Objective. In this chapter, we focus primarily on a classical numerical method (the Euler method) for ordinary differential equations (ODE) after reviewing some basic definitions for ordinary differential equations. In particular, we study linear differential equations of order 1 and 2 so as to compare analytical and numerical solutions.

1 Vector field

A vector field in the plane is an assignment of a vector to each point in the plane. It can be visualised as a collection of arrows with a given magnitude and direction, each attached to a point in the plane. Vector fields are often used to model, for example, the speed and direction of a moving fluid throughout space, or the strength and direction of some force (from Wikipedia).

A vector field is usually represented by a vector-valued function

\[ F : S \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2 \]

\[ (x, y) \mapsto (F_x(x, y), F_y(x, y)) \]

which means that the 2D vector \((F_x(x, y), F_y(x, y))\) is assigned to each 2D point \((x, y)\).

We now describe the two builtin Scilab functions \texttt{champ()} and \texttt{champ1()} regarding the display of a vector field.

- \texttt{champ(x, y, fx, fy)} draws a 2D vector field. The length of the arrows is proportional to the intensity of the field.
  - \texttt{x : vector \([x(1), \ldots, x(i), \ldots, x(m)]\)}
  - \texttt{y : vector \([y(1), \ldots, y(j), \ldots, y(n)]\)}
  - \texttt{fx, fy : (m,n) matrices which describe the components of the vector field.}
  - \texttt{fx(i,j) is the x-component of the vector field at point \([x(i), y(j)]\)}
  - \texttt{fy(i,j) is the y-component of the vector field at point \([x(i), y(j)]\)}

Exercise 1

Experiment the following Scilab script and modify some parameters in the definition of the vector field \(F(x, y)\).
Nx = 21;
Ny = 21;
x = linspace(0, 2, Nx);
y = linspace(0, 2, Ny);
Fx = ones(Nx, Ny);
Fy = Fx;
for i = 1 : Nx
    for j = 1 : Ny
        Fx(i, j) = sin(2 * x(i) * y(j));
        Fy(i, j) = cos(2 * x(i) * y(j));
    end
end
champ(x, y, Fx, Fy);

Vector field \( \mathbf{F}(x, y) = (\sin(2x y), \cos(2x y)) \)

- \texttt{champ1(x, y, fx, fy)} is similar to \texttt{champ()}: \texttt{champ1} draws a 2D vector field with colored arrows. The color of the arrows depends on the intensity of the field.

**Exercise 2**

Complete the following scilab script so as to draw the vector field \( \mathbf{F}(x, y) = (\sin(y), \sin(x)) \) as shown in the two figures below.

// file FigchampEx1.sce

// Vector field \((\sin y, \sin x)\)
Nx = 15;
Ny = 15;
a = %pi/2;
x = linspace(-a, a, Nx);
y = linspace(-a, a, Ny);

scf()
champ(x, y, Fx, Fy);
scf()
champ1(x, y, Fx, Fy);

Vector field \( \mathbf{F}(x, y) = (\sin(y), \sin(x)) \). Left: with \texttt{champ()} — Right: with \texttt{champ1()}. \[ \text{MAP101} \]
Solution to exercise

```matlab
// file FigchampEx1.sce

// Vector field (sin y, sin x)
Nx = 15;
Ny = 15;
a = %pi/2;
x = linspace(-a,a,Nx);
y = linspace(-a,a,Ny);
Fx = ones(Nx,Ny);
Fy = Fx;
for i = 1 : Nx
  for j = 1 : Ny
    Fx(i,j) = sin(y(j));
    Fy(i,j) = sin(x(i));
  end
end
scf();
champ(x,y,Fx,Fy);
scf();
champl(x,y,Fx,Fy);
```

2 Ordinary differential equation

An ordinary differential equation (ODE) is an equation connecting a function of a single independent variable with some of its derivatives. Ordinary differential equations appear in many contexts of mathematics, physics, biology, social sciences, finance... An ODE is usually associated with initial conditions on the unknown function. An ordinary differential equation often models a phenomenon of evolution so that the variable of the function is usually assimilated (by abuse) to a temporal variable.

◊ An explicit ordinary differential equation of order \( n \) is defined by an equation on the form

\[
y^{(n)}(t) = f(t, y(t), y'(t), \ldots, y^{(n-1)}(t)).
\]

◊ An ordinary differential equation not depending on the parameter \( t \) is said to be autonomous:

\[
y^{(n)}(t) = f(y(t), y'(t), \ldots, y^{(n-1)}(t)).
\]

The derivatives of a function \( y(t) \) will be indifferently noted using the Leibniz\(^1\) notation \( dy/dt \), \( d^2y/dt^2 \), \ldots or the Lagrange\(^2\) notation \( y', y'' \), \( y^{(2)} \), \ldots, whereas the Newton\(^3\) notation \( \dot{y} \), \( \ddot{y} \), \ldots is more commonly used in physics for representing low order derivatives with respect to time.

2.1 Examples

We consider some classical differential equations. The first two are fundamental models of population growth.

---

1 Gottfried Wilhelm (von) Leibniz, 1646-1716, was a German mathematician and philosopher.
2 Joseph-Louis Lagrange, 1736-1813, was an Italian mathematician and astronomer.
3 Isaac Newton, 1642-1726, was an English mathematician, astronomer and physicist.
Given a population $N(t)$ with initial condition $N(t_0) = N_0 > 0$ at time $t = t_0$, we first assume that the birth and death rates ($b$ and $d$) are proportional to the population $N$, i.e., $\Delta N/N = (b - d) \Delta t$, which leads to the exponential growth model (Malthus, 1798)

$$\frac{dN(t)}{dt} = (b - d) N(t) \quad \text{with solution} \quad N(t) = N_0 e^{(b-d)(t-t_0)}.$$  

This model does not take into account the ability of the community to support exponential growth.

The logistic growth model (Verhulst, 1836) assume that the growth rate is a decreasing function of the population, depending on an environment variable $K$:

$$\frac{dN(t)}{dt} = (b - d) \left(1 - \frac{N(t)}{K}\right)N(t) \quad \text{with} \quad b - d > 0, \ K > 0.$$  

These two models of population growth are explicit autonomous ODEs of order one.

