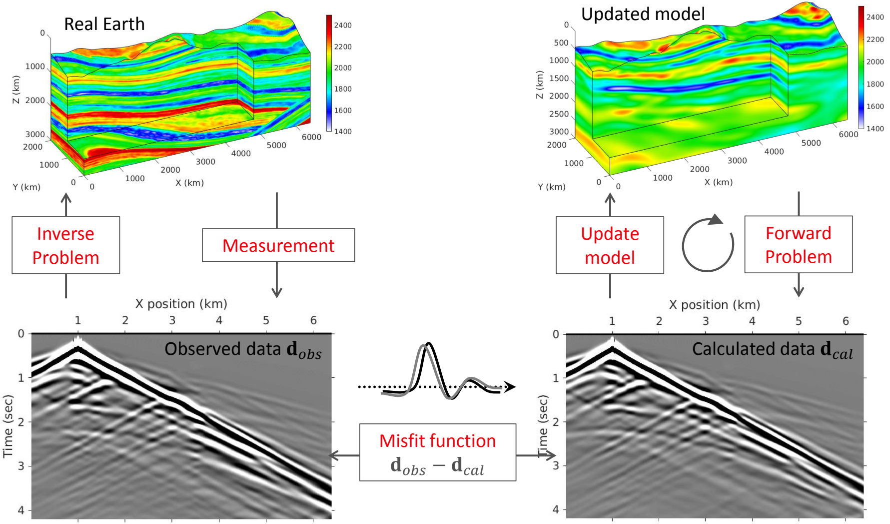
Full waveform inversion principle

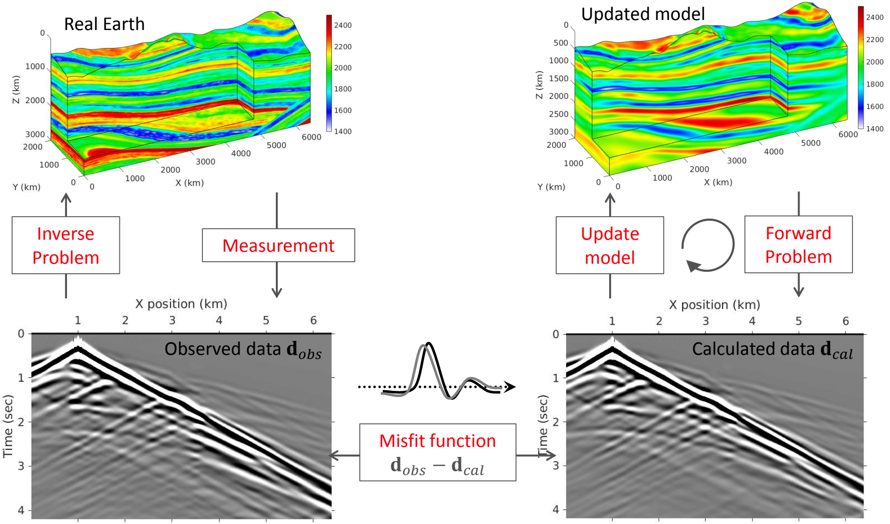


Full waveform inversion principle

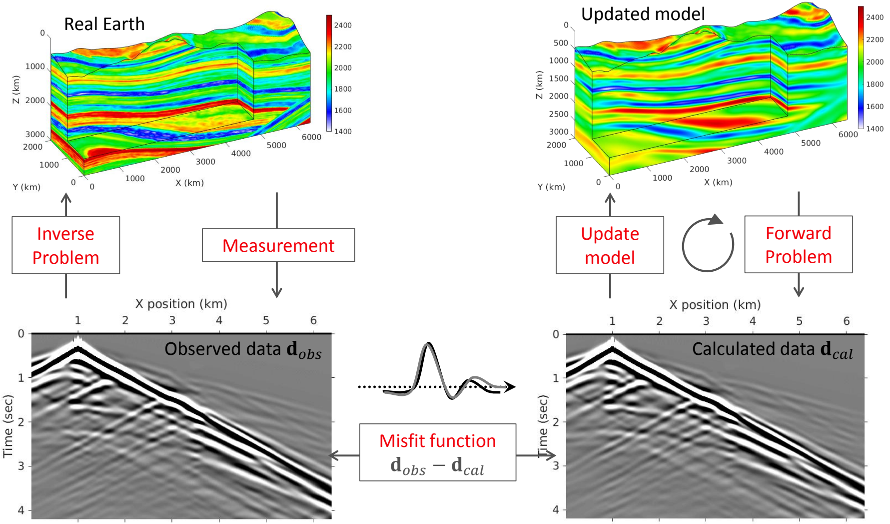


Full waveform inversion principle




Real Earth
Updated model
Z (km)
X (km)
Y (km)
Inverse Problem
Measurement
Update model
Forward Problem
X position (km)
Time (sec)
Observed data $\mathbf{d}_{obs}$
Calculated data $\mathbf{d}_{cal}$
Misfit function
$\mathbf{d}_{obs} - \mathbf{d}_{cal}$

Full waveform inversion principle



Mathematical formulation: a PDE constrained minimization problem

$$\min_{m} J(m), \quad J(m) = \frac{1}{2}\|d_{cal}[m] - d_{obs}\|^2 = \frac{1}{2}\sum_{s=1}^{N_s}\sum_{r=1}^{N_r}\int_0^T |d_{cal,s}[m](x_r,t) - d_{obs,s}(x_r,t)|^2 dt \tag{7}$$

$$\min_m J(m), \quad J(m) = \frac{1}{2}\|d_{cal}[m] - d_{obs}\|^2 = \frac{1}{2}\sum_{s=1}^{N_s}\sum_{r=1}^{N_r}\int_0^T |d_{cal,s}[m](x_r, t) - d_{obs,s}(x_r, t)|^2 dt \quad (7)$$

where

$$d_{cal,s}[m](x_r, t) = Ru_s[m] = \int_\Omega \delta(x - x_r)u_s[m](x, t)dx, \quad (8)$$

$$\min_m J(m), \quad J(m) = \frac{1}{2}\|d_{cal}[m] - d_{obs}\|^2 = \frac{1}{2}\sum_{s=1}^{N_s}\sum_{r=1}^{N_r}\int_0^T |d_{cal,s}[m](x_r,t) - d_{obs,s}(x_r,t)|^2 dt \tag{7}$$

where

$$d_{cal,s}[m](x_r,t) = Ru_s[m] = \int_\Omega \delta(x - x_r)u_s[m](x,t)dx, \tag{8}$$

and

$$A(m)u_s = b_s \tag{9}$$

with $A(m)$ a wave propagation operator, for instance acoustic approximation,

$$A(m)u = \partial_{tt}u - \rho c^2 \mathrm{div}\left(\frac{1}{\rho}\nabla u\right), \quad m = [\rho \;\; c] \tag{10}$$

or elastic approximation

$$A(m)u = \begin{cases} \rho\partial_{tt}u - \mathrm{div}\sigma, \\ \sigma = \frac{1}{2}C(\nabla u + \nabla u^T), \end{cases}, \quad m = [\rho \;\; C] \tag{11}$$

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