Another classic example of ODE is the differential equation (of order two) of the pendulum

$$\ddot{\theta} = -\frac{L}{g} \sin \theta \quad \text{or} \quad \ddot{\theta} = -\frac{L}{g} \theta \quad \text{for small angles}.$$  

The left differential equation is not easily solved. The right one corresponds to the small angle approximation (restriction on the size of the oscillation’s amplitude) and can be easily solved. These two models are explicit autonomous ODEs of order two.

### 2.2 Solution of an ODE

The resolution of an ordinary differential equation consists in finding a (regular) function which satisfies this equation as well as the initial conditions.

Let $I \subset \mathbb{R}$ an open interval and a continuous function $f : I \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \to \mathbb{R}$.

A solution of the explicit ordinary differential equation $y^{(n)}(t) = f(t, y(t), y'(t), \ldots, y^{(n-1)}(t))$ is a couple $(J, u)$ where $J \subset I$ is an open interval and $u : J \to \mathbb{R}$ is a function of class $C^n$ such that $u^{(n)}(t) = f(t, u(t), u'(t), \ldots, u^{(n-1)}(t))$, $\forall t \in J$.

Another solution $(\tilde{J}, \tilde{u})$ with $J \subset \tilde{J} \subset I$ and coinciding with $u$ on $J$ is called an extension of $(J, u)$.

A solution that has no extension is a maximal solution.

A general solution of an ordinary differential equation contains arbitrary independent constants of integration. A particular solution is derived from the general solution by setting these constants to particular values, usually chosen to fulfill a set of initial conditions.

### 2.3 The Cauchy problem

Let $I \subset \mathbb{R}$ an open interval and consider a continuous function $f : I \times \mathbb{R}^d \to \mathbb{R}^d$, $t_0 \in I$, $y_0 \in \mathbb{R}^d$.

The Cauchy problem associated with the condition $(t_0, y_0)$ consists in studying the problem

$$\begin{cases}
    y'(t) = f(t, y(t)), & t \in I \\
    y(t_0) = y_0
\end{cases}$$

(1)
This problem is also referred as the *Cauchy initial value problem*.

- A solution \((J, u)\) of this Cauchy problem is a \(C^1\) function \(u : J \rightarrow \mathbb{R}^d\), with \(t_0 \in J \subset I\) such that \(u'(t) = f(t, u(t)), \forall t \in J\) and \(u(t_0) = y_0\).

- With these hypothesis, if the function \(f : I \times \mathbb{R}^d \rightarrow \mathbb{R}^d\) is \(C^1\), the Cauchy problem admits a *unique* maximal solution.

In the rest of this document, we assume that \(d = 1\) (one-dimensional case), except in Appendix A.

- **Geometric interpretation.**
  The initial value problem can be interpreted geometrically as follows.

1. The function \(f\) assigns a non-zero vector with components \((1, f(t, y))\) to each point \((t, y)\) of the 2D plane. The set of these vectors form a vector field of the \((t, y)\) plane.

2. The graph of any solution \(t \in I \mapsto \hat{y}(t)\) of the differential equation \(y'(t) = f(t, y(t))\) is tangent to this vector field, which means that any point of the graph of \(\hat{y}\) admits the vector \((1, f(t, \hat{y}(t)))\) as a tangent vector.

3. Two different solutions can not cross, otherwise at such a crossing point, we would have 2 different vectors.

4. The solution to the initial value problem is the unique solution which goes through point \((t_0, y_0)\). See section 2.3 for uniqueness.

As an example, consider the following exercise.

**Exercise 3**

Consider the following vector field in the \((t, y)\) plane, where the graph of some solutions with initial conditions \((t_0, y_0)\) (marked by a circle) have been plotted. Complete manually (by hand) this figure with the graph of the solutions starting from points marked with a circle.

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4 Augustin-Louis Cauchy, 1789-1857, was a French mathematician, engineer and physicist.
Solution to exercise 3

3 Linear ODE

Linear differential equation.
An ordinary differential equation of order $n$ is said to be linear if it can be written as a linear
combination of the derivatives of the unknown function \(y\):

\[
y^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t) y^{(i)}(t) = r(t)
\]

where functions \(a_i(t)\) and \(r(t)\) are continuous. The function \(r(t)\) is called the second member of the linear ODE.

\(\diamond\) Homogeneous linear differential equation.
If the second member \(r(t) = 0\), the above linear differential equation is said to be homogeneous. In that case, the function \(y = 0\) is a solution of the linear differential equation.

If \(r(t) \neq 0\), the linear equation is said to be inhomogeneous.

\(\diamond\) We associate to an inhomogeneous linear differential equation \((E)\) its associated homogeneous linear equation \((H)\) by setting \(r(t) = 0\):

\[
y^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t) y^{(i)}(t) = r(t) \quad (E)
\]

\[
y^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t) y^{(i)}(t) = 0 \quad (H)
\]

\(\diamond\) Solutions of a linear ODE.

- Consider an homogeneous linear differential equation of order \(n\):

\[
y^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t) y^{(i)}(t) = 0.
\]

If \(u\) and \(v\) are two solutions on an interval \(I\), the function \(\lambda u + \mu v\) is another solution of this homogeneous linear ODE on the interval \(I\) for any real numbers \(\lambda\) and \(\mu\). In other words, the set of solutions of an homogeneous linear ODE on a given interval is a vector space.

- Consider an inhomogeneous linear differential equation of order \(n\):

\[
y^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t) y^{(i)}(t) = r(t).
\]

If \(u\) and \(v\) are two solutions of this ODE on an interval \(I\), the function \(u - v\) is a solution of the associate homogeneous linear ODE on the interval \(I\).

### 3.1 Linear differential equations of order 1

In this section we review analytic methods for solving linear differential equations of order 1. Precisely, we consider the following linear differential equation \((E_1)\) of order 1, together with its associated homogeneous differential equation \((H_1)\)

\[
y'(t) + a(t) y(t) = r(t) \quad (E_1)
\]

\[
y'(t) + a(t) y(t) = 0 \quad (H_1)
\]

where \(a(t)\) and \(r(t)\) are real valued continuous functions on a given interval \(I\).
Proposition 1

For any values \( t_0 \in I, y_0 \in \mathbb{R} \), the associated Cauchy problem

\[
\begin{align*}
    y'(t) &= -a(t) y(t) + r(t), \quad t \in I \\
    y(t_0) &= y_0
\end{align*}
\]

admits a unique maximal solution.

3.1.1 Resolution of the homogeneous equation \((H_1)\)

<table>
<thead>
<tr>
<th>Homogeneous linear ODE ((H_1))</th>
<th>General solution (y_H(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y'(t) + a(t) y(t) = 0)</td>
<td>(y_H(t) = C e^{-A(t)})</td>
</tr>
</tbody>
</table>

where \(A(t) = \int a(t) \, dt\) is any primitive integral of \(a(t)\) and \(C\) an arbitrary constant of integration.

3.1.2 Resolution of the complete equation \((E_1)\)

<table>
<thead>
<tr>
<th>Complete linear ODE ((E_1))</th>
<th>General solution (y_E(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y'(t) + a(t) y(t) = r(t))</td>
<td>(y_E(t) = y_0(t) + C e^{-A(t)})</td>
</tr>
</tbody>
</table>

where \(y_0(t)\) is any (particular) solution of \((E_1)\).

The problem is therefore to find any particular solution \(y_0(t)\) of the complete equation \((E_1)\). For this purpose, we distinguish two methods.

1. The variation of constant method leads to

\[
y_0(t) = \left( \int r(t) e^{A(t)} \, dt \right) e^{-A(t)}
\]

2. Another method that is often simpler and faster is to look for a particular solution \(y_0(t)\) which is in the form of the second member \(r(t)\).

3.1.3 The Cauchy problem

The resolution of the Cauchy problem \([2]\) consists in determining the unique value of the constant \(C\) such that \(y(t_0) = y_0\).
Exercise 4

Solve the following Cauchy problems.

1. \( y'(t) - 2y(t) = 4 \), \( y(0) = 0 \), \( t \in \mathbb{R} \)
2. \( y'(t) = \frac{y(t) + 1}{t} \), \( y(1) = 0 \), \( t > 0 \)
3. \( y'(t) - 2y(t) = 2t \), \( y(0) = \frac{1}{2} \), \( t \in \mathbb{R} \)
4. \( t^2 y'(t) - (2t - 1) y(t) = t^2 \), \( y(1) = 1 \), \( t > 0 \)
5. \( (t + 1) y'(t) - ty(t) + 1 = 0 \), \( y(0) = 2 \), \( t > -1 \)

Solution to exercise

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( y(t) = 2e^{2t} - 2 )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( y(t) = t - 1 )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( y(t) = \frac{3e^{2t} - 2t + 1}{2} )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( y(t) = t^2 )</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>( y(t) = \frac{e^t + 1}{t + 1} )</td>
<td></td>
</tr>
</tbody>
</table>

3.2 Linear differential equations of order 2 with constant coefficients

In this section we study analytic methods for solving linear differential equations of order 2 with constant coefficients.

Precisely, we consider the following linear differential equation \( E_2 \) of order 2, together with its associated homogeneous differential equation \( H_2 \)

\[
\begin{align*}
  y''(t) + by'(t) + cy(t) &= r(t) \\
  y''(t) + by'(t) + cy(t) &= 0
\end{align*}
\]

where \( b, c \) are two real constants and where the function \( r(t) \) is continuous on a given interval \( I \).

Proposition 2

For any values \( t_0 \in I, y_0, y'_0 \in \mathbb{R} \), the associated Cauchy problem

\[
\begin{align*}
  y''(t) &= -by'(t) - cy(t) + r(t), \quad t \in I \\
  y(t_0) &= y_0 \\
  y'(t_0) &= y'_0
\end{align*}
\]

admits a unique maximal solution.
3.2.1 Resolution of the homogeneous equation \((H_2)\)

<table>
<thead>
<tr>
<th>Homogeneous linear ODE ((H_2))</th>
<th>General solution (y_H(t))</th>
</tr>
</thead>
</table>
| \(y''(t) + b y'(t) + c y(t) = 0\) | Step 1: Resolution of the characteristic equation: 
  \[ r^2 + b r + c = 0 \]
  discriminant \(\Delta = b^2 - 4c\) |

<table>
<thead>
<tr>
<th>Step 2:</th>
</tr>
</thead>
</table>
| if \(\Delta > 0\), we have two distinct roots \(r_1, r_2\) 
  \(y_H(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}\) |
| if \(\Delta = 0\), we have a double root \(r_0\) 
  \(y_H(t) = (C_1 + C_2 t) e^{r_0 t}\) |
| if \(\Delta < 0\), we have two conjugate complex roots 
  \(r_1 = \alpha - i \omega, \quad r_2 = \alpha + i \omega\) 
  \(y_H(t) = (C_1 \cos(\omega t) + C_2 \sin(\omega t)) e^{\alpha t}\) |

where \(C_1, C_2\) are arbitrary constants of integration.

Notice that in each case, the general solution \(y_H(t)\) can be expressed as a linear combination of two independent solutions \(y_1(t)\) and \(y_2(t)\):

\[ y_H(t) = C_1 y_1(t) + C_2 y_2(t) \]

3.2.2 Resolution of the complete equation \((E_2)\)

<table>
<thead>
<tr>
<th>Complete linear ODE ((E_2))</th>
<th>General solution (y_E(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y''(t) + b y'(t) + c y(t) = r(t))</td>
<td>(y_E(t) = y_0(t) + C_1 y_1(t) + C_2 y_2(t))  [y_H(t)]</td>
</tr>
</tbody>
</table>

with the two independent solutions \(y_1(t)\) & \(y_2(t)\) of \((H_2)\) and where \(y_0(t)\) is any (particular) solution of \((E_2)\)

So, we are reduced to find any particular solution \(y_0(t)\) of the complete equation \((E_2)\). For this purpose, again we distinguish two methods.

1. The variation of constants method.
   We look for a particular solution \(y_0(t) = C_1(t) y_1(t) + C_2(t) y_2(t)\) where \(y_1(t)\) and \(y_2(t)\) are the two independent solutions of \((H_2)\) and such that
   \[ \begin{cases} 
   C_1'(t) y_1(t) + C_2'(t) y_2(t) = 0 \\
   C_1'(t) y_1'(t) + C_2'(t) y_2'(t) = r(t) 
   \end{cases} \]

   From this system, we deduce \(C_1'(t)\) and \(C_2'(t)\) and then \(C_1(t)\) and \(C_2(t)\) by integration.

2. But again, it is often simpler and faster to look for a particular solution \(y_0(t)\) which is in the form of the second member \(r(t)\).

3.2.3 The Cauchy problem

The resolution of the Cauchy problem consists in determining the unique values of the constants \(C_1\) and \(C_2\) such that \(y(t_0) = y_0\) and \(y'(t_0) = y'_0\).
Exercise 5

Solve the following Cauchy problems.

1. \( y''(x) - 3y'(x) + 2y(x) = 4x^2 \) with \( y(0) = 1 \) and \( y'(0) = 0 \)
   
   indication: look for a particular solution in the form \( y_0(x) = ax^2 + bx + c \)

2. \( y''(x) + 2y'(x) + y(x) = 4xe^x \) with \( y(0) = 1 \) and \( y'(0) = 2 \)

3. \( y''(x) + y(x) = \cos(x) \) with \( y(0) = 1 \) and \( y'(0) = 1 \)
   
   indication: look for a particular solution in the form \( y_0(x) = ax \sin(x) \)

4. \( y''(x) + y'(x) - 2y(x) = 9e^x \) with \( y(0) = 1 \) and \( y'(0) = 1 \)

5. \( y''(x) + y'(x) = x \) with \( y(0) = 0 \) and \( y'(0) = 2 \)

Solution to exercise 5

1. \( y_E(x) = 2x^2 + 6x + 7 + C_1e^x + C_2e^{2x} \)
   
   Cauchy conditions \( y(0) = 1 \) and \( y'(0) = 0 \) \( \Rightarrow \) \( C_1 = -6 \) and \( C_2 = 0 \)
   
   \[ y(x) = -6e^x + 2x^2 + 6x + 7 \]

2. \( y_H(x) = (C_1x + C_2)e^{-x} \)
   
   We look for a particular solution in the form \( y_0(x) = C_1(x)y_1(x) + C_2(x)y_2(x) \)
   
   with \( y_1(x) = xe^{-x} \) and \( y_2(x) = e^{-x} \), which leads to solve the system
   
   \[
   \begin{align*}
   C_1'(x)y_1(x) + C_2'(x)y_2(x) &= 0 \\
   C_1'(x)y_1(x) + C_2'(x)y_2(x) &= 0
   \end{align*}
   \]

   \[ \Rightarrow \]

   \[
   \begin{align*}
   C_1''(x) &= 4xe^{2x} \\
   C_2''(x) &= -4xe^{2x}
   \end{align*}
   \]

   \[ \Rightarrow \]

   \[
   \begin{align*}
   C_1(x) &= (2x-1)e^{2x} \\
   C_2(x) &= (-2x^2 + 2x - 1)e^{2x}
   \end{align*}
   \]

   \[ \Rightarrow \]

   \[
   \begin{align*}
   y_0(x) &= C_1(x)y_1(x) + C_2(x)y_2(x) = (2x-1)e^{2x}xe^{-x} + (-2x^2 + 2x - 1)e^{2x}e^{-x} \\
   y_E(x) &= (x-1)e^x + (C_1x + C_2)e^{-x}
   \end{align*}
   \]

   Cauchy conditions \( y(0) = 1 \) and \( y'(0) = 2 \) \( \Rightarrow \) \( C_1 = 4 \) and \( C_2 = 2 \)

   \[ y(x) = (x-1)e^x + (4x + 2)e^{-x} \]

3. \( \Delta = -4, r = \pm i \) \( \Rightarrow \) \( y_H(x) = C_1 \cos(x) + C_2 \sin(x) \)

   \[ \frac{x}{2} \sin(x) + C_1 \cos(x) + C_2 \sin(x) \]

   Cauchy conditions \( y(0) = 1 \) and \( y'(0) = 1 \) \( \Rightarrow \) \( C_1 = 1 \) and \( C_2 = 1 \)

   \[ y(x) = \frac{x}{2} \sin(x) + \cos(x) + \sin(x) \]

4. \( y_H(x) = C_1e^{-2x} + C_2e^x \)
   
   We look for a particular solution in the form \( y_0(x) = C_1(x)y_1(x) + C_2(x)y_2(x) \)
   
   with \( y_1(x) = e^{-2x} \) and \( y_2(x) = e^x \), which leads to solve the system
   
   \[
   \begin{align*}
   C_1'(x)y_1(x) + C_2'(x)y_2(x) &= 0 \\
   C_1'(x)y_1(x) + C_2'(x)y_2(x) &= r(x)
   \end{align*}
   \]

   \[ \Rightarrow \]

   \[
   \begin{align*}
   C_1''(x) &= -3e^{3x} \\
   C_2''(x) &= 3
   \end{align*}
   \]

   \[ \Rightarrow \]

   \[
   \begin{align*}
   C_1(x) &= -e^{3x} \\
   C_2(x) &= 3x
   \end{align*}
   \]

   \[ \Rightarrow \]

   \[
   \begin{align*}
   y_0(x) &= -3e^{3x}e^{-2x} + 3xe^x = (3x - 1)e^x \\
   y_E(x) &= (3x - 1)e^x + C_1e^{-2x} + C_2e^x
   \end{align*}
   \]
Cauchy conditions $y(0) = 1$ and $y'(0) = 1 \implies C_1 = 1$ and $C_2 = 1$

$y(x) = (3x - 1)e^x + e^{-2x} + e^x$

5. This ODE can be reduced to a differential equation of order 1 with the change of function $z(x) = y'(x)$ so that $y''(x) + y'(x) = x \iff z'(x) + z(x) = x$

The general solution of the equation $z'(x) + z(x) = x$ is

$z_E(x) = x - 1 + C_1 e^{-x}$

The general solution of the equation $y''(x) + y'(x) = x$ is

$y_E(x) = \frac{x^2}{2} - x + C_2 - C_1 e^{-x}$

Cauchy conditions $y(0) = 0$ and $y'(0) = 2 \implies C_1 = 3$ and $C_2 = 3$

$y(x) = \frac{x^2}{2} - x + 3 - 3 e^{-x}$
4 Numerical methods for ODEs

In this section, we will consider the Taylor-Lagrange development (see also appendix B). Let \( f : [a, b] \to \mathbb{R} \) a function of class \( C^{n+1} \). Then, for any value \( h \) such that \( a \leq a + h \leq b \), we have

\[
f(a + h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f^{(3)}(a) + \cdots + \frac{h^n}{n!} f^{(n)}(a) + O(h^{n+1})
\] (4)

4.1 Objective

We consider the question of determining a numerical approximation of the solution of the following Cauchy (or Cauchy-Lipschitz) problem

\[
\begin{align*}
y'(t) &= f(t, y(t)), \quad t \in I \\
y(t_0) &= y_0
\end{align*}
\] (5)

where \( f : I \times \mathbb{R} \to \mathbb{R} \), \( f \) is a given continuous function, locally Lipschitz continuous with respect to the second variable \( x \), and with \( t_0 \in I \), \( y_0 \in \mathbb{R} \).

So, we must specify what is a numerical approximation of the solution.

1. The solution \( y(t) \) is determined for a finite number of \( t \)-values : \( t_0, t_1, \ldots, t_i, \ldots, t_N \).

2. Each value \( y(t_i) \) of the solution function at time \( t_i \) is approximated by \( y_i \) \((y_i \approx y(t_i))\).

3. \( h_i = t_i - t_{i-1}, \ i = 1, 2, \ldots, N, \) is the \( i \)-th time step of the discretization.

   Usually, the time step is constant : \( h_i = h, \ i = 1, 2, \ldots, N \).

In this chapter,

- the time step is assumed to be constant equal to \( h \), so that \( t_i = t_{i-1} + h \) for all \( i \).

- we assume that the solution \( y(t) \) is sufficiently regular so as to apply the Taylor-Lagrange formula.

4.2 Error

The error breaks down into two parts.

1. A local consistency error produced by the difference between the exact value \( y(t_i) \) and its approximation \( y_i \). This local error is due to the discretization of the differential equation.

2. A stability error due to the error propagation from one time step to another.

A numerical method is said to converge with order \( p \) (or is a method of order \( p \)) if

\[
\max_{1 \leq i \leq N} |y(t_i) - y_i| = O(h^p)
\]

Usually, the order of convergence of a method is one less than its local error.
4.3 Euler method

Assume we know the value \( y(t_{i-1}) \), then the value \( y(t_i) = y(t_{i-1} + h) \) can be deduced from the Taylor-Lagrange formula

\[
y(t_i) = y(t_{i-1}) + h y'(t_{i-1}) + \frac{h^2}{2} y''(t_{i-1}) + O(h^3).
\]

For a small value of \( h \) we neglect second order terms, so that

\[
y(t_i) \approx y(t_{i-1}) + h y'(t_{i-1}) \approx y(t_{i-1}) + h f(t_{i-1}, y(t_{i-1}))
\]

Then, with the approximation \( y_{i-1} \approx y(t_{i-1}) \) we get an approximation \( y_i \) for \( y(t_i) \)

\[
y_i = y_{i-1} + h f(t_{i-1}, y_{i-1})
\]

leading to the explicit Euler Method.

Explicit Euler method:

\[
\text{Input} : \text{ the function } f: [t_0, t_N] \times \mathbb{R} \to \mathbb{R} \\
y_0 \in \mathbb{R}, \ h > 0 \\
\text{Body} : \ y_i = y_{i-1} + h f(t_{i-1}, y_{i-1}) \quad i = 1, 2, \ldots, N
\]

This method is said to be explicit as \( y_i \) is an explicit function of \( y_{i-1} \) and \( t_{i-1} \). It is a one-step method because \( y_i \) is determined from the previous time data \( t_{i-1} \).

Leonhard Euler (1707 – 1783) was a Swiss mathematician, physicist, astronomer, logician and engineer.
We now state the local error of this Euler method. With the Taylor-Lagrange formula and the approximation (7), we have

\[
|y(t_i) - y_i| = \left| (y(t_{i-1}) + hy'(t_{i-1}) + O(h^2)) - \left(y_{i-1} + hf(t_{i-1}, y_{i-1})\right) \right| = O(h^2)
\]
if we assume that \(y_{i-1} = y(t_{i-1})\).

Consequently, the Euler method is of order 1.

The derivative can be evaluated at the end of the interval \([t_{i-1}, t_i]\), leading to the approximation

\[
y_i = y_{i-1} + hf(t_i, y_i)
\]
referred as the implicit Euler method. The unknown data \(y_i\) appears in both sides of this equation, requiring thus to solve an equation.

### 4.4 Linear case of order 1 and 2

1. **Linear ODE of order 1.**

   We consider the Cauchy problem stated in Proposition 1 with notations of section 3.1.

   \[
   \begin{align*}
   y'(t) &= -a(t) y(t) + r(t), \quad t \in I \\
   y(t_0) &= y_0
   \end{align*}
   \]

   The Euler method applies as follows.

   | Input | the two functions \(a, r : [t_0, t_N] \times \mathbb{R} \to \mathbb{R}\)  
   |       | \(y_0 \in \mathbb{R}\)  
   |       | \(h = \frac{t_N - t_0}{N} > 0\)  
   | Body  | \(y_i = y_{i-1} + h \left(r(t_{i-1}) - a(t_{i-1}) y_{i-1}\right)\) \(i = 1, 2, \ldots, N\)

2. **Linear ODE of order 2 with constant coefficients.**

   We consider the Cauchy problem stated in Proposition 2 with notations of section 3.2

   \[
   \begin{align*}
   y''(t) &= -by'(t) - cy(t) + r(t), \quad t \in I \\
   y(t_0) &= y_0 \\
   y'(t_0) &= y'_0
   \end{align*}
   \]

   This Cauchy problem of order 2 and dimension 1 can be rewritten as a vector Cauchy problem of order 1 and dimension 2 as follows

   \[
   \begin{align*}
   \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}' &= \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} + \begin{pmatrix} 0 \\ r(t) \end{pmatrix}, \quad t \in I \\
   \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix} &= \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}
   \end{align*}
   \]

   that we note in vector form

   \[
   \begin{align*}
   Y''(t) &= F(t, Y(t)), \quad t \in I \\
   Y(t_0) &= Y_0
   \end{align*}
   \]

   by introducing
- the vector function $Y : t \mapsto Y(t) = (y(t), y'(t))^T \in \mathbb{R}^2$,
- the 2D vector $Y_0 = (y_0, y'_0)^T \in \mathbb{R}^2$,
- and the vector function $F : I \times \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$(t, X) \mapsto F(t, X) = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} X + \begin{pmatrix} 0 \\ r(t) \end{pmatrix}$$

Then, denoting $z(t) = y'(t)$, and considering the following approximations at time $t_i$

$$y_i \approx y(t_i), \quad z_i \approx z(t_i) = y'(t_i), \quad Y_i = (y_i, z_i)^T \approx Y(t_i),$$

the Euler method leads to

$$Y_i = Y_{i-1} + h F(t_{i-1}, Y_{i-1}), \quad i = 1, 2, \ldots, N$$
and applies as follows.

| Input : the function $r : [t_0, t_N] \times \mathbb{R} \to \mathbb{R}$
| $b, c \in \mathbb{R}$
| $y_0, z_0 = y'_0 \in \mathbb{R}$
| $h = \frac{t_N - t_0}{N} > 0$
| Body : for $i = 1, 2, \ldots, N$
| $y_i = y_{i-1} + h z_{i-1}$
| $z_i = z_{i-1} + h (r(t_{i-1}) - c y_{i-1} - b z_{i-1})$

### 4.5 Implementation

#### 4.5.1 Order 1

**Exercise 6**

Complete the following scilab function so as to implement the Euler method for linear ODE of order 1. Note that this scilab function plots the exact solution defined as $S(t)$.

```scilab
// file EulerMethods.sci

function Euler1(a, r, tmin, tmax, yy0, S, n)
    // S is the exact solution
    h = (tmax - tmin)/n;
    // Euler solution :
    tt(1) = tmin;
    yy(1) = yy0;
    ......

    plot(tt, yy, 'r')
    // Exact solution :
    t = linspace(tmin, tmax, 500);
    plot(t, S(t), 'g')
endfunction
```
Solution to exercise 6

Complete the following scilab function so as to implement the Euler method for linear ODE of order 2. Note that this scilab function plots the exact solution defined as $S(t)$.

```scilab
// file EulerMethods.sci

function Euler1(a,r,tmin,tmax,yy0,S,n)
    // S is the exact solution
    h = (tmax - tmin)/n;
    // Euler solution :
    tt(1) = tmin;
    yy(1) = yy0;
    for k = 1 : n
        tt(k+1) = tt(k) + h;
        yy(k+1) = yy(k) + h * ( r(tt(k)) - a(tt(k)) * yy(k) );
    end
    scf()
    plot(tt,yy,'r')
    // Exact solution :
    t = linspace(tmin,tmax,500);
    plot(t,S(t),'g')
endfunction
```

## 4.5.2 Order 2

**Exercise 7**

Complete the following scilab function so as to implement the Euler method for linear ODE of order 2. Note that this scilab function plots the exact solution defined as $S(t)$.

```scilab
// file EulerMethods.sci

function Euler2(b,c,r,tmin,tmax,yy0,zz0,S,n)
    // S is the exact solution
    h = (tmax - tmin)/n;
    // Euler solution :
    tt(1) = tmin
    yy(1) = yy0;
    zz(1) = zz0;
    ......
    plot(tt,yy,'r')
    // Exact solution :
    t = linspace(tmin,tmax,500);
    plot(t,S(t),'g')
endfunction
```
Solution to exercise 7

```scilab
// file EulerMethods.sci

// Equation y''(t) + b y'(t) + c y(t) = r(t)
// t \in [tmin, tmax]
// y(tmin) = yy0
// y'(tmin) = zz0

function Euler2(b, c, r, tmin, tmax, yy0, zz0, S, n)
    // S is the exact solution
    h = (tmax - tmin)/n;
    // Euler solution :
    tt(1) = tmin;
    yy(1) = yy0;
    zz(1) = zz0;
    for k = 1 : n
        tt(k+1) = tt(k) + h;
        yy(k+1) = yy(k) + h * zz(k);
        zz(k+1) = zz(k) + h * (r(tt(k)) - b * zz(k) - c * yy(k));
    end
    scf();
    plot(tt, yy, 'r')
    // Exact solution :
    t = linspace(tmin, tmax, 500);
    plot(t, S(t), 'g')
endfunction
```

4.5.3 Examples

We consider in this section the following Cauchy problems involving linear ordinary differential equations of order 1 and 2.

<table>
<thead>
<tr>
<th>Order 1</th>
<th>Order 2</th>
</tr>
</thead>
</table>
| \[\begin{align*}
  y'(t) &= \cos(t), & t \in [0, 10\pi] \\
  y(0) &= 0
  \end{align*}\] | \[\begin{align*}
  y''(t) + 2 y'(t) + 26 y(t) &= 0, & t \in [0, 2\pi] \\
  y(0) &= 5 \\
  y'(0) &= 0
  \end{align*}\] |
| \[\begin{align*}
  y'(t) + y(t) &= 3 e^{-t}, & t \in [0, 5] \\
  y(0) &= 1
  \end{align*}\] | \[\begin{align*}
  y''(t) + y(t) &= t/2, & t \in [0, 10\pi] \\
  y(0) &= 1 \\
  y'(0) &= -1/2
  \end{align*}\] |
| \[\begin{align*}
  y'(t) + y(t) &= \frac{1}{5} (\sin(5t) + 5 \cos(5t)), & t \in [0, \frac{3\pi}{2}] \\
  y(0) &= 2
  \end{align*}\] | \[\begin{align*}
  y''(t) - y(t) &= 0, & t \in [-1, 1] \\
  y(0) &= e + 1/e \\
  y'(0) &= -e + 1/e
  \end{align*}\] |

Exercise 8

1. Solve analytically the previous linear ODEs of order 1 and order 2.

2. Use the previous scilab functions to determine the Euler solution of these linear ODE for different values of the integer $n$ as shown in the figures below. On each case, plot on
the same figure the exact analytical solution together with the Euler approximation.
As an example, we give the script for the second linear ODE of order 1.

3. Explain the “shape” of the Euler solutions for \( n = 5, n = 10, n = 20 \) in the case of the linear ODE of order one : \( y' = \cos(t) \). In particular, why do we get a straight line for \( n = 5 \) ?

```scilab
// file EulerTestsOrder1.sce
exec("EulerMethods.sci", -1);

// Equation y' + y = 3 * exp(-t)
deff("y = a(t)","y = 1");
deff("y = r(t)","y = 3 * exp(-t)");
deff("y = S(t)","y = (3 * t + 1) .* exp(-t)");
tmin = 0;
tmax = 5;
yy0 = 1;
n = 20;
Euler1(a,r,tmin,tmax,yy0,S,n)
ntext = 'n = ' + string(n);
h = legend([ntext ; 'exact solution']);

....... Solution to exercise 8 — Order 1 :

// file EulerTestsOrder1.sce
exec("EulerMethods.sci", -1);

// Equation y'(t) + a(t) y(t) = r(t)
// t \in [tmin,tmax]
// y(tmin) = yy0

// Equation y' = \cos(t)
deff("y = a(t)","y = 0");
deff("y = r(t)","y = \cos(t)");
deff("y = S(t)","y = \sin(t)");
tmin = 0;
tmax = 10 * %pi;
yy0 = 0;
n = 10;
Euler1(a,r,tmin,tmax,yy0,S,n)
ntext = 'n = ' + string(n);
h = legend([ntext ; 'exact solution']);

// Equation y' + y = 3 * exp(-t)
deff("y = a(t)","y = 1");
deff("y = r(t)","y = 3 * exp(-t)");
deff("y = S(t)","y = (3 * t + 1) .* exp(-t)");
tmin = 0;
tmax = 5;
yy0 = 1;
n = 20;
Euler1(a,r,tmin,tmax,yy0,S,n)
```
Solution to exercise 8 — Order 2:

```scilab
// file EulerTestsOrder2.sce
exec("EulerMethods.sci", -1);

// Equation y'' + b y' + c y = r(t) in [tmin, tmax]
// y(tmin) = yy0
// y'(tmin) = zz0

// Equation y'' + 2 y' + 26 y = 0
b = 2;
c = 26;
deff("y = r(t)";"y = 0");
deff("y = S(t)";"y = (sin(5*t) + 5*cos(5*t)) .* exp(-t)");
tmin = 0;
tmax = 2 * %pi;
yy0 = 5;
zz0 = 0;
Euler2(b, c, r, tmin, tmax, yy0, zz0, S, n)
n = 100;
teXct = 'n = ' + string(n);
h = legend([nteXct; 'exact solution']);
clear tt
clear yy
clear zz

// Equation y'' + y = t/2
b = 0;
c = 1;
deff("y = r(t)";"y = t / 2");
deff("y = S(t)";"y = t / 2 - sin(t) + cos(t)");
tmin = 0;
tmax = 10 * %pi;
yy0 = 1;
zz0 = -1/2;
n = 1000;
Euler2(b, c, r, tmin, tmax, yy0, zz0, S, n)
teXct = 'n = ' + string(n);
h = legend([nteXct; 'exact solution']);
```


Solution to exercise 8 — Order 2:
// Equation y'' - y = 0
b = 0;
c = -1;
deff("y = r(t)","y = 0");
deff("y = S(t)","y = 2 * cosh(t)");
tmin = -1;
tmax = 1;
yy0 = %e + 1 / %e;
zz0 = -%e + 1 / %e;
n = 100;
Euler2(b,c,r,tmin,tmax,yy0,zz0,S,n)
ntext = 'n = ' + string(n);
h = legend([ntext;'exact solution']);

Order 1: \( y'(t) = \cos(t) \)

Order 1: \( y'(t) + y(t) = 3e^{-t} \)

Order 1: \( y'(t) + y(t) = \frac{1}{5}(\sin(5t) + 5\cos(5t)) \)
Order 2: \( y''(t) + 2y'(t) + 26y(t) = 0 \)

Order 2: \( y''(t) + y(t) = \frac{t}{2} \)

Order 2: \( y''(t) - y(t) = 0 \)
The `ode()` function allows to solve numerically an ordinary differential equation of order one with initial condition as defined in (??).

- The Scilab built-in function `ode()` solves the explicit Ordinary Differential Equations defined by equations (??), where `t0` is the initial time, `y0` is the initial value at time `t0`, `t` is the vector of times at which the solution `y` is computed and `y` is the solution vector `y=[y(t(1)),y(t(2)),...]

  - `y0` : a real value: initial state at time `t0`
  - `t0` : a real scalar, the initial time
  - `t` : vector `[t(1),t(2),...,t(N)]`, the times at which the solution is computed.
  - `f` : a function, the right hand side of the differential equation (??)
  - `y` : solution vector `y=[y(t(1)),y(t(2)),...,y(t(N))]`.

- In the following exercise, we consider the autonomous differential equation `\dot{y} = f(t,y) = C - ay` with `C = a = 5` and with general solution `y(t) = C/a + Ke^{(-at)}`, `K \in \mathbb{R}`.

We first plot the vector field `(t,y) \mapsto (1,f(t,y))` associated with this differential equation.

Then, we determine and plot 4 solutions of the Cauchy problem with initial condition `y(t_0) = y_0`.

**Exercise 9**

Experiment the following script and complete it so as to plot the four solutions as shown in the figure.

```scilab
// file FigchampEx2.sce

a = 5; C = 5;
def('yp = f(t,y)', 'yp=C-a*y');

// Vector field
Nt = 15;
Ny = 15;
t = linspace(0,2,Nt);
y = linspace(-1,2,Ny);
Ft = ones(Nt,Ny);
Fy = Ft;
for i = 1 : Nt
  for j = 1 : Ny
    Fy(i,j) = f(t(i),y(j));
  end
end
champ(t,y,Ft,Fy);

// SOME SOLUTIONS :
t0 = 0.5; y0 = -0.9; // red
//t0 = 0.2; y0 = 1.9; // green
//t0 = 1; y0 = 1.9; // blue
//t0 = 0; y0 = -0.8; // blue
plot(t0,y0,'or','markersize',10)

// SOME SOLUTIONS :
t0 = 0.5; y0 = -0.9; // red
//t0 = 0.2; y0 = 1.9; // green
//t0 = 1; y0 = 1.9; // blue
//t0 = 0; y0 = -0.8; // blue
plot(t0,y0,'or','markersize',10)

// SOME SOLUTIONS :
t0 = 0.5; y0 = -0.9; // red
//t0 = 0.2; y0 = 1.9; // green
//t0 = 1; y0 = 1.9; // blue
//t0 = 0; y0 = -0.8; // blue
plot(t0,y0,'or','markersize',10)
```

- In the following exercise we consider the logistic growth (or Verhulst law) presented as an example in section 2.1.
Exercise 10

Modify the script file FigchampEx2.sce of the previous exercise so as to reproduce the figure at the right associated with the Cauchy problem

\[
\begin{align*}
    y'(t) &= f(t, y(t)) = r \left(1 - \frac{y}{K}\right) y \\
    y(t_0) &= y_0
\end{align*}
\]

with \( r = 1, \ K = 12, \ t_0 = 0, \ y_0 = 1, \)

and with \( t \in [0, 8] \) and \( y \in [0, 15] \).

Solution to exercise

```matlab
// file FigchampEx3.sce
r = 1;
K = 12;
def('yp = fV(t,y)', 'yp = r * (1 - y / K) .* y');

// Domain :
tmax = 8; ymax = 15;
Nt = 15; Ny = 15;
t = linspace(0, tmax, Nt);
y = linspace(0, ymax, Ny);

// Vector field
Ft = ones(Nt, Ny);
Fy = Ft;
for i = 1 : Nt
    for j = 1 : Ny
        Fy(i, j) = fV(t(i), y(j));
    end
end
champ(t, y, Ft, Fy);

// Solution
t0 = 0; y0 = 1; // blue
plot(t0, y0, 'ob', 'markersize', 12)
tt = linspace(t0, tmax, 400);
yy = ode(y0, t0, tt, fV);
plot(tt, yy, 'b');
```

```matlab```
6 Appendix A : Cauchy problem in two dimension

Notice that in the one dimensional case, the vector field associated with the differential equation
\[ y'(t) = f(t, y(t)) \] is on the form \((t, y(t)) \mapsto (1, f(t, y))\), that is the first component is equal to one.

Consider now the Cauchy initial value problem in two dimensions
\[
\begin{align*}
Y'(t) &= f(t, Y(t)), & t \in I \\
Y(t_0) &= Y_0 & t_0 \in I
\end{align*}
\]
(8)

with a continuous function \( f : I \times \mathbb{R}^2 \to \mathbb{R}^2 \) and \( Y_0 = (x_0, y_0) \in \mathbb{R}^2 \).

A solution of this Cauchy problem is a \( C^1 \) vector function \( Y : t \in J \subset I \mapsto Y(t) = (x(t), y(t)) \) such that \( Y(t_0) = (x_0, y_0) \).

Such a solution is geometrically a 2D parametric curve which admits at each point \( Y(t) = (x(t), y(t)) \) the tangent vector \((x'(t), y'(t)) = f(t, (x(t), y(t)))\).

As in the one-dimensional case, we now consider some examples of implementation with Scilab. For this purpose we will use the builtin function \texttt{fchamp()} and again the builtin function \texttt{ode()}

\begin{itemize}
    \item \texttt{fchamp(f, t, x, y)} draws the direction field of a 2D first order ODE defined by the external function \( f \).

        \begin{itemize}
            \item \( x \) : vector \([x(1),\ldots,x(i),\ldots,x(m)]\)
            \item \( y \) : vector \([y(1),\ldots,y(j),\ldots,y(n)]\)
            \item \( t \) : the selected time
            \item \( f \) : function of type \( v = f(t, xy) \) which returns a column vector \( v = [vx, vy] \) of size 2 which gives the value of the direction field \( f \) at point \( xy = [x,y] \) and at time \( t \). Note that even if argument \( t \) is useless, it must be given.
        \end{itemize}

    \item \texttt{Y = ode(Y0, t0, t, f)} solves the explicit Ordinary Different Equations defined by equations \([8]\), where \( Y_0 \) is the vector of initial conditions, \( t_0 \) is the initial time, \( t \) is the vector of times at which the solution \( Y \) is computed and \( Y = [x, y] \) is matrix of solution vectors \( Y = [Y(t(1)),Y(t(2)),\ldots] \).

        \begin{itemize}
            \item \( Y_0 \) : a real vector or matrix: initial state at time \( t_0 \)
            \item \( t_0 \) : a real scalar, the initial time
            \item \( t \) : vector \([t(1),\ldots,t(N)]\), the times at which the solution is computed.
            \item \( f \) : a function, the right hand side of the differential equation \([8]\)
            \item \( Y = [x, y] \) : matrix of solution vectors \( Y = [Y(t(1)),\ldots,Y(t(N))] \), with
                \begin{align*}
                    Y(1,:) &= [x(t(1)),\ldots,x(t(N))] \\
                    Y(2,:) &= [y(t(1)),\ldots,y(t(N))]
                \end{align*}
        \end{itemize}
\end{itemize}

\begin{itemize}
    \item A first example.
    \end{itemize}

\textbf{Exercise 11}

\begin{itemize}
    \item Experiment the following scilab script and complete it so as to draw additional solutions, as shown in the figure below.
\end{itemize}
function y = f(t,p)  
    y=[p(2), sin(p(1))- cos(p(2))*p(1)]
endfunction
x = -5:0.8:5;
y = -4:0.8:4;
fchamp(f,0,x,y);

// two solutions :
t = 0:0.01:20;
x0 = -5; y0 = 1;
y = ode([x0;y0],0,t,f);
plot(y(1,:),y(2,:),'r');
t1 = 0:0.01:6;
x0 = -5; y0 = 4;
y = ode([x0;y0],0,t1,f);
plot(y(1,:),y(2,:),'b');

Vector field $F(x,y) = (y, \sin(x) - x \cos(y))$
with some solutions (parametric curves).

- Influence of the initial condition.

**Exercise 12**

Experiment the following script and add the solution with initial value $x_0 = -5; y_0 = 3.7$
7 Appendix B : Taylor formulas

We recall here some Taylor formulas.

⋄ The Taylor-Laplace formula (Taylor with integral remainder).
Let \( f : [a, b] \rightarrow \mathbb{R} \) a function of class \( C^{n+1} \). Then, \( \forall x \in [a, b] \),

\[
f(x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + \frac{1}{n!} \int_{a}^{x} (x-t)^n f^{(n+1)}(t) \, dt. \tag{9}
\]

The proof comes by induction and integrating by part the integral remainder
\[
R_n(f) = \frac{1}{n!} \int_{a}^{x} (x-t)^n f^{(n+1)}(t) \, dt.
\]

⋄ The Taylor-Lagrange formula.
Let \( f : [a, b] \rightarrow \mathbb{R} \) a function of class \( C^{n+1} \). Then, \( \forall x \in [a, b] \), there exists \( c \in ]a, x[ \) such that

\[
f(x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c). \tag{10}
\]

⋄ The Taylor-Young formula.
Let \( f \) a function of class \( C^n \) on an interval \( I \). Let \( a \in I \). Then there exists a function \( \epsilon : I \rightarrow \mathbb{R} \) with \( \lim_{x \rightarrow a} \epsilon(x) = 0 \), such that for any \( x \in I \),

\[
f(x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + (x-a)^n \epsilon(x). \tag{11}
\]